

EXPRESS LETTER

A single-sided homogeneous Green's function representation for holographic imaging, inverse scattering, time-reversal acoustics and interferometric Green's function retrieval

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SUMMARY

Green's theorem plays a fundamental role in a diverse range of wavefield imaging applications, such as holographic imaging, inverse scattering, time-reversal acoustics and interferometric Green's function retrieval. In many of those applications, the homogeneous Green's function (i.e. the Green's function of the wave equation without a singularity on the right-hand side) is represented by a closed boundary integral. In practical applications, sources and/or receivers are usually present only on an open surface, which implies that a significant part of the closed boundary integral is by necessity ignored. Here we derive a homogeneous Green's function representation for the common situation that sources and/or receivers are present on an open surface only. We modify the integrand in such a way that it vanishes on the part of the boundary where no sources and receivers are present. As a consequence, the remaining integral along the open surface is an accurate single-sided representation of the homogeneous Green's function. This single-sided representation accounts for all orders of multiple scattering. The new representation significantly improves the aforementioned wavefield imaging applications, particularly in situations where the first-order scattering approximation breaks down.

Key words: Interferometry; Controlled source seismology; Wave scattering and diffraction.

1 INTRODUCTION

In optical, acoustic and seismic imaging, the central process is the retrieval of the wavefield inside the medium from experiments carried out at the boundary of that medium. Once the wavefield is known inside the medium, it can be used to form an image of the interior of that medium. The process to obtain the wavefield inside the medium is in essence a form of optical, acoustic or seismic holography (Porter 1970; Lindsey & Braun 2004). At the basis of these holographic methods lies Green's theorem, often cast in the form of a homogeneous Green's function representation or variants thereof. Although this representation is formulated as a closed boundary integral, measurements are generally available only on an open boundary. Despite this limitation, imaging methods based on the holographic principle work quite well in practice as long as the effects of multiple scattering are negligible. The same applies to linear inverse source problems (Porter & Devaney 1982) and linearized inverse scattering methods (Oristaglio 1989). However, in strongly inhomogeneous media the effects of multiple scattering can be quite severe. In these cases, approximating the closed boundary representation of the homogeneous Green's function by an open boundary integral leads to unacceptable errors in the homogeneous Green's function and, as a conse-

quence, to significant artefacts in the image of the interior of the medium.

In the field of time-reversal acoustics, the response to a source inside a medium is recorded at the boundary of the medium, reversed in time and emitted back from the boundary into the medium. Because of the time-reversal invariance of the wave equation, the time-reversed field obeys the same wave equation as the original field and therefore focuses at the position of the source. The back-propagated field can be quantified by the homogeneous Green's function representation (Fink 2008). Time-reversed wavefield imaging (McMechan 1983) uses the same principle, except that here the time-reversed field is propagated numerically through a model of the medium. Time-reversal acoustics suffers from the same limitations as holographic imaging and inverse scattering: when the original field is recorded on an open boundary only, the back-propagated field is no longer accurately described by the homogeneous Green's function.

In the field of interferometric Green's function retrieval, the recordings of a wavefield at two receivers are mutually cross-correlated. Under specific conditions (equipartitioning of the wavefield, etc.), the time-dependent cross-correlation function converges to the response at one of the receivers to a virtual source at the position of the other, that is, the Green's function (Larose *et al.* 2006;

Schuster 2009). The method is related to time-reversed acoustics and hence the retrieved Green's function can be described by the homogeneous Green's function representation (Wapenaar & Fokkema 2006). When the positions of the primary sources are restricted to an open boundary, the retrieved Green's function may become very inaccurate.

The aim of this paper is to derive a single-sided homogeneous Green's function representation which circumvents the approximations inherent to the absence of sources/receivers on a large part of the closed boundary. We show that with our single-sided representation it is possible to obtain the complete response to a virtual source anywhere inside the medium, observed by virtual receivers anywhere inside the medium, from measurements on a single boundary (note that in our earlier work on the Marchenko method the response to the virtual source was only obtained for receivers at the surface).

2 THE CLASSICAL HOMOGENEOUS GREEN'S FUNCTION REPRESENTATION AND ITS APPLICATIONS

For the closed-boundary configuration of Fig. 1(a), the homogeneous Green's function representation for an arbitrary inhomogeneous lossless medium reads (Porter 1970; Oristaglio 1989; Wapenaar & Fokkema 2006)

$$G_h(\mathbf{x}_A, \mathbf{x}_B, \omega) = \oint_{\partial\mathbb{D}} \frac{-1}{j\omega\rho(\mathbf{x})} \{G^*(\mathbf{x}, \mathbf{x}_A, \omega)\partial_i G(\mathbf{x}, \mathbf{x}_B, \omega) - \partial_i G^*(\mathbf{x}, \mathbf{x}_A, \omega)G(\mathbf{x}, \mathbf{x}_B, \omega)\}n_i d^2\mathbf{x}, \quad (1)$$

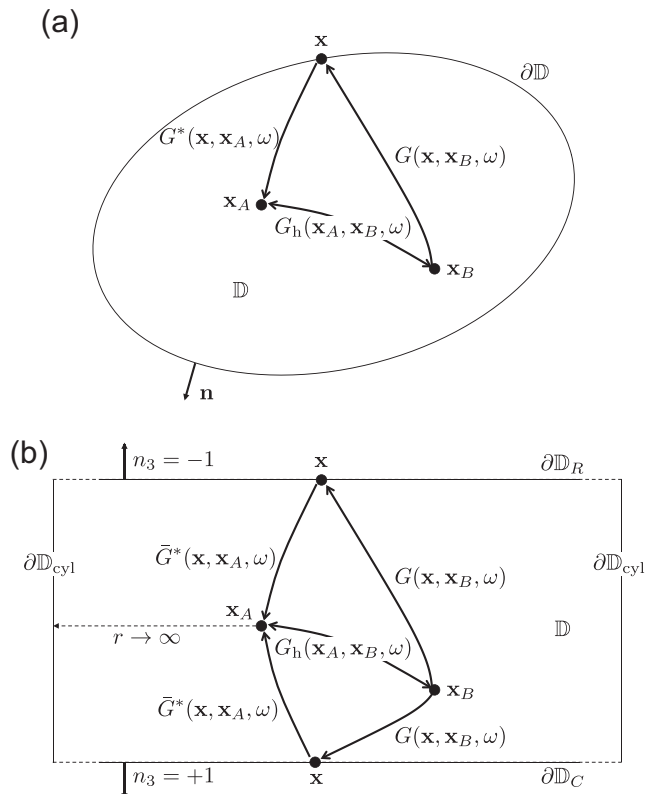


Figure 1. (a) Visualization of the homogeneous Green's function representation (eq. 1). Note that the rays in this figure represent the full responses between the source and receiver points, including multiple scattering. (b) Configuration for the modified representation. When the integrals along $\partial\mathbb{D}_C$ and $\partial\mathbb{D}_{\text{cyl}}$ vanish, a single-sided representation remains.

where Green's function $G(\mathbf{x}, \mathbf{x}_B, \omega)$ is the frequency-domain response to a unit source at \mathbf{x}_B , observed at \mathbf{x} (with ω denoting angular frequency), $G^*(\mathbf{x}, \mathbf{x}_A, \omega)$ (with the asterisk superscript denoting complex conjugation) is a back-propagating Green's function, and $G_h(\mathbf{x}_A, \mathbf{x}_B, \omega) = G(\mathbf{x}_A, \mathbf{x}_B, \omega) + G^*(\mathbf{x}_A, \mathbf{x}_B, \omega) = 2\Re\{G(\mathbf{x}_A, \mathbf{x}_B, \omega)\}$ (with \Re denoting the real part) is the homogeneous Green's function. Furthermore, $\rho(\mathbf{x})$ is the mass density, j the imaginary unit, ∂_i denotes differentiation with respect to x_i , and $\partial\mathbb{D}$ is a closed boundary with outward pointing normal vector $\mathbf{n} = (n_1, n_2, n_3)$; the domain enclosed by $\partial\mathbb{D}$ is denoted as \mathbb{D} . Einstein's summation convention applies to repeated subscripts. Another common form of the homogeneous Green's function is $\mathcal{G}_h(\mathbf{x}_A, \mathbf{x}_B, \omega) = \mathcal{G}(\mathbf{x}_A, \mathbf{x}_B, \omega) - \mathcal{G}^*(\mathbf{x}_A, \mathbf{x}_B, \omega)$, with $\mathcal{G} = G/j\omega$. Further details about the derivation and different forms of the classical homogeneous Green's function representation can be found in the Supporting Information.

In imaging and inverse scattering applications, $G(\mathbf{x}, \mathbf{x}_B, \omega)$ in eq. (1) stands for measurements at the boundary $\partial\mathbb{D}$, $G^*(\mathbf{x}, \mathbf{x}_A, \omega)$ back-propagates these measurements to \mathbf{x}_A inside the medium, and $G_h(\mathbf{x}_A, \mathbf{x}_B, \omega)$ (fixed \mathbf{x}_B , variable \mathbf{x}_A) quantifies the resolution of the image around \mathbf{x}_B . For sufficiently large $\partial\mathbb{D}$ and a homogeneous medium outside $\partial\mathbb{D}$, eq. (1) can be approximated in the time domain by (Wapenaar & Fokkema 2006; Fink 2008)

$$G_h(\mathbf{x}_B, \mathbf{x}_A, t) \approx \frac{2}{\rho c} \oint_{\partial\mathbb{D}} G(\mathbf{x}_B, \mathbf{x}, t) * G(\mathbf{x}, \mathbf{x}_A, -t) d^2\mathbf{x}, \quad (2)$$

where t denotes time, c is the propagation velocity and the inline asterisk denotes temporal convolution. In time-reversal acoustics, $G(\mathbf{x}, \mathbf{x}_A, -t)$ is the time-reversed field injected from the boundary into the medium, $G(\mathbf{x}_B, \mathbf{x}, t)$ propagates this field to \mathbf{x}_B inside the medium and $G_h(\mathbf{x}_B, \mathbf{x}_A, t)$ (fixed \mathbf{x}_A , variable \mathbf{x}_B) describes the time-dependent evolution of the injected field through the medium. In interferometric Green's function retrieval, $G(\mathbf{x}_B, \mathbf{x}, t) * G(\mathbf{x}_A, \mathbf{x}, -t)$ describes the cross-correlation of measurements at \mathbf{x}_B and \mathbf{x}_A of responses to sources at the boundary, and the causal part of $G_h(\mathbf{x}_B, \mathbf{x}_A, t)$ is the time-dependent response to a virtual source at \mathbf{x}_A , observed at \mathbf{x}_B .

3 AN AUXILIARY FUNCTION

In many practical cases, the medium of investigation can be approached from one side only. Hence, the exact closed boundary integral in eq. (1) is by necessity approximated by an open boundary integral, which leads to severe errors in the homogeneous Green's function, particularly when the medium is strongly inhomogeneous so that multiple scattering cannot be ignored. We consider a closed boundary $\partial\mathbb{D}$ which consists of three parts, according to $\partial\mathbb{D} = \partial\mathbb{D}_R \cup \partial\mathbb{D}_C \cup \partial\mathbb{D}_{\text{cyl}}$, see Fig. 1(b). Here $\partial\mathbb{D}_R$ is the accessible boundary of the medium where the measurements take place. For simplicity we will assume it is a horizontal boundary, defined by $x_3 = x_{3,R}$. The second part of the closed boundary, $\partial\mathbb{D}_C$, is a horizontal boundary somewhere inside the medium, at which no measurements are done. This boundary is defined by $x_3 = x_{3,C}$, with $x_{3,C} > x_{3,R}$ (the positive x_3 -axis is pointing downward). It is chosen sufficiently deep so that both \mathbf{x}_A and \mathbf{x}_B lie between $\partial\mathbb{D}_R$ and $\partial\mathbb{D}_C$. Finally, $\partial\mathbb{D}_{\text{cyl}}$ is a cylindrical boundary with a vertical axis through \mathbf{x}_A and infinite radius. This cylindrical boundary exists between $\partial\mathbb{D}_R$ and $\partial\mathbb{D}_C$ and closes the boundary $\partial\mathbb{D}$. The contribution of the integral over $\partial\mathbb{D}_{\text{cyl}}$ vanishes (but for another reason than Sommerfeld's radiation condition, Wapenaar *et al.* (1989)).

We modify eq. (1) for this configuration as follows

$$\begin{aligned}
& G(\mathbf{x}_A, \mathbf{x}_B, \omega) + \bar{G}^*(\mathbf{x}_B, \mathbf{x}_A, \omega) \\
&= \int_{\partial\mathbb{D}_R} \frac{1}{j\omega\rho} \{\bar{G}_A^* \partial_3 G_B - \partial_3 \bar{G}_A^* G_B\} d^2\mathbf{x} \\
&\quad - \int_{\partial\mathbb{D}_C} \frac{1}{j\omega\rho} \{\bar{G}_A^* \partial_3 G_B - \partial_3 \bar{G}_A^* G_B\} d^2\mathbf{x}, \quad (3)
\end{aligned}$$

where we used $\mathbf{n} = (0, 0, -1)$ on $\partial\mathbb{D}_R$ and $\mathbf{n} = (0, 0, +1)$ on $\partial\mathbb{D}_C$. \bar{G}_A and G_B are short-hand notations for $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ and $G(\mathbf{x}, \mathbf{x}_B, \omega)$, respectively. Note that we replaced $G(\mathbf{x}, \mathbf{x}_A, \omega)$ by a reference Green's function $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$, to be distinguished from the Green's function $G(\mathbf{x}, \mathbf{x}_B, \omega)$ in the actual medium. Both Green's functions obey the same wave equation in \mathbb{D} (with different source positions), but at and outside $\partial\mathbb{D} = \partial\mathbb{D}_R \cup \partial\mathbb{D}_C$ the medium parameters for these Green's functions may be different (Wapenaar *et al.* 1989). For the Green's function $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ we choose a reference medium which is identical to the actual medium below $\partial\mathbb{D}_R$, but homogeneous at and above $\partial\mathbb{D}_R$.

Next, at $\partial\mathbb{D}_C$ we choose boundary conditions in such a way that the integral along $\partial\mathbb{D}_C$ vanishes. Imposing either a Dirichlet or a Neumann boundary condition is not sufficient because when $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ is zero on $\partial\mathbb{D}_C$ then $\partial_3 \bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ is not, and vice versa. Hence, $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ cannot obey Dirichlet and Neumann conditions simultaneously. To deal with this problem, we introduce an auxiliary function $\Gamma(\mathbf{x}, \omega)$ which we subtract from the reference Green's function, according to

$$\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega) \rightarrow \bar{G}(\mathbf{x}, \mathbf{x}_A, \omega) - \Gamma(\mathbf{x}, \omega). \quad (4)$$

The function $\Gamma(\mathbf{x}, \omega)$ is defined in the reference medium and obeys the same wave equation as $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$, but without the singularity at \mathbf{x}_A . As a consequence, $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega) - \Gamma(\mathbf{x}, \omega)$ obeys the same wave equation as $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$, with the singularity at \mathbf{x}_A . Hence, in eq. (3) we may replace $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ by $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega) - \Gamma(\mathbf{x}, \omega)$, according to

$$\begin{aligned}
& G(\mathbf{x}_A, \mathbf{x}_B, \omega) + \{\bar{G}(\mathbf{x}_B, \mathbf{x}_A, \omega) - \Gamma(\mathbf{x}_B, \omega)\}^* \\
&= \int_{\partial\mathbb{D}_R} \frac{1}{j\omega\rho} \{(\bar{G}_A - \Gamma)^* \partial_3 G_B - \partial_3 (\bar{G}_A - \Gamma)^* G_B\} d^2\mathbf{x} \\
&\quad - \int_{\partial\mathbb{D}_C} \frac{1}{j\omega\rho} \{(\bar{G}_A - \Gamma)^* \partial_3 G_B - \partial_3 (\bar{G}_A - \Gamma)^* G_B\} d^2\mathbf{x}. \quad (5)
\end{aligned}$$

When a function $\Gamma(\mathbf{x}, \omega)$ can be found such that $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega) - \Gamma(\mathbf{x}, \omega)$ obeys the Cauchy boundary condition (i.e. simultaneous Dirichlet and Neumann boundary conditions) on $\partial\mathbb{D}_C$, then the integral along $\partial\mathbb{D}_C$ vanishes.

Introducing auxiliary functions is a common approach to manipulate the boundary conditions (Morse & Feshbach 1953; Berkhout 1982). In fact it has been previously proposed for the integral in eq. (5) (Weglein *et al.* 2011), but a straightforward way to find a $\Gamma(\mathbf{x}, \omega)$ that obeys the conditions for an arbitrary inhomogeneous medium has, to the knowledge of the authors, not been presented yet. Recent work of the authors (Wapenaar *et al.* 2014) concerns the generalization of the single-sided 1-D Marchenko method for inverse scattering (Marchenko 1955) and autofocusing (Rose 2002; Brogini & Snieder 2012) to the 3-D situation. We show with intuitive arguments that the so-called focusing functions, developed for the single-sided 3-D Marchenko method, provide a means to find $\Gamma(\mathbf{x}, \omega)$. For a more precise derivation we refer to the Supporting Information.

Fig. 2(a) shows a focusing function $f_1^+(\mathbf{x}, \mathbf{x}_A, \omega)$ (downward pointing red rays), which is emitted from the homogeneous up-

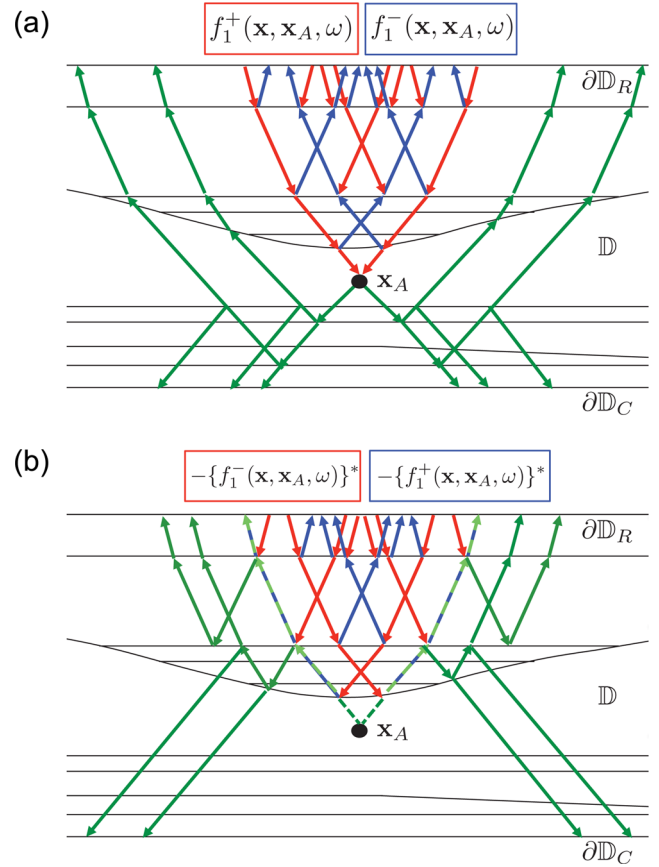


Figure 2. Visualization of the auxiliary function $\Gamma(\mathbf{x}, \omega)$. It consists of the focusing functions $f_1^\pm(\mathbf{x}, \mathbf{x}_A, \omega)$ and $-\{f_1^\mp(\mathbf{x}, \mathbf{x}_A, \omega)\}^*$ (red and blue rays) and the Green's function $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ (green rays). By subtracting this auxiliary function from the Green's function (eq. 4), the field in the half-space below \mathbf{x}_A vanishes and hence obeys the Cauchy boundary condition at $\partial\mathbb{D}_C$.

per half-space into the medium to focus at \mathbf{x}_A . Because there is no sink at \mathbf{x}_A to annihilate the focused field $f_1^+(\mathbf{x}_A, \mathbf{x}_A, \omega)$, the field continues to propagate as if there were a source for down-going waves at \mathbf{x}_A (indicated by the green rays). The response to this virtual downward radiating source mimics a part of the Green's function $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$. We now discuss how the remaining part of the Green's function is obtained. Before reaching the focus, a part of the focusing function is reflected upward and is called $f_1^-(\mathbf{x}, \mathbf{x}_A, \omega)$ (upward pointing blue rays in Fig. 2(a)). Fig. 2(b) visualizes the emission of the back-propagating focusing function $-\{f_1^-(\mathbf{x}, \mathbf{x}_A, \omega)\}^*$ into the medium (downward pointing red rays). Its response consists of $-\{f_1^+(\mathbf{x}, \mathbf{x}_A, \omega)\}^*$ (upward pointing blue rays), and a field apparently originating from a source for up-going waves at \mathbf{x}_A (indicated by the green rays). The response to this virtual upward radiating source mimics the remaining part of the Green's function $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$. Figs 2(a) and (b) together visualize the auxiliary function $\Gamma(\mathbf{x}, \omega)$. It consists of the Green's function $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ (the green rays in both figures) and, above the focal point, the focusing function $f_1(\mathbf{x}, \mathbf{x}_A, \omega) - \{f_1(\mathbf{x}, \mathbf{x}_A, \omega)\}^*$, with $f_1(\mathbf{x}, \mathbf{x}_A, \omega) = f_1^+(\mathbf{x}, \mathbf{x}_A, \omega) + f_1^-(\mathbf{x}, \mathbf{x}_A, \omega)$ (the red and blue rays). Hence,

$$\Gamma(\mathbf{x}, \omega) = \bar{G}(\mathbf{x}, \mathbf{x}_A, \omega) + H(x_{3,A} - x_3) 2j\Im\{f_1(\mathbf{x}, \mathbf{x}_A, \omega)\}, \quad (6)$$

where $H(x_3)$ is the Heaviside step function and \Im denotes the imaginary part. With this definition, $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega) - \Gamma(\mathbf{x}, \omega)$ vanishes in

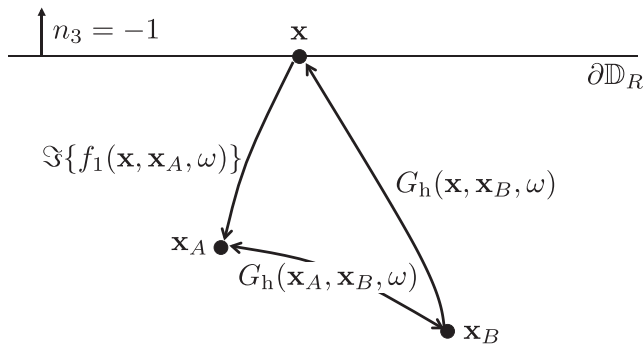


Figure 3. Visualization of the single-sided homogeneous Green's function representation (eq. 8). Similar as in Fig. 1, the rays in this figure represent the full responses between the source and receiver points, including multiple scattering.

the half-space below \mathbf{x}_A . Because this function is zero in an entire half-space, its derivative is zero as well and hence it obeys the Cauchy boundary condition at $\partial\mathbb{D}_C$.

4 THE SINGLE-SIDED HOMOGENEOUS GREEN'S FUNCTION REPRESENTATION

Substitution of eq. (6) into eq. (5) gives

$$\begin{aligned} G(\mathbf{x}_A, \mathbf{x}_B, \omega) + H(x_{3,A} - x_{3,B})2j\Im\{f_1(\mathbf{x}_B, \mathbf{x}_A, \omega)\} \\ = \int_{\partial\mathbb{D}_R} \frac{2}{\omega\rho(\mathbf{x})} \left(\Im\{f_1(\mathbf{x}, \mathbf{x}_A, \omega)\} \partial_3 G(\mathbf{x}, \mathbf{x}_B, \omega) \right. \\ \left. - \Im\{\partial_3 f_1(\mathbf{x}, \mathbf{x}_A, \omega)\} G(\mathbf{x}, \mathbf{x}_B, \omega) \right) d^2\mathbf{x}. \end{aligned} \quad (7)$$

Taking the real part of both sides of this equation gives

$$\begin{aligned} G_h(\mathbf{x}_A, \mathbf{x}_B, \omega) \\ = \int_{\partial\mathbb{D}_R} \frac{2}{\omega\rho(\mathbf{x})} \left(\Re\{f_1(\mathbf{x}, \mathbf{x}_A, \omega)\} \partial_3 G_h(\mathbf{x}, \mathbf{x}_B, \omega) \right. \\ \left. - \Re\{\partial_3 f_1(\mathbf{x}, \mathbf{x}_A, \omega)\} G_h(\mathbf{x}, \mathbf{x}_B, \omega) \right) d^2\mathbf{x}. \end{aligned} \quad (8)$$

This is the main result of this paper. The homogeneous Green's function $G_h(\mathbf{x}_A, \mathbf{x}_B, \omega)$, with both \mathbf{x}_A and \mathbf{x}_B inside the medium, is represented by an integral along the acquisition boundary $\partial\mathbb{D}_R$ only (Fig. 3).

Note that the Green's function $G_h(\mathbf{x}, \mathbf{x}_B, \omega)$ under the integral can be obtained from a similar representation. With some simple replacements (see Supporting Information for details) we obtain

$$\begin{aligned} G_h(\mathbf{x}, \mathbf{x}_B, \omega) \\ = \int_{\partial\mathbb{D}_S} \frac{2}{\omega\rho(\mathbf{x}')} \left(\Re\{f_1(\mathbf{x}', \mathbf{x}_B, \omega)\} \partial'_3 G_h(\mathbf{x}, \mathbf{x}', \omega) \right. \\ \left. - \Re\{\partial'_3 f_1(\mathbf{x}', \mathbf{x}_B, \omega)\} G_h(\mathbf{x}, \mathbf{x}', \omega) \right) d^2\mathbf{x}', \end{aligned} \quad (9)$$

with \mathbf{x} on $\partial\mathbb{D}_R$ and \mathbf{x}' on $\partial\mathbb{D}_S$, just above $\partial\mathbb{D}_R$. Note that $G_h(\mathbf{x}, \mathbf{x}', \omega)$ stands for the reflection response at the surface. Hence, eqs (8) and (9) can be used to retrieve $G_h(\mathbf{x}_A, \mathbf{x}_B, \omega)$ from $G(\mathbf{x}, \mathbf{x}', \omega)$ in a data-driven way. The complete procedure is as follows. Define the initial estimate of the focusing function $f_1(\mathbf{x}', \mathbf{x}_B, \omega)$ by the time-reversed direct arrivals between \mathbf{x}_B and \mathbf{x}' at the boundary. Retrieve the complete focusing function $f_1(\mathbf{x}', \mathbf{x}_B, \omega)$ from its initial estimate and the reflection response $G(\mathbf{x}, \mathbf{x}', \omega)$ at the surface, using the iterative Marchenko method (Wapenaar *et al.* 2014). Use

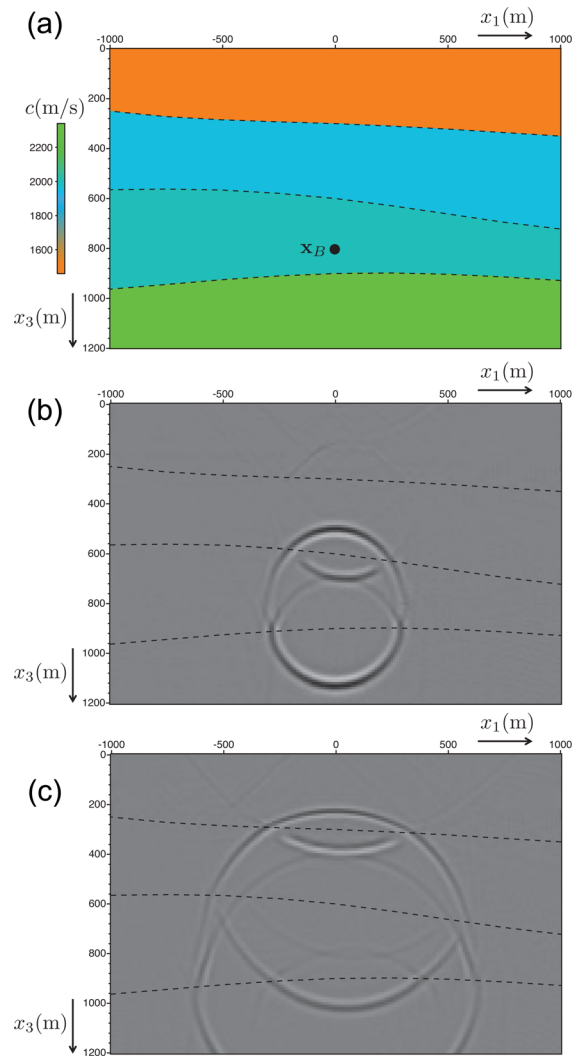


Figure 4. Numerical example, illustrating the application of the single-sided homogeneous Green's function representations (eqs 8 and 9). (a) Inhomogeneous medium. (b) Snapshot of $G(\mathbf{x}_A, \mathbf{x}_B, t) + G(\mathbf{x}_A, \mathbf{x}_B, -t)$ at $t = 0.15$ s, for fixed $\mathbf{x}_B = (0, 800)$ and variable \mathbf{x}_A . (c) Idem, for $t = 0.30$ s.

eq. (9) to obtain $G_h(\mathbf{x}, \mathbf{x}_B, \omega)$ from $G_h(\mathbf{x}, \mathbf{x}', \omega)$. This step brings the sources down from \mathbf{x}' on $\partial\mathbb{D}_S$ to \mathbf{x}_B . Next, in a similar way use eq. (8) to obtain $G_h(\mathbf{x}_A, \mathbf{x}_B, \omega)$ from $G_h(\mathbf{x}, \mathbf{x}_B, \omega)$. This step brings the receivers down from \mathbf{x} on $\partial\mathbb{D}_R$ to \mathbf{x}_A .

Recall that the Green's functions without bars are defined in the actual medium, which may be inhomogeneous above $\partial\mathbb{D}_R$. For example, similar as discussed by Singh *et al.* (2015), there may be a free boundary just above $\partial\mathbb{D}_R$, in which case the second term under the integral in eqs (7)–(9) vanishes. In the following example, however, the half-space above $\partial\mathbb{D}_R$ is homogeneous. Fig. 4(a) shows a 2D inhomogeneous medium. We modelled the reflection response $G(\mathbf{x}, \mathbf{x}', \omega)$ for 600 sources and 600 receivers, with a horizontal spacing of 10 m, at the upper boundary. The central frequency of the band-limited source function is 30 Hz. Using the process described above we obtain $G_h(\mathbf{x}_A, \mathbf{x}_B, \omega)$, or in the time domain $G_h(\mathbf{x}_A, \mathbf{x}_B, t) = G(\mathbf{x}_A, \mathbf{x}_B, t) + G(\mathbf{x}_A, \mathbf{x}_B, -t)$. The Supporting Information contains a movie of $G_h(\mathbf{x}_A, \mathbf{x}_B, t)$ for $t \geq 0$. Figs 4(b)–(c) show ‘snapshots’ of this function for $t = 0.15$ s and $t = 0.30$ s, respectively, each time for fixed $\mathbf{x}_B = (0, 800)$ and variable \mathbf{x}_A . Note that the movie and snapshots nicely mimic the response to a source at $\mathbf{x}_B = (0, 800)$, including scattering at the interfaces between

layers with different propagation velocities. It is remarkable that this virtual response is obtained from the reflection response at the upper boundary plus estimates of the direct arrivals, but no information about the positions and shapes of the scattering interfaces has been used. Yet the virtual response clearly shows how scattering occurs at the interfaces.

5 DISCUSSION

Unlike the classical homogeneous Green's function representation (eq. 1), the single-sided representation of eq. (8) can be applied in situations in which the medium of investigation is accessible from one side only. We foresee many interesting applications, which we briefly indicate below.

Eq. (8) will find its most prominent applications in holographic imaging and inverse scattering in strongly inhomogeneous media. As illustrated in the previous section, the two-step procedure described by eqs (8) and (9) brings sources and receivers down from the surface to arbitrary positions in the subsurface. For weakly scattering media (ignoring multiples), a similar two-step process is known in exploration seismology as source–receiver redatuming (Berkhout 1982; Berryhill 1984). For strongly scattering media (including multiple scattering) a similar two-step process, called source–receiver interferometry, has previously been formulated in terms of closed-boundary representations for the homogeneous Green's function (Halliday & Curtis 2010). Our method replaces the closed boundary representations in the latter method by single-sided representations. Once $G_h(\mathbf{x}_A, \mathbf{x}_B, \omega)$ is obtained, an image can be formed by setting \mathbf{x}_A equal to \mathbf{x}_B . However, $G_h(\mathbf{x}_A, \mathbf{x}_B, t)$ for variable and independent virtual sources and receivers contains a wealth of additional information about the interior of the medium, as can be witnessed from Fig. 4. The advantages of the two-step process for holographic imaging and inverse scattering will be further explored. Results like that in Fig. 4 could for example also be used to predict the propagation of microseismic signals through an unknown subsurface.

For the field of time-reversal acoustics, the inverse Fourier transform of eq. (7) forms an alternative to eq. (2). It shows that, instead of physically injecting $G(\mathbf{x}, \mathbf{x}_A, -t)$ from a closed boundary into the medium, the function $f_1(\mathbf{x}, \mathbf{x}_A, t) - f_1(\mathbf{x}, \mathbf{x}_A, -t)$ should be injected into the medium when it is accessible only from one side. The injected field will focus at \mathbf{x}_A and subsequently the focused field will act as a virtual source.

The application of eq. (8) for interferometric Green's function retrieval is very similar to the redatuming procedure described above. However, in the field of seismic interferometry the Green's functions $G(\mathbf{x}_A, \mathbf{x}, t)$ and $G(\mathbf{x}_B, \mathbf{x}, t)$ usually stand for measured data. This has the potential to obtain a more accurate estimate of the focusing function $f_1(\mathbf{x}, \mathbf{x}_A, t)$. Substituting its Fourier transform into eq. (8), together with that of the measured response $G(\mathbf{x}_B, \mathbf{x}, t)$, may yield an even more accurate recovery of the homogeneous Green's function.

We foresee that the single-sided representation of the homogeneous Green's function will lead to many more applications in holographic imaging, inverse scattering, time-reversal acoustics and interferometric Green's function retrieval.

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SUPPORTING INFORMATION

Additional Supporting Information may be found in the online version of this paper:

Appendix 1. Derivation of the classical homogeneous Green's function representation.

Appendix 2. Derivation of the auxiliary function.

Appendix 3. Alternative derivation of the single-sided representation.

Movie 1. The homogeneous Green's function $G_h(\mathbf{x}_A, \mathbf{x}_B, t)$ for $t \geq 0$, obtained from the reflection response at the upper boundary.

Movie 2. As a reference, the Green's function $G(\mathbf{x}_A, \mathbf{x}_B, t)$ obtained by direct modelling.

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A single-sided homogeneous Green’s function representation for holographic imaging, inverse scattering, time-reversal acoustics and interferometric Green’s function retrieval

Appendices

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1 DERIVATION OF THE CLASSICAL HOMOGENEOUS GREEN’S FUNCTION REPRESENTATION

We consider the scalar wave equation for the acoustic pressure in an arbitrary inhomogeneous lossless medium,

$$\partial_i(\rho^{-1}\partial_i p) - \kappa\partial_t^2 p = -\partial_t q. \quad (1)$$

Here $p = p(\mathbf{x}, t)$ is the acoustic pressure, with $\mathbf{x} = (x_1, x_2, x_3)$ denoting position in a Cartesian coordinate system (with the positive x_3 -axis pointing downward) and t denoting time. Further, $\kappa = \kappa(\mathbf{x})$ and $\rho = \rho(\mathbf{x})$ are the compressibility and mass density of the medium, respectively, and $q = q(\mathbf{x}, t)$ is the source function in terms of volume-injection rate density. Differentiation with respect to time is denoted by ∂_t and differentiation with respect to x_i by ∂_i . Einstein’s summation convention applies to repeated subscripts (except for subscript t). The propagation velocity $c = c(\mathbf{x})$ of the medium is related to κ and ρ via $c = (\kappa\rho)^{-1/2}$. Although we consider here specifically the acoustic wave equation, all that follows applies to other scalar wave equations by making the appropriate substitutions.

The Green’s function $G(\mathbf{x}, \mathbf{x}_A, t)$ is defined as the response to an impulsive source at $\mathbf{x}_A = (x_{1,A}, x_{2,A}, x_{3,A})$ and $t = 0$, observed at \mathbf{x} . It obeys the following wave equation

$$\partial_i(\rho^{-1}\partial_i G) - \kappa\partial_t^2 G = -\delta(\mathbf{x} - \mathbf{x}_A)\partial_t\delta(t). \quad (2)$$

To get the causal solution of equation (2), we impose the initial condition $G(\mathbf{x}, \mathbf{x}_A, t) = 0$ for all $t < 0$. This corresponds to the physical radiation condition of outgoing waves at infinity. Because wave equation (2) is time-symmetric (except for the source function, which is anti-symmetric in time), the time-reversed Green’s function $G(\mathbf{x}, \mathbf{x}_A, -t)$ is a solution of the same wave equation, but with opposite sign for the source on the right-hand side. This time-reversed solution is acausal and obeys the final condition $G(\mathbf{x}, \mathbf{x}_A, -t) = 0$ for all $t > 0$. This corresponds to the non-physical radiation condition of incoming waves at infinity. The homogeneous Green’s function $G_h(\mathbf{x}, \mathbf{x}_A, t)$ is defined as

the sum of the causal and acausal Green’s functions, hence $G_h(\mathbf{x}, \mathbf{x}_A, t) = G(\mathbf{x}, \mathbf{x}_A, t) + G(\mathbf{x}, \mathbf{x}_A, -t)$. It obeys the homogeneous wave equation (i.e., the wave equation without the singularity on the right-hand side). We define the temporal Fourier transform of a space- and time-dependent quantity $p(\mathbf{x}, t)$ as

$$p(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} p(\mathbf{x}, t) \exp(-j\omega t) dt, \quad (3)$$

where ω is the angular frequency and j the imaginary unit. To keep the notation simple, we denote quantities in the time and frequency domain by the same symbol. In the frequency domain, the homogeneous Green’s function is defined as $G_h(\mathbf{x}, \mathbf{x}_A, \omega) = G(\mathbf{x}, \mathbf{x}_A, \omega) + G^*(\mathbf{x}, \mathbf{x}_A, \omega) = 2\Re\{G(\mathbf{x}, \mathbf{x}_A, \omega)\}$, where $G(\mathbf{x}, \mathbf{x}_A, \omega)$ is the Fourier transform of $G(\mathbf{x}, \mathbf{x}_A, t)$, the asterisk superscript denotes complex conjugation, and \Re stands for the real part. We derive a representation for $G_h(\mathbf{x}, \mathbf{x}_A, \omega)$ from Rayleigh’s reciprocity theorem (Rayleigh 1878; de Hoop 1988; Fokkema & van den Berg 1993) for two acoustic states A and B in a domain \mathbb{D} enclosed by boundary $\partial\mathbb{D}$ with outward pointing normal $\mathbf{n} = (n_1, n_2, n_3)$. In the frequency domain this theorem reads

$$\int_{\mathbb{D}} \{q_A p_B - p_A q_B\} d^3\mathbf{x} = \oint_{\partial\mathbb{D}} \frac{1}{j\omega\rho} \{p_A \partial_i p_B - (\partial_i p_A) p_B\} n_i d^2\mathbf{x}. \quad (4)$$

Here q_A and p_A are the source and acoustic pressure in state A , whereas q_B and p_B are these quantities in state B . The medium parameters for state A and B are identical in \mathbb{D} , hence, the two states obey the same wave equation in \mathbb{D} , but at and outside $\partial\mathbb{D}$ the medium parameters may be different. Because of the time-reversal invariance of the wave equation, the complex conjugates $-q_A^*$ and p_A^* obey the same wave equation as q_A and p_A . Making this replacement in equation (4) we obtain a second form of Rayleigh’s reciprocity theorem (Bojarski 1983),

$$\int_{\mathbb{D}} \{q_A^* p_B + p_A^* q_B\} d^3\mathbf{x} = \oint_{\partial\mathbb{D}} \frac{-1}{j\omega\rho} \{p_A^* \partial_i p_B - (\partial_i p_A^*) p_B\} n_i d^2\mathbf{x}. \quad (5)$$

Substituting $q_A = \delta(\mathbf{x} - \mathbf{x}_A)$, $p_A = G(\mathbf{x}, \mathbf{x}_A, \omega) =$

$G(\mathbf{x}_A, \mathbf{x}, \omega)$, $q_B = \delta(\mathbf{x} - \mathbf{x}_B)$ and $p_B = G(\mathbf{x}, \mathbf{x}_B, \omega)$ into equation (5), with \mathbf{x}_A and \mathbf{x}_B both situated in \mathbb{D} , it follows that the homogeneous Green's function can be represented as (Porter 1970; Oristaglio 1989; Wapenaar & Fokkema 2006)

$$G_h(\mathbf{x}_A, \mathbf{x}_B, \omega) = \oint_{\partial\mathbb{D}} \frac{-1}{j\omega\rho(\mathbf{x})} \{G^*(\mathbf{x}, \mathbf{x}_A, \omega)\partial_i G(\mathbf{x}, \mathbf{x}_B, \omega) - \partial_i G^*(\mathbf{x}, \mathbf{x}_A, \omega)G(\mathbf{x}, \mathbf{x}_B, \omega)\} n_i d^2\mathbf{x}. \quad (6)$$

This is equation (1) for the configuration of Figure 1(a) in the main paper.

Another form of the homogeneous Green's function representation exists for a modified Green's function $\mathcal{G}(\mathbf{x}, \mathbf{x}_A, t)$, obeying the wave equation

$$\partial_i(\rho^{-1}\partial_i\mathcal{G}) - \kappa\partial_t^2\mathcal{G} = -\delta(\mathbf{x} - \mathbf{x}_A)\delta(t). \quad (7)$$

In comparison with equation (2), the time-derivative of the delta function on the right-hand side is absent. The solutions of both equations are related via $G(\mathbf{x}, \mathbf{x}_A, t) = \partial_t\mathcal{G}(\mathbf{x}, \mathbf{x}_A, t)$. For this modified Green's function, the homogeneous Green's function is given by $\mathcal{G}_h(\mathbf{x}, \mathbf{x}_A, t) = \mathcal{G}(\mathbf{x}, \mathbf{x}_A, t) - \mathcal{G}(\mathbf{x}, \mathbf{x}_A, -t)$ or, in the frequency domain, $\mathcal{G}_h(\mathbf{x}, \mathbf{x}_A, \omega) = \mathcal{G}(\mathbf{x}, \mathbf{x}_A, \omega) - \mathcal{G}^*(\mathbf{x}, \mathbf{x}_A, \omega) = 2j\Im\{\mathcal{G}(\mathbf{x}, \mathbf{x}_A, \omega)\}$, where \Im denotes the imaginary part. Using $G(\mathbf{x}, \mathbf{x}_A, \omega) = j\omega\mathcal{G}(\mathbf{x}, \mathbf{x}_A, \omega)$, it easily follows from equation (6) that the modified homogeneous Green's function can be represented as

$$\mathcal{G}_h(\mathbf{x}_A, \mathbf{x}_B, \omega) = \oint_{\partial\mathbb{D}} \frac{1}{\rho(\mathbf{x})} \{G^*(\mathbf{x}, \mathbf{x}_A, \omega)\partial_i\mathcal{G}(\mathbf{x}, \mathbf{x}_B, \omega) - \partial_i G^*(\mathbf{x}, \mathbf{x}_A, \omega)\mathcal{G}(\mathbf{x}, \mathbf{x}_B, \omega)\} n_i d^2\mathbf{x}. \quad (8)$$

This equation, with $\mathcal{G}_h(\mathbf{x}_A, \mathbf{x}_B, \omega) = 2j\Im\{\mathcal{G}(\mathbf{x}_A, \mathbf{x}_B, \omega)\}$, is the more common form of the classical homogeneous Green's function representation (Porter 1970; Oristaglio 1989). Nevertheless, we prefer to continue with the representation of equation (6) for the following reasons. First, the Green's function $G(\mathbf{x}, \mathbf{x}_A, t)$, obeying wave equation (2), has a clear physical meaning: it represents the response in terms of acoustic pressure to an impulsive point source of volume-injection rate density (this is easily seen by comparing equation (2) with equation (1)). Second, unlike $\mathcal{G}_h(\mathbf{x}_A, \mathbf{x}_B, t)$, which is anti-symmetric in time and therefore vanishes at $t = 0$, the homogeneous Green's function $G_h(\mathbf{x}_A, \mathbf{x}_B, t)$ does not vanish at $t = 0$, which makes it better suited for imaging applications. Third, the representation of equation (6) is consistent with earlier work of the authors on seismic interferometry (Wapenaar & Fokkema 2006) and Marchenko imaging (Wapenaar et al. 2014).

For the configuration of Figure 1(b) in the main paper we modify equation (6) to

$$G(\mathbf{x}_A, \mathbf{x}_B, \omega) + \{\bar{G}(\mathbf{x}_B, \mathbf{x}_A, \omega) - \Gamma(\mathbf{x}_B, \omega)\}^* \\ = \int_{\partial\mathbb{D}_R} \frac{1}{j\omega\rho} \{(\bar{G}_A - \Gamma)^* \partial_3 G_B - \partial_3(\bar{G}_A - \Gamma)^* G_B\} d^2\mathbf{x} \\ - \int_{\partial\mathbb{D}_C} \frac{1}{j\omega\rho} \{(\bar{G}_A - \Gamma)^* \partial_3 G_B - \partial_3(\bar{G}_A - \Gamma)^* G_B\} d^2\mathbf{x}, \quad (9)$$

where \bar{G}_A and G_B are short-hand notations for $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ and $G(\mathbf{x}, \mathbf{x}_B, \omega)$, respectively. $\partial\mathbb{D}_R$ and $\partial\mathbb{D}_C$ are horizontal boundaries, defined by $x_3 = x_{3,R}$ and $x_3 = x_{3,C}$, respectively (with $x_{3,C} > x_{3,R}$). We replaced $G(\mathbf{x}, \mathbf{x}_A, \omega)$

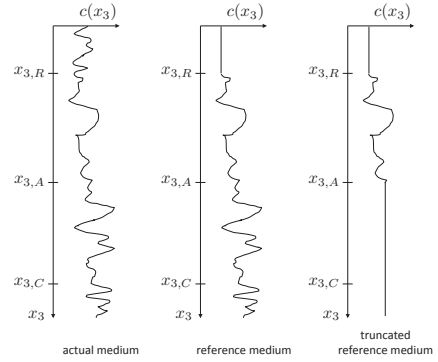


Figure S.1. Illustration of a vertical cross-section of one of the medium parameters ($c(x_3)$), for the actual medium, the reference medium (homogeneous above $x_{3,R}$ and below $x_{3,A}$) and the truncated reference medium (homogeneous above $x_{3,R}$ and below $x_{3,A}$). $G(\mathbf{x}, \mathbf{x}_B, \omega)$ is defined in the actual medium, $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ and $\Gamma(\mathbf{x}, \omega)$ in the reference medium, and $f_1(\mathbf{x}, \mathbf{x}_A, \omega)$ in the truncated reference medium.

by a reference Green's function $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$, to be distinguished from the Green's function $G(\mathbf{x}, \mathbf{x}_B, \omega)$ in the actual medium. Both Green's functions obey the same wave equation in \mathbb{D} (with different source positions), but at and outside $\partial\mathbb{D} = \partial\mathbb{D}_R \cup \partial\mathbb{D}_C$ the medium parameters for these Green's functions may be different. For the Green's function $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ we choose a reference medium which is identical to the actual medium below $\partial\mathbb{D}_R$, but homogeneous at and above $\partial\mathbb{D}_R$. This is illustrated in Figure S.1 for a vertical cross-section through the medium. The auxiliary function $\Gamma(\mathbf{x}, \omega)$ is defined in the reference medium and obeys the same wave equation as $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega)$, but without the singularity at \mathbf{x}_A . In the next section we derive a function $\Gamma(\mathbf{x}, \omega)$ such that $\bar{G}(\mathbf{x}, \mathbf{x}_A, \omega) - \Gamma(\mathbf{x}, \omega)$ obeys the Cauchy boundary condition (i.e., simultaneous Dirichlet and Neumann boundary conditions) on $\partial\mathbb{D}_C$.

2 DERIVATION OF THE AUXILIARY FUNCTION

We start by introducing a focusing function $f_1(\mathbf{x}, \mathbf{x}_A, \omega)$, where \mathbf{x}_A denotes a focal point in \mathbb{D} (hence, between $\partial\mathbb{D}_R$ and $\partial\mathbb{D}_C$). We truncate the reference medium at the depth level $\partial\mathbb{D}_A$ of the focal point \mathbf{x}_A : below $\partial\mathbb{D}_A$ the truncated reference medium is homogeneous. This is illustrated in Figure S.1 for a vertical cross-section. Recall that the reference medium is the actual medium with the half-space above $\partial\mathbb{D}_R$ replaced by a homogeneous medium. Hence, the truncated reference medium is the actual medium between $\partial\mathbb{D}_R$ and $\partial\mathbb{D}_A$, sandwiched between homogeneous half-spaces. The focusing function $f_1(\mathbf{x}, \mathbf{x}_A, \omega)$ is defined in the truncated reference medium. We explicitly write this function as a superposition of its downgoing and upgoing constituents, according to

$$f_1(\mathbf{x}, \mathbf{x}_A, \omega) = f_1^+(\mathbf{x}, \mathbf{x}_A, \omega) + f_1^-(\mathbf{x}, \mathbf{x}_A, \omega), \quad (10)$$

where superscript $+$ stands for downgoing and $-$ for upgoing, both at observation point \mathbf{x} (Figure S.2). The downgoing field $f_1^+(\mathbf{x}, \mathbf{x}_A, \omega)$ is incident to the truncated reference medium from the homogeneous upper half-space ($x_3 < x_{3,R}$). This field is shaped such that at the focal depth

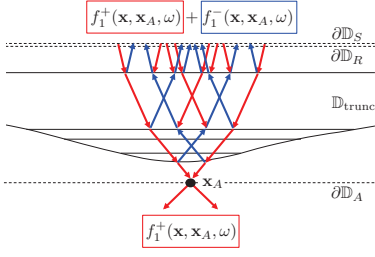


Figure S.2. Focusing function $f_1(\mathbf{x}, \mathbf{x}_A, \omega) = f_1^+(\mathbf{x}, \mathbf{x}_A, \omega) + f_1^-(\mathbf{x}, \mathbf{x}_A, \omega)$ in the truncated reference medium.

$x_3 = x_{3,A}$ the following conditions are obeyed (Wapenaar et al. 2014; Slob et al. 2014)

$$\partial_3 f_1^+(\mathbf{x}, \mathbf{x}_A, \omega)|_{x_3=x_{3,A}} = -\frac{1}{2}j\omega\rho(\mathbf{x})\delta(\mathbf{x}_H - \mathbf{x}_{H,A}), \quad (11)$$

$$\partial_3 f_1^-(\mathbf{x}, \mathbf{x}_A, \omega)|_{x_3=x_{3,A}} = 0. \quad (12)$$

Here \mathbf{x}_H stands for the horizontal components of the coordinate vector, hence, $\mathbf{x}_H = (x_1, x_2)$ and $\mathbf{x}_{H,A} = (x_{1,A}, x_{2,A})$. The factor $-\frac{1}{2}j\omega\rho(\mathbf{x})$ is chosen for convenience and for consistency with previous work. At and below the focal depth there is no upgoing field because the truncated reference medium is homogeneous below the truncation depth. A focusing function which exactly obeys condition (11) is unstable in the evanescent field. In the following we exclude evanescent wave components and tacitly assume that the spatial delta function in equation (11) is band limited. Note that, despite the focusing condition formulated in equation (11), the focusing function $f_1(\mathbf{x}, \mathbf{x}_A, \omega)$ obeys the wave equation without a singularity on the right-hand side. Hence, the focal point \mathbf{x}_A is not a sink; below the focal depth the function $f_1^+(\mathbf{x}, \mathbf{x}_A, \omega)$ continues propagating downward.

Next we derive a relation between the focusing function in the truncated reference medium and the Green's function in the reference medium (without the truncation). To this end we apply the two forms of Rayleigh's reciprocity theorem, equations (4) and (5), to the truncated domain $\mathbb{D}_{\text{trunc}}$. The lower boundary of this domain is $\partial\mathbb{D}_A$, defined by $x_3 = x_{3,A}$. For the upper boundary we choose $\partial\mathbb{D}_S$ (defined as $x_3 = x_{3,S}$), just above $\partial\mathbb{D}_R$. Furthermore, because we decomposed the focusing function, we modify the boundary integrals in equations (4) and (5) for decomposed wave fields, according to (Wapenaar et al. 2014)

$$\begin{aligned} \int_{\mathbb{D}_{\text{trunc}}} \{q_A p_B - p_A q_B\} d^3 \mathbf{x} = & \quad (13) \\ \int_{\partial\mathbb{D}_S} \frac{2}{j\omega\rho} \{(\partial_3 p_A^+) p_B^- + (\partial_3 p_A^-) p_B^+\} d^2 \mathbf{x} \\ - \int_{\partial\mathbb{D}_A} \frac{2}{j\omega\rho} \{(\partial_3 p_A^+) p_B^- + (\partial_3 p_A^-) p_B^+\} d^2 \mathbf{x}, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{D}_{\text{trunc}}} \{q_A^* p_B + p_A^* q_B\} d^3 \mathbf{x} = & \quad (14) \\ - \int_{\partial\mathbb{D}_S} \frac{2}{j\omega\rho} \{(\partial_3 p_A^+)^* p_B^- + (\partial_3 p_A^-)^* p_B^-\} d^2 \mathbf{x} \\ + \int_{\partial\mathbb{D}_A} \frac{2}{j\omega\rho} \{(\partial_3 p_A^+)^* p_B^- + (\partial_3 p_A^-)^* p_B^-\} d^2 \mathbf{x}, \end{aligned}$$

respectively, with $p_A = p_A^+ + p_A^-$ and $p_B = p_B^+ + p_B^-$. Equation

(13) is exact, whereas in equation (14) evanescent wave components are neglected. We decompose the reference Green's function into downgoing and upgoing waves at observation point \mathbf{x} , according to

$$\bar{G}(\mathbf{x}, \mathbf{x}', \omega) = \bar{G}^+(\mathbf{x}, \mathbf{x}', \omega) + \bar{G}^-(\mathbf{x}, \mathbf{x}', \omega), \quad (15)$$

$$\bar{G}(\mathbf{x}, \mathbf{x}', \omega)|_{x_3=x_{3,S}} = \bar{G}^-(\mathbf{x}, \mathbf{x}', \omega)|_{x_3=x_{3,S}}, \quad (16)$$

with \mathbf{x}' anywhere below $\partial\mathbb{D}_S$, hence $x'_3 > x_{3,S}$. Equation (16) states that the reference Green's function is purely upgoing at $\partial\mathbb{D}_S$. This is because the reference medium is homogeneous above $\partial\mathbb{D}_S$, whereas the source of the Green's function is chosen below $\partial\mathbb{D}_S$. Substituting $p_A^\pm(\mathbf{x}, \omega) = f_1^\pm(\mathbf{x}, \mathbf{x}_A, \omega)$, $q_A(\mathbf{x}, \omega) = 0$, $p_B^\pm(\mathbf{x}, \omega) = \bar{G}^\pm(\mathbf{x}, \mathbf{x}', \omega)$ and $q_B(\mathbf{x}, \omega) = \delta(\mathbf{x} - \mathbf{x}')$ into equations (13) and (14), using equations (10) – (12), (15) and (16), gives

$$\bar{G}^-(\mathbf{x}_A, \mathbf{x}', \omega) + H(x_{3,A} - x'_3) f_1(\mathbf{x}', \mathbf{x}_A, \omega) \quad (17)$$

$$= -2 \int_{\partial\mathbb{D}_S} \bar{G}^-(\mathbf{x}, \mathbf{x}', \omega) \frac{\partial_3 f_1^+(\mathbf{x}, \mathbf{x}_A, \omega)}{j\omega\rho(\mathbf{x})} d^2 \mathbf{x}$$

and

$$\bar{G}^+(\mathbf{x}_A, \mathbf{x}', \omega) - H(x_{3,A} - x'_3) f_1^*(\mathbf{x}', \mathbf{x}_A, \omega) \quad (18)$$

$$= 2 \int_{\partial\mathbb{D}_S} \bar{G}^-(\mathbf{x}, \mathbf{x}', \omega) \frac{\{\partial_3 f_1^-(\mathbf{x}, \mathbf{x}_A, \omega)\}^*}{j\omega\rho(\mathbf{x})} d^2 \mathbf{x},$$

respectively, where $H(x_3)$ is the Heaviside step function. Summing these equations, using $\bar{G}^-(\mathbf{x}, \mathbf{x}', \omega) = \bar{G}(\mathbf{x}, \mathbf{x}', \omega) = \bar{G}(\mathbf{x}', \mathbf{x}, \omega)$ for $x_3 = x_{3,S}$, gives

$$\bar{G}(\mathbf{x}', \mathbf{x}_A, \omega) + H(x_{3,A} - x'_3) 2j\Im\{f_1(\mathbf{x}', \mathbf{x}_A, \omega)\} \quad (19)$$

$$= \int_{\partial\mathbb{D}_S} \bar{G}(\mathbf{x}', \mathbf{x}, \omega) f(\mathbf{x}, \mathbf{x}_A, \omega) d^2 \mathbf{x},$$

with

$$f(\mathbf{x}, \mathbf{x}_A, \omega) = -\frac{2}{j\omega\rho(\mathbf{x})} \partial_3 \left(f_1^+(\mathbf{x}, \mathbf{x}_A, \omega) - \{f_1^-(\mathbf{x}, \mathbf{x}_A, \omega)\}^* \right). \quad (20)$$

The right-hand side of equation (19) describes propagation of the function $f(\mathbf{x}, \mathbf{x}_A, \omega)$ from the boundary $\partial\mathbb{D}_S$ to any point \mathbf{x}' in the medium below $\partial\mathbb{D}_S$. Note that the ‘‘propagator’’ $\bar{G}(\mathbf{x}', \mathbf{x}, \omega)$, evaluated as a function of \mathbf{x}' below $\partial\mathbb{D}_S$, obeys the wave equation without a singularity at $\mathbf{x}' = \mathbf{x}_A$ on the right-hand side (nor anywhere else below $\partial\mathbb{D}_S$). Hence, the expression on the left-hand side of equation (19) also obeys the wave equation without a singularity on the right-hand side. This is one of the conditions for the auxiliary function Γ . We define $\Gamma(\mathbf{x}, \omega)$ by the expression on the left-hand side of equation (19) (with \mathbf{x}' replaced by \mathbf{x})

$$\Gamma(\mathbf{x}, \omega) = \bar{G}(\mathbf{x}, \mathbf{x}_A, \omega) + H(x_{3,A} - x_3) 2j\Im\{f_1(\mathbf{x}, \mathbf{x}_A, \omega)\}. \quad (21)$$

This is equation (6) in the main paper.

Upon substitution of equation (21) into equation (9) we obtain

$$G(\mathbf{x}_A, \mathbf{x}_B, \omega) + H(x_{3,A} - x_{3,B}) 2j\Im\{f_1(\mathbf{x}_B, \mathbf{x}_A, \omega)\} \quad (22)$$

$$= \int_{\partial\mathbb{D}_R} \frac{2}{\omega\rho(\mathbf{x})} \left(\Im\{f_1(\mathbf{x}, \mathbf{x}_A, \omega)\} \partial_3 G(\mathbf{x}, \mathbf{x}_B, \omega) \right.$$

$$\left. - \Im\{\partial_3 f_1(\mathbf{x}, \mathbf{x}_A, \omega)\} G(\mathbf{x}, \mathbf{x}_B, \omega) \right) d^2 \mathbf{x}.$$

Taking the real part of both sides of this equation gives

$$G_h(\mathbf{x}_A, \mathbf{x}_B, \omega) = \int_{\partial\mathbb{D}_R} \frac{2}{\omega\rho(\mathbf{x})} \left(\Im\{f_1(\mathbf{x}, \mathbf{x}_A, \omega)\} \partial_3 G_h(\mathbf{x}, \mathbf{x}_B, \omega) - \Im\{\partial_3 f_1(\mathbf{x}, \mathbf{x}_A, \omega)\} G_h(\mathbf{x}, \mathbf{x}_B, \omega) \right) d^2\mathbf{x}. \quad (23)$$

This is equation (8) in the main paper. Note that the Green's function $G_h(\mathbf{x}, \mathbf{x}_B, \omega)$ in the right-hand side can be obtained from a similar representation. To see this, replace in the right-hand side of equation (23) $\partial\mathbb{D}_R$ by $\partial\mathbb{D}_S$ just above $\partial\mathbb{D}_R$, replace \mathbf{x} on $\partial\mathbb{D}_R$ by \mathbf{x}' on $\partial\mathbb{D}_S$, \mathbf{x}_B inside the medium by \mathbf{x} on $\partial\mathbb{D}_R$ and \mathbf{x}_A by \mathbf{x}_B . This gives a representation for $G_h(\mathbf{x}_B, \mathbf{x}, \omega)$. Using $G_h(\mathbf{x}_B, \mathbf{x}, \omega) = G_h(\mathbf{x}, \mathbf{x}_B, \omega)$ and $G(\mathbf{x}', \mathbf{x}, \omega) = G(\mathbf{x}, \mathbf{x}', \omega)$ we finally get

$$G_h(\mathbf{x}, \mathbf{x}_B, \omega) = \int_{\partial\mathbb{D}_S} \frac{2}{\omega\rho(\mathbf{x}')} \left(\Im\{f_1(\mathbf{x}', \mathbf{x}_B, \omega)\} \partial'_3 G_h(\mathbf{x}, \mathbf{x}', \omega) - \Im\{\partial'_3 f_1(\mathbf{x}', \mathbf{x}_B, \omega)\} G_h(\mathbf{x}, \mathbf{x}', \omega) \right) d^2\mathbf{x}'. \quad (24)$$

This is equation (9) in the main paper.

3 ALTERNATIVE DERIVATION OF THE SINGLE-SIDED REPRESENTATION

Equation (22) has been obtained by inserting an auxiliary function into the standard homogeneous Green's function representation (equation 9). This auxiliary function is defined in such a way that the closed-boundary integral reduces to a single-sided representation. Here we present an alternative derivation of equation (22), which misses the clear link with the standard homogeneous Green's function representation, but which is more direct.

Our starting point is again formed by equations (4) and (5). We define the boundary $\partial\mathbb{D}$ as the combination of $\partial\mathbb{D}_R$ (upper boundary) and $\partial\mathbb{D}_A$ (lower boundary). The domain between these boundaries is denoted as $\mathbb{D}_{\text{trunc}}$. Unlike in the previous section, this time we only modify the boundary integral along $\partial\mathbb{D}_A$ for decomposed wave fields, according to

$$\begin{aligned} & \int_{\mathbb{D}_{\text{trunc}}} \{q_A p_B - p_A q_B\} d^3\mathbf{x} = \\ & - \int_{\partial\mathbb{D}_R} \frac{1}{j\omega\rho} \{p_A \partial_3 p_B - (\partial_3 p_A) p_B\} d^2\mathbf{x} \\ & - \int_{\partial\mathbb{D}_A} \frac{2}{j\omega\rho} \{(\partial_3 p_A^+) p_B^- + (\partial_3 p_A^-) p_B^+\} d^2\mathbf{x}, \end{aligned} \quad (25)$$

and

$$\begin{aligned} & \int_{\mathbb{D}_{\text{trunc}}} \{q_A^* p_B + p_A^* q_B\} d^3\mathbf{x} = \\ & \int_{\partial\mathbb{D}_R} \frac{1}{j\omega\rho} \{p_A^* \partial_3 p_B - (\partial_3 p_A^*) p_B\} d^2\mathbf{x} \\ & + \int_{\partial\mathbb{D}_A} \frac{2}{j\omega\rho} \{(\partial_3 p_A^+)^* p_B^- + (\partial_3 p_A^-)^* p_B^-\} d^2\mathbf{x}, \end{aligned} \quad (26)$$

Our aim is to derive a relation between the focusing function in the truncated reference medium and the Green's function in the actual medium (which may be inhomogeneous above $\partial\mathbb{D}_R$). Substituting $p_A^\pm(\mathbf{x}, \omega) = f_1^\pm(\mathbf{x}, \mathbf{x}_A, \omega)$, $q_A(\mathbf{x}, \omega) = 0$, $p_B^\pm(\mathbf{x}, \omega) = G^\pm(\mathbf{x}, \mathbf{x}_B, \omega)$ and $q_B(\mathbf{x}, \omega) = \delta(\mathbf{x} - \mathbf{x}_B)$ into equations (25) and (26), using equations (10)

– (12) and $G(\mathbf{x}, \mathbf{x}_B, \omega) = G^+(\mathbf{x}, \mathbf{x}_B, \omega) + G^-(\mathbf{x}, \mathbf{x}_B, \omega)$, gives

$$\begin{aligned} & G^-(\mathbf{x}_A, \mathbf{x}_B, \omega) + H(x_{3,A} - x_{3,B}) f_1(\mathbf{x}_B, \mathbf{x}_A, \omega) \\ & = \int_{\partial\mathbb{D}_R} \frac{1}{j\omega\rho(\mathbf{x})} \left(f_1(\mathbf{x}, \mathbf{x}_A, \omega) \partial_3 G(\mathbf{x}, \mathbf{x}_B, \omega) - \{\partial_3 f_1(\mathbf{x}, \mathbf{x}_A, \omega)\} G(\mathbf{x}, \mathbf{x}_B, \omega) \right) d^2\mathbf{x} \end{aligned} \quad (27)$$

and

$$\begin{aligned} & G^+(\mathbf{x}_A, \mathbf{x}_B, \omega) - H(x_{3,A} - x_{3,B}) f_1^*(\mathbf{x}_B, \mathbf{x}_A, \omega) \\ & = - \int_{\partial\mathbb{D}_R} \frac{1}{j\omega\rho(\mathbf{x})} \left(f_1^*(\mathbf{x}, \mathbf{x}_A, \omega) \partial_3 G(\mathbf{x}, \mathbf{x}_B, \omega) - \{\partial_3 f_1^*(\mathbf{x}, \mathbf{x}_A, \omega)\} G(\mathbf{x}, \mathbf{x}_B, \omega) \right) d^2\mathbf{x}. \end{aligned} \quad (28)$$

Summing these two equations yields equation (22).

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