Doi: 10.1007/s11589-010-0739-1

Equipartitioning is not sufficient for Green's function extraction*

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Abstract The extraction of the Earth's Green's function from field fluctuations is a rapidly growing area of research. The principle of Green's function extraction is often related to the requirement of equipartitioning, which stipulates that the energy of field fluctuations is distributed evenly in some sense. We show the meaning of equipartitioning for a variety of different formulations for Green's function retrieval. We show that equipartitioning is not a sufficient condition, and provide several examples that illustrate this point. We discuss the implications of lack of equipartitioning for various schemes for the reconstruction of the Green's function in seismology. The theory for Green's function extraction is usually based on a statistical theory that relies on ensemble averages. Since there is only one Earth, one usually replaces the ensemble average with a time average. We show that such a replacement only makes sense when attenuation is taken into account, and show how the theory for Green's function extraction for oscillating systems can be extended to incorporate attenuation.

Key words: **seismic interferometry; Green's function retrieval; equipartitioning**
CLC number: P315.01 **Document code**: A

1 Introduction

The extraction of the elastic Earth response from field fluctuations is an area of research that has spectacularly grown over the last decade (Larose et al., 2006; Curtis et al., 2006; Wapenaar et al., 2008; Schuster, 2009). This line of research was spurred to a large extent by the seminal work of Lobkis and Weaver (2001). In seismology, the principle of Green's function retrieval from field fluctuations is known by other names that include *Green's function extraction* and *seismic interferometry*.

The principle of Green's function retrieval relies on various formulations of wave theory that all require in one form or another field fluctuations to be "evenly" distributed in space. As we show in section 2, the precise meaning of this requirement differs for different formulations, but in all cases the required distribution of sources implies that the energy of the field fluctuations is evenly distributed in space. Using classical mechanics terminology, this principle is referred to as *equipartitioning* (Goldstein, 1980). Since the energy of strongly scattered waves diffuses through space, the expression *diffuse waves* is also used (e.g., Campillo and Paul, 2003; Malcolm et al., 2004; Weaver and Lobkis, 2006; Sánchez-Sesma et al., 2008). The concepts of equipartitioning and diffuse waves have received so much attention that one might think that the Green's function can be retrieved whenever the wavefield is equipartitioned, but we show in this work that this is not the case.

In section 2 we review different formulations of equipartitioning and discuss what the requirement of equipartitioning means for the distribution of sources of field fluctuations. We provide in section 3 a number of examples of equipartitioned fields that do not allow for Green's function extraction. We discuss in sections 4 and 5 the implications of a lack of adequate sources of field fluctuations for Green's function retrieval in the Earth, and argue that attenuation plays an essential role in practical Green's function extraction of the Earth response. In appendix A we show how attenuation can be incorporated in Green's function retrieval of damped oscillating systems.

* **Received** 15 April 2010; **accepted in revised form** 19 July 2010; **published** 10 October 2010.
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2 Various requirements for equipartitioning

The retrieval of the Green's function has been linked to the equipartitioning of waves (e.g., Lobkis and Weaver, 2001; Campillo and Paul, 2003; Snieder et al., 2007), which sometimes is referred to as the wavefield being *diffuse*. The latter term indicates that the waves are strongly scattered and propagate with equal strength in each direction. In different studies, the equipartitioning requirement is presented in different ways. In the next subsections we review the meaning of equipartitioning as presented in different studies.

2.1 An acoustic source

The first explanation of equipartitioning, as given by Snieder et al. (2007), is heuristic. Consider an explosive point source in an acoustic medium at location *A*, as shown in Figure 1a. The medium can be inhomogeneous, but if the medium is locally homogeneous in the vicinity of the source, the waves radiate isotropically as shown in Figure 1a. Because of the isotropic character of the source, the waves propagate towards points *B* and *C* with equal amplitude.

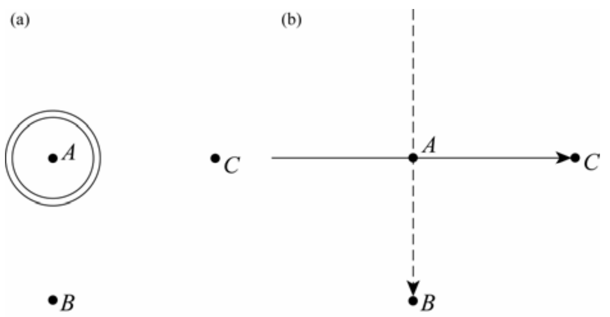

Figure 1 (a) A point source in a homogeneous acoustic medium at point *A* that emits equal amounts of energy toward points *B* and *C*; (b) Equal energy transport along the dashed and solid arrows is needed to retrieve the Green's function for the propagation from the source point *A* to the points *B* and *C*.

Consider next the situation shown in Figure 1b where the source at location *A* is replaced by a receiver acting as a virtual source. The waves radiated toward points *B* and *C* can be retrieved by cross-correlating the waves recorded at *A* with the waves recorded at locations *B* and *C*, respectively. In this situation one can only hope to retrieve the waves propagating to point *B* if there is a physical wave propagating from *A* to *B*, as indicated by the dashed arrow in Figure 1b, because no amount of data processing can produce a wave that does not propagate through the physical system. Similarly, a wave propagating along the solid arrow must be present to give, after cross-correlation, the wave propagating from *A* to *C*. Because in a real medium the waves propagating from an isotropic source at point *A* towards *B* and *C* have equal amplitude, the waves obtained from cross-correlation in Figure 1b must also propagate from point *A* in all directions with the same amplitude if the true Green's function is to be retrieved. This means that the wave propagating along the dashed and solid arrows in Figure 1b must have the same intensity. This only happens when the energy of the waves propagating through the point *A* is independent of direction. In other words, the wavefield must be equipartitioned in the sense that the energy propagation is independent of direction.

2.2 Normal modes

The theory and practice of Green's function extraction was spurred by the seminal work of Lobkis and Weaver (2001). Their derivation applies to any closed and undamped scalar system that has normal modes $u_n(\boldsymbol{r})$ that are not degenerate, but the theory can be extended to include degenerate modes (Tsai, 2010). We review their derivation because the role of equipartitioning, which in this case stipulates that different normal modes carry equal energy, is particularly clear. The Green's function can be expressed in the normal modes (Snieder, 2004b) as

$$G(\boldsymbol{r},\boldsymbol{r}',t)=\sum_m \frac{u_m(\boldsymbol{r})u_m(\boldsymbol{r}')}{\omega_m}\sin(\omega_m t)H(t), \tag{1}$$

where ω_m is the angular frequency of mode m and $H(t)$ is the Heaviside function.

We consider a state of the wavefield where the modal coefficients a_m and b_m are random variables:

$$u(\boldsymbol{r},t)=\sum_m \frac{1}{\omega_m}(a_m\cos\omega_m t+b_m\sin\omega_m t)u_m(\boldsymbol{r}). \tag{2}$$

The modal coefficients are assumed to have zero mean

$$\langle a_m\rangle=\langle b_m\rangle=0, \tag{3}$$

and covariance given by

$$\langle a_n a_m\rangle=\langle b_n b_m\rangle=S^2\delta_{nm} \quad \text{and} \quad \langle a_n b_m\rangle=0. \tag{4}$$

In this work $\langle\cdots\rangle$ denotes the expectation value. The presence of the $1/\omega_m$ term in expression (2) can be understood as follows. The time derivative of equation (2) is given by $\dot{u}(\boldsymbol{r},t)=\sum_m(-a_m\sin\omega_m t+b_m\cos\omega_m t)u_m(\boldsymbol{r})$. From condition (4) and the fact that the modes are normalized, it follows that the different modes in equation

(2) have equal kinetic energy. Because the kinetic energy of a normal mode of a linear system is equal to the potential energy (Goldstein, 1980), condition (4) implies that the normal modes carry equal energy; they are equipartitioned.

We next define the cross-correlation as

$$C_{AB}(\tau)=\langle u(\boldsymbol{r}_A,t)u(\boldsymbol{r}_B,t+\tau)\rangle. \tag{5}$$

Note that in practice one often cannot carry out an ensemble average because one only has one realization. In that case one can consider the time averaged cross-correlation

$$C_{AB}^{\text{time}}(\tau)=\frac{1}{T_0}\int_0^{T_0}u(\boldsymbol{r}_A,t)u(\boldsymbol{r}_B,t+\tau)\,\mathrm{d}t, \tag{6}$$

where T_0 is the length of the employed time interval. It is not evident that expressions (5) and (6) always give the same result. We first follow the derivation of Lobkis and Weaver (2001) and consider the ensemble average (5). Inserting the normal mode expansion (2) gives

$$C_{AB}(\tau)=\sum_{n,m}\frac{u_n(\boldsymbol{r}_A)u_m(\boldsymbol{r}_B)}{\omega_n\omega_m}\{\langle a_na_m\rangle\cos\omega_nt\cos\omega_m(t+\tau)+\langle a_nb_m\rangle\cos\omega_nt\sin\omega_m(t+\tau)+\langle b_na_m\rangle\sin\omega_nt\cos\omega_m(t+\tau)+\langle b_nb_m\rangle\sin\omega_nt\sin\omega_m(t+\tau)\}. \tag{7}$$

Because of expression (4), the second and third terms on the right hand side vanish and the crossterms between different modes are zero so that

$$C_{AB}(\tau)=S^2\sum_m\frac{u_m(\boldsymbol{r}_A)u_m(\boldsymbol{r}_B)}{\omega_m^2}\times\{\cos\omega_mt\cos\omega_m(t+\tau)+\sin\omega_mt\sin\omega_m(t+\tau)\}. \tag{8}$$

Since $\cos\omega_mt\cos\omega_m(t+\tau)+\sin\omega_mt\sin\omega_m(t+\tau)=\cos\omega_m\tau$, the cross-correlation is given by

$$C_{AB}(\tau)=S^2\sum_m\frac{u_m(\boldsymbol{r}_A)u_m(\boldsymbol{r}_B)}{\omega_m^2}\cos\omega_m\tau. \tag{9}$$

Note that this expression does not depend on the absolute time t, but only on the lag-time τ. This may seem gratifying because only the lag time τ has physical meaning. Since t is not present in expression (9) one might think that the ensemble average can be replaced by a time average, but we show in section 4 that this conclusion is not correct when the modal coefficients have fixed values. An average over time t of equation (9) does not change that expression because the right hand side is independent of t. This means that the time average and the ensemble average are equal only when the time averaging implies an averaging over different values of the modal coefficients. This is the case when the system is damped and a random excitation continuously excites the modes. As an example of such a system we treat the kicked damped oscillator in appendix A.

The cross-correlation in expression (9) is not equal to the Green's function in equation (1), and instead we consider the time-derivative:

$$\frac{\mathrm{d}C_{AB}(\tau)}{\mathrm{d}\tau}=-S^2\sum_m\frac{u_m(\boldsymbol{r}_A)u_m(\boldsymbol{r}_B)}{\omega_m}\sin\omega_m\tau. \tag{10}$$

This result is valid for all τ. For positive τ the right hand side is equal to $-S^2G(\boldsymbol{r}_A, \boldsymbol{r}_B, \tau)$, while for negative τ the right hand side is equal to $S^2G(\boldsymbol{r}_A, \boldsymbol{r}_B, -\tau)$, hence

$$\frac{\mathrm{d}C_{AB}(\tau)}{\mathrm{d}\tau}=-S^2[G(\boldsymbol{r}_A,\boldsymbol{r}_B,\tau)-G(\boldsymbol{r}_A,\boldsymbol{r}_B,-\tau)]. \tag{11}$$

The time-derivative of the cross-correlation is thus equal to the superposition of the Green's function and its time-reversed counterpart. This derivation hinges on equipartitioning as defined in equation (4). Note that this definition of equipartitioning is different from the one given in section 2.1, and it is not obvious that these definitions are equivalent.

2.3 Waves generated on a bounding surface

We next treat the case of acoustic waves that are excited on a surface surrounding receivers. Waves are assumed to propagate outward through this surface, the system is open. One can define normal modes of such a system but the modal eigenfunctions and normal mode frequencies are, in general, complex (Maupin, 1996; Yamamura and Kawakatsu, 1998; Lognonné et al., 1998). The orthogonality relation used in the previous section must also be generalized to include the energy loss due to radiation (Lognonné et al., 1998). The treatment of the previous section can thus not be used for open systems without modification. It is therefore easier to use a formulation based on representation theorems.

The following treatment is valid in the frequency domain using the following Fourier convention:

$$f(t) = \int F(\omega) \exp(-\mathrm{i}\omega t)\, \mathrm{d}\omega.$$

Assuming that the surface is a large sphere ∂V, the radiation boundary condition on this sphere is

$$\frac{\partial u}{\partial n} = \mathrm{i}ku, \tag{12}$$

where k is the wavenumber and n the distance orthogonal to ∂V. In this case Green's function retrieval is expressed as the following surface integral (Derode et al., 2003; Wapenaar et al., 2005; Snieder et al., 2007):

$$G(\boldsymbol{r}_A, \boldsymbol{r}_B, \omega) - G^*(\boldsymbol{r}_A, \boldsymbol{r}_B, \omega) = 2\mathrm{i}\omega \oint_{\partial V} \frac{1}{\rho(\boldsymbol{r})c(\boldsymbol{r})} G(\boldsymbol{r}_A, \boldsymbol{r}, \omega) G^*(\boldsymbol{r}_B, \boldsymbol{r}, \omega)\, \mathrm{d}S, \tag{13}$$

with ρ the mass density and c the wave velocity. Equation (13) differs slightly from equivalent expressions published elsewhere (Derode et al., 2003; Wapenaar et al., 2005) because of different Fourier conventions. Time-reversal corresponds, in the frequency domain, to complex conjugation, hence G–G^* in the left hand side of equation (13) accounts for the difference of the causal Green's function and its time-reversed counterpart, as shown in equation (11).

To complete the theory for Green's function retrieval one assumes uncorrelated sources $q(\boldsymbol{r}, \omega)$ of field fluctuations on the surface ∂V that satisfy

$$\left\langle q(\boldsymbol{r}_1, \omega) q^*(\boldsymbol{r}_2, \omega) \right\rangle = \frac{1}{\rho(\boldsymbol{r}_1)c(\boldsymbol{r}_1)} \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) \,|\, S(\omega) \,|^2, \tag{14}$$

where $|S(\omega)|^2$ is the power spectrum of these sources. The factor $1/\rho c$ can be explained as follows. The power flux in an acoustic medium is given by pv^* (Morse and Ingard, 1968), with p pressure and v velocity. The velocity is related to the pressure through the impedance: $v = p/\rho c$, hence the power flux is given by $|p|^2/\rho c$. The factor $1/\rho c$ in equation (13) and the definition (14) for the sources thus ensures that all sources on the surface radiate the same amount of energy. Any wave that propagates back to the surface radiates outward by virtue of the radiation boundary condition (12). All sources on the surface radiate an equal power flux into the surface, and because of the lack of attenuation, the energy is ultimately radiated out off the surface. This is yet another example of equipartitioning.

Note that the treatment in this section treats each frequency independent of other frequencies. As a result the power spectrum $|S(\omega)|^2$ may vary with frequency, and thus there is no reason why there should be equipartitioning between different frequencies. This treatment is different from the use of equipartitioning in the normal-mode treatment of the previous section, where expression (4) stipulates that all modes do carry the same energy.

2.4 Attenuating waves

The Green's function can also be extracted for attenuating acoustic media. The normal mode analysis of section 2.2 is not applicable in the presence of attenuation because in that case the normal mode frequencies are complex, and one needs to use adjoint modes in the orthogonality relation (Park and Gilbert, 1986). In the time domain, attenuation can be described by a time-dependent compressibility κ that accounts for the relaxation of the acoustic medium (Dahlen and Tromp, 1998). This corresponds to a frequency-dependent compressibility. Because of the Kramers-Kronig relations (Aki and Richards, 2002) the imaginary component of the compressibility is nonzero in the case of attenuation. For the case where the field or its normal derivative vanishes at the bounding surface, Green's function extraction can be expressed, in the frequency domain, as (Snieder, 2007)

$$G(\boldsymbol{r}_A, \boldsymbol{r}_B, \omega) - G^*(\boldsymbol{r}_A, \boldsymbol{r}_B, \omega) = 2\omega \int_V [\mathrm{Im}\, \kappa(\boldsymbol{r}, \omega)] G(\boldsymbol{r}_A, \boldsymbol{r}, \omega) G^*(\boldsymbol{r}_B, \boldsymbol{r}, \omega) \mathrm{d}V, \tag{15}$$

where Im denotes the imaginary part. The Green's function can be retrieved by cross-correlating field fluctuations that are excited by uncorrelated sources throughout the volume that satisfy (Snieder, 2007)

$$\left\langle q(\boldsymbol{r}_1, \omega) q^*(\boldsymbol{r}_2, \omega) \right\rangle = [\mathrm{Im}\, \kappa(\boldsymbol{r}, \omega)] \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) \,|\, S(\omega) \,|^2 \,. \tag{16}$$

The source strength prescribed by expression (16) can be related to the attenuation. The potential energy of an acoustic wave p is equal to $\kappa|p|^2$ (Morse and Ingard, 1968). The time derivative of the potential energy corresponds, in the frequency domain, to

$$-\mathrm{i}\omega\kappa \,|\, p \,|^2 = -\mathrm{i}\omega(\mathrm{Re}\, \kappa) \,|\, p \,|^2 + \omega(\mathrm{Im}\, \kappa) \,|\, p \,|^2, \tag{17}$$

where Re denotes the real part. The first term in the right hand side denotes the transfer between potential and kinetic energy. This transfer is periodic and does not lead to an energy loss. The last term in expression (17) accounts for attenuation. The term $\mathrm{Im}\kappa$ in the source strength (16) ensures that at every point in the medium the sources supply the same amount of energy that is locally dissipated. The source strength required for Green's function extraction of attenuating acoustic waves thus requires a balance between the injected energy and dissipated energy throughout the volume. This is another formulation of the equipartitioning requirement.

2.5 Potential fields

The principle of Green's function extraction can also be applied to potential fields. When the electrostatic potential V or the normal component of the electric field $-\partial V/\partial n$ vanishes at the boundary, Green's function extraction is expressed as (Snieder et al., 2010)

$$G(\boldsymbol{r}_A, \boldsymbol{r}_B) = \int \varepsilon(\boldsymbol{r})[\nabla G(\boldsymbol{r}_A, \boldsymbol{r}) \cdot \nabla G(\boldsymbol{r}_B, \boldsymbol{r})]\mathrm{d}^3 r, \quad (18)$$

where $\varepsilon(\boldsymbol{r})$ is the electrical permittivity. Since the potential field is static, the Green's function does not depend on frequency. The electrostatic potential is real, which explains the absence of complex conjugates in this expression.

The Green's function for the electrostatic potential follows (Snieder et al., 2010) by averaging over quasi-static field fluctuations excited by random electric dipoles $\boldsymbol{p}$ that are spatially uncorrelated and satisfy

$$\left\langle p_i(\boldsymbol{r}_1) p_j(\boldsymbol{r}_2) \right\rangle = |S|^2\, \varepsilon(\boldsymbol{r}_1)\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\delta_{ij}. \quad (19)$$

We next consider the energy of the field excited by such sources. The energy E of an electric field $\boldsymbol{E}$ is given by Griffiths (1999) as

$$E = \frac{1}{2}\int \varepsilon(\boldsymbol{E}\cdot\boldsymbol{E})\mathrm{d}^3 r = \frac{1}{2}\int \varepsilon(\nabla V \cdot \nabla V)\mathrm{d}^3 r. \quad (20)$$

The last term in this expression has the same form as the right hand side of equation (18). Using field fluctuations excited by dipoles with a strength proportional to ε ensures that the electrostatic energy is constant throughout space. This is yet another expression of the principle of equipartitioning.

3 Equipartitioning may not be sufficient

The examples of the previous section show that in all cases the sources of the field fluctuations used for Green's function extraction must have a position and strength such that the energy density in the system is constant. This principle is formulated in different ways in various examples. Because equipartitioning is such a unifying principle, one might think that the Green's function can be extracted whenever the field is equipartitioned. In this section we present counterexamples to show that even though equipartitioning may be necessary, it is not a sufficient condition for Green's function extraction.

3.1 An acoustic source

We first treat the case of acoustic waves excited on a spherical surface, as treated in section 2.1. We consider sources on a large sphere with radius R as shown in Figure 2 that radiate in phase with a common frequency spectrum $S(\omega)$. The sources radiate inward from all points on the surface with the same intensity, and the associated wave field is certainly equipartitioned. The wavefield recorded at location $\boldsymbol{r}_B$ is given by

$$u(\boldsymbol{r}_B) = -S(\omega)\oint \frac{\mathrm{e}^{\mathrm{i}kR_B}}{4\pi R_B}\mathrm{d}S, \quad (21)$$

where the distance R_B is defined in Figure 2. A similar expression holds for the waves recorded at location $\boldsymbol{r}_A$. The integral can be evaluated exactly (Martin, 2006), but for simplicity we approximate it assuming that the radius R of the sphere is large compared to r_A and r_B. This approximation is exact in the limit $R\to\infty$. In that case we can use the approximation $R_B = R - r_B\cos\theta$ in the exponent of equation (21), and we replace R_B by R in the denominator. This gives

$$u(\boldsymbol{r}_B) = -S(\omega)\oint \frac{\mathrm{e}^{\mathrm{i}k(R - r_B\cos\theta)}}{4\pi R}\mathrm{d}S =$$
$$-S(\omega)\frac{\mathrm{e}^{\mathrm{i}kR}}{4\pi R}\oint \mathrm{e}^{-\mathrm{i}k r_B\cos\theta}\mathrm{d}S = -S(\omega)\frac{\mathrm{e}^{\mathrm{i}kR}}{R}\mathrm{sinc}(k r_B). \quad (22)$$

A similar expression holds for $u(\boldsymbol{r}_A)$, hence

$$\left\langle u(\boldsymbol{r}_A)u^*(\boldsymbol{r}_B)\right\rangle = \frac{|S(\omega)|^2}{R^2}\mathrm{sinc}(k r_A)\mathrm{sinc}(k r_B). \quad (23)$$

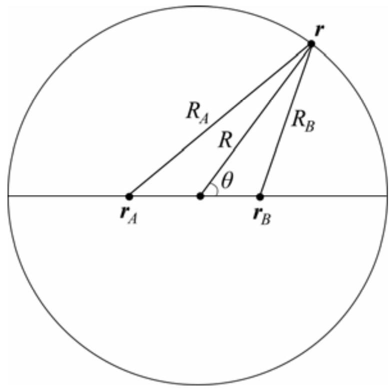

Figure 2 Geometry of the problem where sources are distributed on a sphere radiate waves in phase.

According to expression (13), exact Green's function extraction should give

$$G(\boldsymbol{r}_A, \boldsymbol{r}_B, \omega) - G^*(\boldsymbol{r}_A, \boldsymbol{r}_B, \omega) = 2\,\mathrm{i}\,\mathrm{Im}[G(\boldsymbol{r}_A, \boldsymbol{r}_B, \omega)] = \frac{\mathrm{i}k}{2\pi}\mathrm{sinc}(k\,|\,\boldsymbol{r}_A - \boldsymbol{r}_B\,|), \qquad (24)$$

where we used that for the employed homogeneous medium $G(\boldsymbol{r}_A, \boldsymbol{r}_B, \omega) = -\exp(\mathrm{i}k|\boldsymbol{r}_A - \boldsymbol{r}_B|)/4\pi(k|\boldsymbol{r}_A - \boldsymbol{r}_B|)$. Expressions (23) and (24) have different space dependencies, and thus describe different physical functions. This means that the cross-correlation of fields excited by simultaneous sources on the sphere does not give the Green's function, despite the fact that the wave field is equipartitioned. This should not be surprising, because we violated condition (14) which calls for uncorrelated sources. This example shows that while equipartitioning may be a necessary condition, it is not a sufficient condition.

In this particular example, one may construct the Green's function for a given pair of receivers without equipartitioning. In order to extract the wave propagating from point A to point C in Figure 3 it suffices to have sources in the grey area (Snieder, 2004a). Waves excited in the grey area provide a cross-correlation that is, to first order, independent of the source location. The grey area is called the *stationary phase region*. Sources outside this region do not give a net contribution to the cross-correlation, and hence for the wave that propagates from A to C one only needs sources in the stationary phase region. Equipartitioning is thus not even necessary if one only seeks to retrieve the direct wave that propagates from A to C. In fact, there are several studies that indicate that the surface wave extracted from the cross-correlation of microseismic noise propagates preferentially away from coastlines (e.g., Stehly et al., 2006). This is an indication that although the surface waves are not equipartitioned, it is still possible to extract surface waves from the cross-correlation of such measurements. Alternatively, one can pre-process data that are not equipartitioned to make the energy flux less dependent on direction (Mulargia and Castellaro, 2008; Curtis and Halliday, 2010).

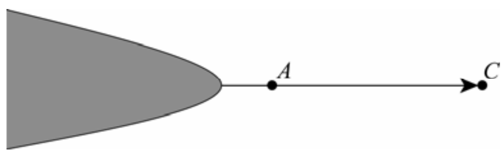

Figure 3 Sources in the stationary phase zone, indicated as the grey shaded area suffice for the extraction of the wave that propagates from A to C.

3.2 Normal modes

Let us next consider a closed undamped system with normal modes as treated in section 2.2. We assume that all the modes carry the same energy, but we drop the condition that the excitation coefficients of the modes are uncorrelated. This means that instead of equation (4) we assume that

$$a_n = b_n = \frac{S}{\omega_n}. \qquad (25)$$

This state is clearly equipartitioned. Repeating the steps leading to equation (10) gives for these modal coefficients

$$\frac{\mathrm{d}C_{AB}(\tau)}{\mathrm{d}\tau} = -S^2 \sum_{n,m} \frac{u_n(\boldsymbol{r}_A)u_m(\boldsymbol{r}_B)}{\omega_n} \times \{\sin[(\omega_m - \omega_n)t + \omega_m\tau] - \cos[(\omega_m + \omega_n)t + \omega_m\tau]\}. \qquad (26)$$

Note that this expression is different from expression (10) in two ways: first, it contains a double sum over modes rather than a single sum; second, it depends on τ *and t* instead of τ only. Both differences are due to the fact that we used correlated modal coefficients. This example also illustrates that equipartitioning is not a sufficient condition for Green's function retrieval.

3.3 Waves generated on a bounding surface

In the previous examples we violated the requirement for Green's function retrieval in trivial ways. In this section we show a more subtle example of a wave state that is equipartitioned, but that does not give the correct Green's function. We consider an open acoustic system, as discussed in section 2.3, and present a numerical example in two space dimensions (Fan and Snieder, 2009) with the geometry shown in Figure 4. In an area of 80 m by 80 m, 200 strong isotropic point scatterers are located. The geometry and parameter choice in this example are not representative for the Earth, but the example serves to make a general point. We computed synthetic seismograms using a variation of Foldy's method (Groenenboom and Snieder, 1995). Sources are placed on a circle with a radius of 90 m. Two receivers are placed in the center of the scattering region at a distance of 20 m. The distance between both receivers and the edge of the region with scatterers is 30 m. We show the true Green's function within the employed frequency band by the dotted lines in both panels of Figure 5. The Green's function estimated from equation (13) using 300 sources placed uniformly on a circle with a radius of 90 m is indicated by the solid lines in Figure 5b. In this case the Green's function is retrieved accurately.

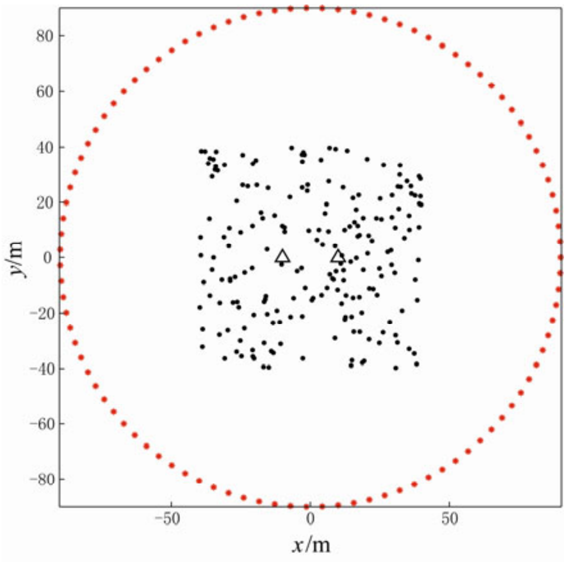

Figure 4 Geometry for the numerical example of 2D scattering by 200 scatterers (dots) of waves excited by sources on a circle. Receivers are indicated by triangles. The scattering cross section of each scatterer, which is a length in two dimensions, is much larger than the radius of each dot.

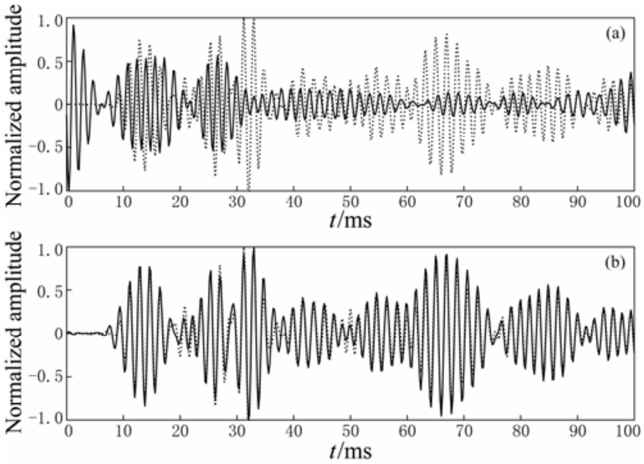

Figure 5 (a) True Green's function (dotted line) and the Green's function estimated from the cross correlation of waves excited by a single source (solid line); (b) True Green's function (dotted line) and the Green's function estimated from the cross correlation of waves excited by 300 sources placed uniformly on a circle (solid line).

In the numerical example, the transport scattering mean free path is given by $l_*=1/N\sigma$ (Sheng, 1995), where σ is the scattering cross section and N is the scatterer density. The transport mean free path is defined as the distance over which the direction of wave propagation is randomized by scattering (Sheng, 1995). For the employed scatterer configuration, $\sigma= 1.6$ m, $N = 200/(80\ \text{m})^2$, hence $l_*=5$ m. Both receivers are separated by a distance of 30 m from the edge of the scattering domain, which means that any wave recorded at the receivers has propagated over at least six times the transport mean free path. The wavefield therefore is for all practical purposes equipartitioned.

We next consider the case where one source is placed at a distance of 90 m from the center of the scattering array. Since this wave field is equipartitioned because of strong scattering, one might think that computing the cross-correlation of the waves generated by this single source would give the Green's function. The cross-correlation of these waves is shown by the solid line in Figure 5a. This cross-correlation does not resemble the true Green's function (indicated in dotted line) at all. The cross-correlation of waves excited by a single source does not give the Green's function because equation (13) requires sources everywhere on a closed surface; replacing this by a single source on that surface is not an adequate discretization of the surface integral. Yet the wave state generated by a single source is for all practical purposes equipartitioned after it has propagated over at least six times the transport mean free path to the receivers. This is another example of an equipartitioned wave state that does not suffice to give the Green's function.

4 What does this mean for the Earth?

One might wonder which of the schemes for Green's function retrieval presented in section 2 is applicable to the Earth. The short answer is that none of these methods is applicable for all types of wave propagation at all frequencies in the Earth. The different schemes each have different restrictions that we discuss below.

First consider the model of sources that generate waves propagating in all possible directions, as discussed in section 2.1. Conceptually, this model is applicable to surface waves that are excited by uncorrelated noise sources near the Earth's surface. The extraction of surface wave from the cross-correlation of noise has been very successful in the microseismic band with periods between 5 s and 10 s (Campillo and Paul, 2003). The path coverage obtained by cross-correlating waves recorded at numerous receiver pairs has fundamentally changed crustal tomography (Shapiro et al., 2005; Sabra et al., 2005). The extraction of the surface waves from noise has also been applied to longer periods (Shapiro and Campillo, 2004; Nishida et al., 2009). The station density currently offered by US Array makes it possible to even retrieve the full surface wave field propagating from one receiver through the array (Lin et al., 2009). In practice, the microseismic noise is not generated with the same strength at all locations, and the retrieved surface waves are strongest in directions propagating away from oceanic regions with large wave activity. When the isotropy in the microseismic noise is a limiting factor in the retrieval of the surface waves, one can cross-correlate the coda of extracted surface waves again in order to utilize surface waves that are better equipartitioned (Stehly et al., 2008). One can retrieve the different elements of the Green's tensor by cross-correlating different pairs of components of the recorded ground motion (Campillo and Paul, 2003; Snieder, 2004a).

The work of Lobkis and Weaver (2001) had a huge impact because the theory is very elegant and the employed formulation in normal modes is natural for global seismologists. Although the derivation is correct, it is strictly speaking not applicable to the Earth for a number of reasons. First, the modes of the Earth certainly are not equipartitioned. Since most of the noise is generated near the surface of the Earth, the modes that are confined to the near-surface carry more energy than the modes that penetrate deep into the Earth. In practice, the fundamental mode surface wave carries the most energy, and it is for this reason that estimates of the Earth's Green's function usually are dominated by the fundamental mode surface wave. The second reason why the method is not applicable to the Earth is that the formalism truly calls for an ensemble average. The employed model assumes there is no attenuation, which means that once the modes are excited, they oscillate forever without any change in the modal excitation. This means that a time average is not equivalent to an ensemble average, and one would need an ensemble of earths to implement the theory. We show in appendix A that for a damped oscillator that is periodically kicked for $t>0$, the cross-correlation $C(\tau)=\langle x(t)\,x(t+\tau)\rangle$ is given by

$$\frac{\mathrm{d}C(\tau)}{\mathrm{d}\tau}=-\frac{\langle F^2\rangle}{4m^2T\gamma}[G(\tau)-G(-\tau)]. \tag{27}$$

In this expression T is the time between kicks, $\langle F^2\rangle$ is the average of the square of the forces during the kicks, and γ is the damping parameter. This equation shows that expression (11) can be generalized to include attenuation. Third, for an *undamped* oscillator that is excited by random forces with zero mean, the cross-correlation grows linearly with time

$$\frac{\mathrm{d}C(\tau)}{\mathrm{d}\tau}=-\frac{\langle F^2\rangle}{2m^2T}t(G(\tau)-G(-\tau)). \tag{28}$$

(The change from expression (27) to (28) follows from taking the limit: $\lim_{\gamma\to 0}[1-\exp(-2\gamma t)]/\gamma=2t$.) Because of this secular growth, such a system cannot be in equilibrium. This growth is due to the fact that the energy of an undamped kicked oscillator grows linearly with time, even when the average of the forces vanishes (Snieder et al., 2010). These last two points are mostly academic, because in practice there is attenuation, and when the Earth is continuously excited, the modal coefficients are effectively "reset" at a time equal to the attenuation time of waves in the Earth. For thermal fluctuations this can be described by using time-dependent modal coefficients that satisfy in the notation of this paper $\langle a_n(t)a_m(t')\rangle=\delta_{nm}2k_\mathrm{B}T\exp(-\gamma|t-t'|)$, where $k_\mathrm{B}T$ is the thermal energy (Weaver and Lobkis, 2003). As we show in expression (27), the theory of Lobkis and Weaver (2001) can be extended to include attenuation and an explicit description of the force that excites field fluctuations.

As discussed in section 2.3, the Green's function can be retrieved from the cross-correlation of field fluctuations excited by sources on a bounding surface. This principle can be extended to elastic waves without including sources at the Earth's surface (Wapenaar, 2004). In practice, however, there are insufficient sources in the interior of the Earth to provide the required excitation for field fluctuations on the closed surface surrounding the receivers. This hurdle has been overcome (Bostock, 2002; Bostock et al., 2002; Kumar and Bostock, 2006; Tonegawa et al., 2009) by using teleseismic waves that impinge on the crust from below, as shown in Figure 6. In that case the teleseismic waves propagating through the dashed surface in Figure 6 replace the sources on

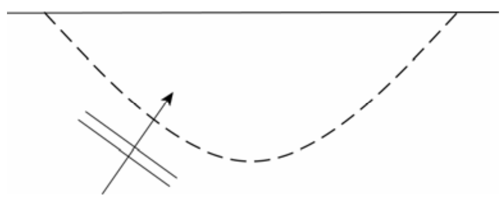

Figure 6 Teleseismic waves propagating through a bounding surface illuminating the crust from below.

that surface. The reflections of these waves by the Earth's free surface can be used to image the crust. This principle was theoretically shown earlier by Claerbout (1968) who demonstrated that the reflection response of a layered system can be retrieved from the cross-correlation of the transmission response.

In reality the Earth is attenuating, and one may wonder whether the formalism for attenuating waves presented in section 2.4 can be the basis for Green's function extraction. In order to use this theory one must have sources that are proportional to the attenuation. For a damped system in equilibrium, the excitation must balance the attenuation; otherwise the system would not be in equilibrium. This principle was shown originally by Nyquist (1928) for voltage fluctuations in a resistor and was later generalized as the *fluctuation dissipation theorem* (Greene and Callen, 1951; Callen and Welton, 1951; Weber, 1956). According to this theorem, the response of a system in thermal equilibrium can be extracted from field fluctuations. The condition of equilibrium implies that thermal fluctuations must balance the dissipation. Unfortunately, thermal fluctuations in the Earth are extremely weak. Boltzman's constant is given by $k_B = 1.4\times10^{-23}$ J/K, and the thermal energy for a mantle temperature of 600 K is equal to $k_B T = 8.4\times10^{-21}$ J. In comparison, a child with a mass of 50 kg jumping from a table 1 m high releases an energy equal to 500 J. It is thus clear that thermal fluctuations and the fluctuation dissipation theorem are irrelevant for observable field fluctuations in the Earth. Thermal fluctuations are more relevant for electrical fields where the accuracy of voltmeters is getting close to the voltage generated by thermal fluctuations (Slob et al., 2010).

5 Discussion

The different examples in this work require equipartitioning in some sense for Green's function retrieval, but it is not clear to what extent these different requirements are equivalent. For example, does the condition that the modes of a closed system carry the same energy imply that the energy density is homogeneous in space and that energy propagates equally in all directions? Questions like this require further research. As we show in several examples in section 3, the requirement of equipartitioning is not sufficient for Green's function retrieval.

As shown in the previous section, none of the sources of field fluctuations in the Earth is adequate to provide the full Green's function. In general, the surface waves in the retrieved Green's function are strongest (e.g., Shapiro and Campillo, 2004; Halliday et al., 2008). The body waves are usually under-represented, and, with the exception of studies based on teleseismic body waves (Bostock, 2002; Bostock et al., 2002; Kumar and Bostock, 2006; Tonegawa et al., 2009), the number of studies that report extracting body waves is modest (Roux et al., 2005; Gerstoft et al., 2006; Draganov et al., 2007, 2009). The reason for the under-representation of body waves is that for the retrieval of the direct surface wave it suffices to have sources anywhere near the Earth surface in a region that straddles a line through the used receivers, as sketched in Figure 3. This is a relatively weak condition that is readily satisfied. Forghani and Snieder (2010) show in more detail why it is more difficult to extract body waves from field fluctuations than it is for surface waves. The dominance of the fundamental mode surface waves in Green's function extraction has led to spectacular advances in surface wave tomography (e.g., Shapiro et al., 2005; Sabra et al., 2005). The extension of this principle beyond the microseismic band (Shapiro and Campillo, 2004; Nishida et al., 2009) holds promise for the determination of mantle structure as well. In most applications of Green's function extraction the surface wave is very strong. For this reason Draganov et al. (2009) process recorded noise extensively before cross-correlation in order to suppress surface waves. The over-representation of surface waves in Green's function retrieval can be advantageous when using the retrieved surface wave for adaptive ground-roll removal in exploration seismology (Halliday et al., 2008; Xue et al., 2009).

Green's function extraction has also been successful in quasi one-dimensional problems, because for waves propagating along a line one needs only sources on the line on both sides of the receivers. In fact, in the presence of an open boundary one needs only a source on one side of the receivers. This has successfully been applied to Green's function retrieval for buildings that are shaken at the base (Snieder and Şafak, 2006; Kohler et al., 2007). Another example is Green's function retrieval for the shallow subsurface (Trampert et al., 1993; Mehta et al., 2007; Sawazaki et al., 2009). In this application, the low wave velocity in the near surface results in near-vertical wave propagation. In combination with a locally layered Earth structure, this makes the wave propagation quasi one-dimensional. In these quasi 1D applications, deconvolution rather than correlation is used because it removes the spectral variations of the excitation without introducing unwanted artifacts (Vasconcelos and Snieder, 2008).

Much research on Green's function extraction was spurred by the work of Lobkis and Weaver (2001). Their normal mode formulation is only relevant for the long-wavelength motion of the Earth that is described well by a superposition of normal modes. Although the theory of Lobkis and Weaver (2001) is correct for the low-frequency motion, it is not directly applicable to the Earth because the lack of attenuation used in their theory implies one of two things: either the Earth is excited once and the modal coefficients keep their value in time, or the Earth is excited continuously and the elastic energy in the Earth grows linearly with time. In the former case the ensemble average needed in their theory cannot be taken because we have only one realization. In the latter case, the Earth is not in equilibrium. These inconsistencies are resolved by accounting for the (weak) attenuation that is present in the Earth. Because of attenuation, the vibrations of the Earth damp out over a typical attenuation time. This causes the ambient vibrations of the Earth to be in equilibrium, and the modal coefficients are effectively "reset" after the characteristic attenuation time. We have shown that attenuation can be incorporated in Green's function extraction based on normal modes, which explains why the theory of Lobkis and Weaver (2001) not only inspired the seismological community; it also provided a recipe for Green's function extraction (by cross-correlation) that is for practical purposes correct.

Acknowledgements We appreciate the thoughtful comments of Simon Luo, Francesco Perrone, and two anonymous reviewers.

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Appendix A Retrieving the Green's function of a kicked oscillator by cross-correlation

As a prototype of the behavior of an excited mode we consider the response of a kicked damped oscillator that satisfies

$$\ddot{x}+2\gamma\dot{x}+\omega_0^2 x=\frac{F(t)}{m}, \tag{29}$$

where $F(t)$ is the excitation, γ the damping, and m the mass. The Green's function of the oscillator is the solution to a force $F(t)=\delta(t)$ and is given by

$$G(t)=\frac{1}{\omega}\mathrm{e}^{-\gamma t}\sin(\omega t)H(t), \tag{30}$$

where $H(t)$ is the Heaviside function and

$$\omega=\sqrt{\omega_0^2-\gamma^2}. \tag{31}$$

We consider a forcing that for $t>0$ consists of a sequence of kicks at times $t_n=nT$:

$$F(t)=\sum_{n=0}^{\infty}F_n\delta(t-nT). \tag{32}$$

The kicks F_n are random with zero mean, while the different kicks are uncorrelated

$$\langle F_n\rangle=0, \qquad \langle F_nF_m\rangle=\langle F^2\rangle\delta_{nm}, \tag{33}$$

where $\langle\cdots\rangle$ denotes an ensemble average. The response to this forcing is given by

$$x(t)=\frac{1}{m}\sum_{0<nT<t}G(t-nT)F_n. \tag{34}$$

Because of the random nature of the kicks, $x(t)$ is a random function. Since the kicks have zero mean, the expectation value of $x(t)$ vanishes

$$\langle x(t)\rangle=0. \tag{35}$$

We define the cross-correlation as

$$C(\tau)=\langle x(t)x(t+\tau)\rangle. \tag{36}$$

Using expression (34) the covariance is given by

$$C(\tau)=\frac{1}{m^2}\sum_{0<nT<t}\sum_{0<mT<t+\tau}G(t-nT)G(t+\tau-mT)\langle F_nF_m\rangle. \tag{37}$$

We consider the case $\tau>0$ first. Using expression (33) for the expectation value of the kicks one can reduce the double sum in equation (37) to a single sum where only the kicks at times $0<nT<t$ contribute

$$C(\tau)=\frac{\langle F^2\rangle}{m^2}\sum_{0<nT<t}G(t-nT)G(t+\tau-nT). \tag{38}$$

We assume that the time interval between kicks is much less than the period ($\omega T\ll 1$). In that case the sum over kicks in expression (38) can be replaced by an integral using

$$\sum_{0<nT<t}f(t-nT)\rightarrow\frac{1}{T}\int_0^t f(t-t')\mathrm{d}t'=\frac{1}{T}\int_0^t f(t')\mathrm{d}t', \tag{39}$$

where the change of variable $t'\rightarrow t-t'$ is used in the last identity. Using this in equation (38) gives

$$C(\tau)=\frac{\langle F^2\rangle}{m^2T}\int_0^t G(t')G(t'+\tau)\mathrm{d}t'. \tag{40}$$

Inserting expression (30) in the right hand side, evaluating the time integral and using equation (32) gives

$$C(\tau)=\frac{\langle F^2\rangle}{4m^2\omega_0^2T\gamma}\mathrm{e}^{-\gamma\tau}\left(\cos\omega\tau+\frac{\gamma}{\omega}\sin\omega\tau\right)-\frac{\langle F^2\rangle}{4m^2\omega^2T\gamma}\mathrm{e}^{-2\gamma t}\mathrm{e}^{-\gamma\tau}\left[\cos\omega\tau-\frac{\gamma^2}{\omega_0^2}\cos\omega(2t+\tau)+\frac{\gamma\omega}{\omega_0^2}\sin\omega(2t+\tau)\right]. \tag{41}$$

Note the different frequencies in the denominator of the two terms in this expression. The second term in this expression decays exponentially with time because of the term $\exp(-2\gamma t)$, this term accounts for the transients associated with starting the kicks. For long times ($\gamma t\rightarrow\infty$) only the first term remains. This term does not depend on t, and hence the average over time t in the long time limit is equal to the expectation value. Taking the derivative of the first term with respect to τ, and using equations (30) and (32), gives in the long time limit

($\gamma t \to \infty$)

$$\frac{\mathrm{d}C(\tau)}{\mathrm{d}\tau} = -\frac{\langle F^2 \rangle}{4m^2 T\gamma} G(\tau) \qquad \text{for } \tau > 0. \tag{42}$$

For $\tau<0$ the derivation is analogous, except that now only the kicks at times $0<nT<t+\tau$ contribute, and that the substitution (39) should be modified into

$$\sum_{0<nT<t+\tau} f(t-nT) \to \frac{1}{T}\int_0^{t+\tau} f(t-t')\mathrm{d}t' = \frac{1}{T}\int_{-\tau}^{t} f(t')\mathrm{d}t'. \tag{43}$$

Using this, and taking the same steps as in the derivation of expression (41), gives for the long-time behavior of the cross-correlation

$$C(\tau) = \frac{\langle F^2 \rangle}{4m^2\omega_0^2 T\gamma} \mathrm{e}^{\gamma\tau}\left(\cos\omega\tau - \frac{\gamma}{\omega}\sin\omega\tau\right) + O(\mathrm{e}^{-2\gamma t}). \tag{44}$$

Taking the derivative and using equations (30) and (32) gives in the long-time limit

$$\frac{\mathrm{d}C(\tau)}{\mathrm{d}\tau} = \frac{\langle F^2 \rangle}{4m^2 T\gamma} G(-\tau) \qquad \text{for } \tau > 0. \tag{45}$$

Expressions (42) and (45) can be combined to the expression (27) that is valid for all τ. Note that the factor that multiplies the Green's function does not depend on frequency, this means that the kicked oscillator delivers an equal energy to the system regardless of the frequency, in the multimode system analyzed by Lobkis and Weaver (2001) this implies equipartitioning of energy among modes.

Let us next consider what happens when the attenuation is switched off. Mathematically this is achieved by taking the lime limit $\gamma \to 0$. We take this limit for the case $\tau>0$ and apply it to expression (41). In the limit $\gamma \to 0$ the transient terms in the last line of equation (41) do not decay exponentially with time. Expanding all terms in γ and using that according to equation (32) ω and ω_0 are equal to each other to first order in γ, gives

$$C(\tau) = \frac{\langle F^2 \rangle}{4m^2\omega_0^2 T}\left\{2t\cos\omega_0 t + \frac{1}{\omega_0}[\sin\omega_0\tau - \sin\omega_0(\tau+2t)]\right\}. \tag{46}$$

Note that the first term grows linearly with time. This means that for the kicked undamped oscillator the correlation does not approach a constant value for long times, and that the ensemble average cannot be replaced by a time average. Physically this is due to the fact that the energy of a kicked undamped oscillator grows linearly with time, even when the mean of the kicks vanishes (Snieder et al., 2010). For long times ($\omega_0 t \gg 1$) the first term in equation (46) dominates, and a comparison with expression (30) shows that

$$\frac{\mathrm{d}C(\tau)}{\mathrm{d}\tau} = -\frac{\langle F^2 \rangle}{2m^2 T} tG(\tau) \qquad \text{for } \tau > 0. \tag{47}$$

Apart from the secular growth term (t), the cross-correlation does produce the correct Green's function. A similar analysis for $\tau<0$ leads to equation (28).