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One-way representations of seismic data

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SUMMARY

A general one-way representation of seismic data can be obtained by substituting a Green's one-way wavefield matrix into a reciprocity theorem of the convolution type for one-way wavefields. From this general one-way representation, several special cases can be derived.

By introducing a Green's one-way wavefield matrix for *primaries,* a generalized Bremmer series representation is obtained. Terminating this series after the first-order term yields a *primary* representation of seismic reflection data. According to this representation, primary seismic reflection data are proportional to a reflection operator, 'modified' by primary propagators for downgoing and upgoing waves. For seismic imaging, these propagators need to be inverted. Stable inverse primary propagators can easily be obtained from a one-way reciprocity theorem of the correlation type.

By introducing a Green's one-way wavefield matrix for *generalized primaries,* an alternative representation is obtained in which multiple scattering is organized quite differently (in comparison with the generalized Bremmer series representation). According to the *generalized primary* representation, full seismic reflection data are proportional to a reflection operator, 'modified' by generalized primary propagators for downgoing and upgoing waves. Internal multiple scattering is fully included in the generalized primary propagators (either via a series expansion or in a parametrized way). Stable inverse generalized primary propagators can be obtained from the one-way reciprocity theorem of the correlation type. These inverse propagators are the nucleus for seismic imaging techniques that take the angle-dependent dispersion effects due to fine-layering into account.

Key words: Green's functions, inverse problem, wave equation.

INTRODUCTION

A representation theorem expresses a wavefield quantity at some point in a medium in terms of boundary and volume integrals over the wavefield, the source distribution, and the contrast function (i.e. the difference between a reference medium and the actual medium). In general, a representation theorem can be obtained by substituting a Green's function into a reciprocity theorem. Acoustic representations were introduced by Lord Rayleigh (1878); early references for the elastodynamic situation are Knopoff (1956) and de Hoop (1958).

In this paper we follow a similar procedure, using as a starting point the reciprocity theorems for *one-way* wavefields, as derived in a companion paper (Wapenaar & Grimbergen 1996; hereafter referred to as Paper A). The contrast function in these reciprocity theorems is defined in terms of one-way *operators.* These operators distinguish explicitly between propagation (downward/upward) and scattering (reflection/transmission).

The reciprocity theorem of the convolution type will be used to derive various one-way representations in terms of a volume integral over a scattering operator. Since the scattering operator is proportional to the *vertical variations* of the medium parameters, this representation is very well suited for the seismic situation, where the vertical variations (due to layering) are much more pronounced than the horizontal variations. In particular we will derive a 3-D 'generalized primary representation', which accounts in a systematic way for propagation and reflection in continuous 3-D finely layered media.

The reciprocity theorem of the correlation type will be used to derive inverse propagators that are the nucleus of seismic reflection imaging techniques, amongst others for continuous 3-D finely layered media.

GENERAL REPRESENTATION FOR ONE-WAY WAVEFIELDS

The aim of this section is to derive a general representation for one-way wavefields in terms of boundary and volume integrals.

To this end we first briefly review the one-way reciprocity theorems that were derived in Paper A. Next we introduce a Green's matrix for one-way wavefields. Finally we derive the general one-way representation by substituting the Green's one-way wavefield matrix into the one-way reciprocity theorem.

One-way reciprocity theorems

In this subsection we briefly review the one-way wave equation and the one-way reciprocity theorems for downgoing and upgoing waves in a lossless inhomogeneous fluid medium and in a lossless inhomogeneous anisotropic solid medium. We use a general notation that applies to the acoustic as well as to the elastodynamic situation.

The medium parameters are infinitely differentiable functions of position, and time-invariant. The Cartesian position coordinates are denoted by the vector $\mathbf{x} = (x_1, x_2, x_3)$ and the x_3-axis is pointing downwards. The time coordinate is denoted by t, and the angular frequency by ω. The Fourier transforms from t to ω and vice versa are given by eqs (1) and (2) in Paper A. Throughout this paper all functions are in the frequency domain; the ω-dependence is not explicitly denoted.

We introduce a one-way wave vector $\mathbf{P}$ and a one-way source vector $\mathbf{S}$, according to

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}^+ \\ \mathbf{P}^- \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} \mathbf{S}^+ \\ \mathbf{S}^- \end{pmatrix}. \tag{1}$$

The superscripts + and − stand for downgoing and upgoing, respectively. In the elastodynamic situation, $\mathbf{P}^+$ and $\mathbf{P}^-$ are 3×1 vectors that can be subdivided as follows:

$$\mathbf{P}^+ = \begin{pmatrix} \Phi^+ \\ \Psi^+ \\ \Upsilon^+ \end{pmatrix} \quad \text{and} \quad \mathbf{P}^- = \begin{pmatrix} \Phi^- \\ \psi^- \\ \Upsilon^- \end{pmatrix}, \tag{2}$$

where $\Phi^\pm$, $\Psi^\pm$ and $\Upsilon^\pm$ represent the (flux-normalized) downgoing and upgoing quasi-P, quasi-$S1$ and quasi-$S2$ waves, respectively. A similar subdivision applies to the 3×1 vectors $\mathbf{S}^+$ and $\mathbf{S}^-$. In the acoustic situation, $\mathbf{P}^+$, $\mathbf{P}^-$, $\mathbf{S}^+$ and $\mathbf{S}^-$ are all scalars (i.e. P^+, P^-, S^+, and S^-).

In the space–frequency domain, the one-way wave equation reads

$$\partial_3 \mathbf{P} - \hat{\mathbf{B}}\mathbf{P} = \mathbf{S}, \tag{3}$$

where the one-way operator matrix $\hat{\mathbf{B}}$ is defined as

$$\hat{\mathbf{B}} = -j\omega\hat{\Lambda} + \hat{\Theta} \tag{4}$$

(j is the imaginary unit), with

$$\hat{\Theta} = -\hat{\mathbf{L}}^{-1}\partial_3\hat{\mathbf{L}}. \tag{5}$$

$\hat{\mathbf{B}}$ is a 2×2 (acoustic) or 6×6 (elastodynamic) pseudo-differential operator matrix (the circumflex denotes an operator containing ∂_1 and ∂_2). In eq. (4), $-j\omega\hat{\Lambda}$ accounts for (downward/upward) *propagation* and $\hat{\Theta}$ for *scattering* due to the vertical variations of the medium parameters. Note that both $\hat{\Lambda}$ and $\hat{\Theta}$ also account implicitly for the scattering due to the horizontal variations of the medium parameters. In the remainder of this paper 'scattering' stands for 'scattering due to the vertical variations'.

The explicit distinction between propagation and scattering *will be exploited in this paper in the derivation of various* one-way representations. $\hat{\Lambda}$ and $\hat{\Theta}$ can be partitioned as follows:

$$\hat{\Lambda} = \begin{pmatrix} \hat{\Lambda}^+ & \mathbf{O} \\ \mathbf{O} & -\hat{\Lambda}^- \end{pmatrix} \quad \text{and} \quad \hat{\Theta} = \begin{pmatrix} \hat{\mathbf{T}}^+ & \hat{\mathbf{R}}^- \\ -\hat{\mathbf{R}}^+ & -\hat{\mathbf{T}}^- \end{pmatrix} \tag{6}$$

(the + and − signs are chosen for later convenience). $\hat{\Lambda}^+$ and $\hat{\Lambda}^-$ are the 1×1 or 3×3 vertical slowness operators for downgoing and upgoing waves; $\hat{\mathbf{R}}^\pm$ and $\hat{\mathbf{T}}^\pm$ are the 1×1 or 3×3 reflection and transmission operators. $\mathbf{O}$ is a null matrix of appropriate size. From eqs (5) and (6) it follows that the reflection and transmission operators are proportional to the vertical variations of the medium parameters appearing in the composition operator $\hat{\mathbf{L}}$.

Using the one-way wave equation, we derived in Paper A for the two states in Table 1 a reciprocity theorem of the convolution type:

$$\int_{\partial\mathcal{V}} \mathbf{P}_A^{\mathrm{T}}\mathbf{N}\mathbf{P}_B n_3 d^2\mathbf{x}_{\mathrm{H}} = \int_{\mathcal{V}} \mathbf{P}_A^{\mathrm{T}}\mathbf{N}\hat{\Delta}\mathbf{P}_B d^3\mathbf{x} + \int_{\mathcal{V}} \{\mathbf{P}_A^{\mathrm{T}}\mathbf{N}\mathbf{S}_B + \mathbf{S}_A^{\mathrm{T}}\mathbf{N}\mathbf{P}_B\}\, d^3\mathbf{x} \tag{7}$$

[T denotes transposition; $\mathbf{x}_{\mathrm{H}} = (x_1, x_2)$ denotes the horizontal coordinates], and a reciprocity theorem of the correlation type:

$$\int_{\partial\mathcal{V}} \mathbf{P}_A^{\mathrm{H}}\mathbf{J}\mathbf{P}_B n_3 d^2\mathbf{x}_{\mathrm{H}} \approx \int_{\mathcal{V}} \mathbf{P}_A^{\mathrm{H}}\mathbf{J}\hat{\Delta}\mathbf{P}_B d^3\mathbf{x} + \int_{\mathcal{V}} \{\mathbf{P}_A^{\mathrm{H}}\mathbf{J}\mathbf{S}_B + \mathbf{S}_A^{\mathrm{H}}\mathbf{J}\mathbf{P}_B\}\, d^3\mathbf{x} \tag{8}$$

(H denotes transposition and complex conjugation). Here the contrast operator $\hat{\Delta}$ is defined as

$$\hat{\Delta} = \hat{\mathbf{B}}_B - \hat{\mathbf{B}}_A; \tag{9}$$

moreover,

$$\mathbf{N} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} \end{pmatrix}, \tag{10}$$

where $\mathbf{I}$ is an identity matrix of appropriate size. In eqs (7) and (8) $\mathcal{V}$ is a volume enclosed by two infinite parallel surfaces normal to the x_3-axis, see Fig. 1. These surfaces need not be physical boundaries. The combination of these surfaces is denoted by $\partial\mathcal{V}$ and the outward-pointing normal vector by

Table 1. States in the reciprocity theorems.

	State A	State B
Wavefield	$\mathbf{P}_A(\mathbf{x})$	$\mathbf{P}_B(\mathbf{x})$
Operator	$\hat{\mathbf{B}}_A(\mathbf{x})$	$\hat{\mathbf{B}}_B(\mathbf{x})$
Source	$\mathbf{S}_A(\mathbf{x})$	$\mathbf{S}_B(\mathbf{x})$

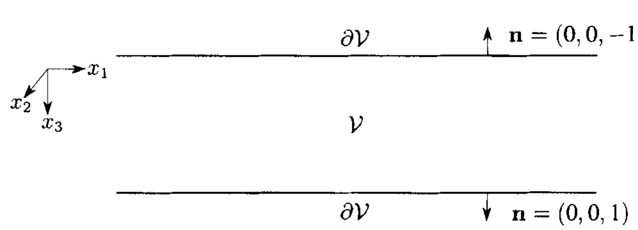

Figure 1. The configuration for the reciprocity theorems.

$\mathbf{n}=(0,0,n_3)$, with $n_3=-1$ at the upper surface and $n_3=+1$ at the lower surface.

Eq. (7) was rigorously derived in Paper A for the acoustic situation. It has been shown that this theorem also applies to the elastodynamic situation when the (non-unique) decomposition is done in such a way that the elastodynamic one-way operators satisfy similar symmetry properties to the acoustic one-way operators. Throughout this paper we assume that this condition is satisfied. Eq. (7) will be used as the basis for the derivation of a general representation of one-way wavefields.

Eq. (8) was derived in Paper A under the condition that evanescent wave modes can be ignored. In this paper we use this theorem for the derivation of various stable inverse propagators for arbitrary inhomogeneous media. In the remainder of this paper we replace the approximation sign $\approx$ by $=$ whenever the only approximation concerns the neglect of evanescent wave modes.

Green's one-way wavefield matrix

For the derivation of the one-way representations we need to replace one of the states in the reciprocity theorems by a one-way Green's function and the other one by the actual one-way wavefield. This will be done in the next subsection. Here we introduce a 2×2 (acoustic) or 6×6 (elastodynamic) Green's matrix $\mathbf{G}$ which satisfies the following one-way wave equation:

$$\partial_3\mathbf{G}(\mathbf{x},\mathbf{x}')-\hat{\bar{\mathbf{B}}}(\mathbf{x})\mathbf{G}(\mathbf{x},\mathbf{x}')=\mathbf{I}\delta(\mathbf{x}-\mathbf{x}'), \tag{11}$$

with $\delta(\mathbf{x})=\delta(x_1)\delta(x_2)\delta(x_3)$. $\hat{\bar{\mathbf{B}}}$ is some reference one-way operator (not necessarily related to a reference medium, see the next main section). The two or six columns of $\mathbf{G}(\mathbf{x},\mathbf{x}')$ represent two or six *independent* Green's one-way wavefields at observation point $\mathbf{x}$, related to two or six independent one-way sources at source point $\mathbf{x}'$. Note that $\mathbf{G}$ can be partitioned as follows:

$$\mathbf{G}(\mathbf{x},\mathbf{x}')=\begin{pmatrix}\mathbf{G}^{+,+}(\mathbf{x},\mathbf{x}') & \mathbf{G}^{+,-}(\mathbf{x},\mathbf{x}')\\ \mathbf{G}^{-,+}(\mathbf{x},\mathbf{x}') & \mathbf{G}^{-,-}(\mathbf{x},\mathbf{x}')\end{pmatrix}, \tag{12}$$

where the superscripts refer to the propagation direction at $\mathbf{x}$ and at $\mathbf{x}'$, respectively.

We derive a reciprocity relation for $\mathbf{G}(\mathbf{x},\mathbf{x}')$ by applying reciprocity theorem (7) to the states in Table 2. We choose both the Green's source points $\mathbf{x}'$ and $\mathbf{x}''$ in $\mathscr{V}$. Moreover, we choose a homogeneous isotropic reference medium outside $\mathscr{V}$. Then in both states only the *outward*-propagating terms of $\mathbf{G}$ are non-zero at $\partial\mathscr{V}$. Hence, the interaction quantity $\mathbf{G}^{\mathrm{T}}(\mathbf{x},\mathbf{x}')\mathbf{N}\mathbf{G}(\mathbf{x},\mathbf{x}'')$ vanishes at $\partial\mathscr{V}$ (see also Fig. 1 in Paper A). Eq. (7) thus yields

$$\mathbf{O}=\int_{\mathscr{V}}\{\mathbf{G}^{\mathrm{T}}(\mathbf{x},\mathbf{x}')\mathbf{N}\delta(\mathbf{x}-\mathbf{x}'')+\delta(\mathbf{x}-\mathbf{x}')\mathbf{N}\mathbf{G}(\mathbf{x},\mathbf{x}'')\}\,d^3\mathbf{x}, \tag{13}$$

or

$$\mathbf{G}(\mathbf{x}',\mathbf{x}'')=-\mathbf{N}^{-1}\mathbf{G}^{\mathrm{T}}(\mathbf{x}'',\mathbf{x}')\mathbf{N}. \tag{14}$$

Using the notation of eq. (12) we obtain

$$\begin{pmatrix}\mathbf{G}^{+,+}(\mathbf{x}',\mathbf{x}'') & \mathbf{G}^{+,-}(\mathbf{x}',\mathbf{x}'')\\ \mathbf{G}^{-,+}(\mathbf{x}',\mathbf{x}'') & \mathbf{G}^{-,-}(\mathbf{x}',\mathbf{x}'')\end{pmatrix} =\begin{pmatrix}-\{\mathbf{G}^{-,-}(\mathbf{x}'',\mathbf{x}')\}^{\mathrm{T}} & \{\mathbf{G}^{+,-}(\mathbf{x}'',\mathbf{x}')\}^{\mathrm{T}}\\ \{\mathbf{G}^{-,+}(\mathbf{x}'',\mathbf{x}')\}^{\mathrm{T}} & -\{\mathbf{G}^{+,+}(\mathbf{x}'',\mathbf{x}')\}^{\mathrm{T}}\end{pmatrix}. \tag{15}$$

For the elastodynamic situation, we can write the elements in for example, the lower left 3×3 matrix in this equation as

$$G^{-,+}_{i,j}(\mathbf{x}',\mathbf{x}'')=G^{-,+}_{j,i}(\mathbf{x}'',\mathbf{x}'). \tag{16}$$

Here the subscripts refer to the wave type at the observation point and at the source point, respectively ($i=1,2,3$ stands for quasi-P, quasi-$S1$ or quasi-$S2$, respectively). For the acoustic situation, eq. (16) reduces to $G^{-,+}(\mathbf{x}',\mathbf{x}'')=G^{-,+}(\mathbf{x}'',\mathbf{x}')$; see also Fig. 5 in Paper A.

General representation of the convolution type

Consider a one-way wavefield vector $\mathbf{P}$, related to a one-way source distribution $\mathbf{S}$. Our aim is to find a representation for $\mathbf{P}$ at some observation point $\mathbf{x}'$. To this end, we apply reciprocity theorem (7) to the two states in Table 3 and we make use of reciprocity relation (14).

We thus obtain

$$\chi(\mathbf{x}')\mathbf{P}(\mathbf{x}')=\int_{\mathscr{V}}\mathbf{G}(\mathbf{x}',\mathbf{x})\mathbf{S}(\mathbf{x})\,d^3\mathbf{x} -\int_{\partial\mathscr{V}}\mathbf{G}(\mathbf{x}',\mathbf{x})\mathbf{P}(\mathbf{x})n_3(\mathbf{x})\,d^2\mathbf{x}_{\mathrm{H}} +\int_{\mathscr{V}}\mathbf{G}(\mathbf{x}',\mathbf{x})\hat{\Delta}(\mathbf{x})\mathbf{P}(\mathbf{x})\,d^3\mathbf{x}, \tag{17}$$

where the characteristic function χ is defined as

$$\chi(\mathbf{x}')=\begin{cases}1 & \text{for } \mathbf{x}'\in\mathscr{V}\\ 1/2 & \text{for } \mathbf{x}'\in\partial\mathscr{V}\\ 0 & \text{for } \mathbf{x}'\notin\mathscr{V}\cup\partial\mathscr{V}\end{cases} \tag{18}$$

and the contrast operator as

$$\hat{\Delta}(\mathbf{x})=\hat{\mathbf{B}}(\mathbf{x})-\hat{\bar{\mathbf{B}}}(\mathbf{x}). \tag{19}$$

Note that the right-hand side of eq. (17) contains, respectively,

Table 2. States for deriving a reciprocity relation for $\mathbf{G}$.

	State A	State B
Wavefield	$\mathbf{G}(\mathbf{x},\mathbf{x}')$	$\mathbf{G}(\mathbf{x},\mathbf{x}'')$
Operator	$\hat{\bar{\mathbf{B}}}(\mathbf{x})$	$\hat{\bar{\mathbf{B}}}(\mathbf{x})$
Source	$\mathbf{I}\delta(\mathbf{x}-\mathbf{x}')$	$\mathbf{I}\delta(\mathbf{x}-\mathbf{x}'')$

Table 3. States for the general representation.

	State A	State B
Wavefield	$\mathbf{G}(\mathbf{x},\mathbf{x}')$	$\mathbf{P}(\mathbf{x})$
Operator	$\hat{\bar{\mathbf{B}}}(\mathbf{x})$	$\hat{\mathbf{B}}(\mathbf{x})$
Source	$\mathbf{I}\delta(\mathbf{x}-\mathbf{x}')$	$\mathbf{S}(\mathbf{x})$

a direct wave contribution, a boundary integral over the interaction quantity **GP**, and a volume integral over the contrast operator $\hat{\mathbf{\Delta}}$.

From this general one-way representation, several well-known representations can be derived as special cases.

When $\mathscr{V}$ is source-free and $\hat{\hat{\mathbf{B}}}$ equals $\hat{\mathbf{B}}$ throughout $\mathscr{V}$, then the two volume integrals vanish. If in addition one of the half-spaces outside $\mathscr{V}$ is chosen to be source-free and homogeneous, then only one boundary integral over a single surface remains. The resulting expression thus describes one-way wavefield extrapolation, as it is commonly named in the literature on seismic exploration (see for example, for the acoustic situation, Claerbout 1976; Berkhout 1982; Berkhout & Wapenaar 1989; and, for the elastodynamic situation, Wapenaar & Berkhout 1989). This will not be discussed further in this paper.

On the other hand, the boundary integral in eq. (17) vanishes when $\mathscr{V}$ is taken equal to $\mathbb{R}^3$. The remaining volume integral representation reduces to some well-known results when a special choice is made for the reference operator $\hat{\hat{\mathbf{B}}}$. This will be extensively discussed in the next main section. In the section after that, a new one-way representation will be introduced, based on an alternative choice for the reference operator $\hat{\hat{\mathbf{B}}}$.

NON-LINEAR AND LINEARIZED REPRESENTATIONS

The aim of this section is to derive non-linear and linearized volume integral representations from eq. (17). To this end we first introduce a Green's matrix for primaries. Next we use this matrix in the derivation of a generalized Bremmer series representation from which the primary representation follows immediately as a special case. Finally we derive inverse primary propagators, which form the basis for seismic reflection imaging techniques.

Green's matrix for primaries

In analogy with eq. (4), the reference operator $\hat{\hat{\mathbf{B}}}$ in eq. (11) is defined as $\hat{\hat{\mathbf{B}}} = -j\omega\hat{\hat{\mathbf{\Lambda}}} + \hat{\hat{\mathbf{\Theta}}}$. This operator allows an independent choice of $\hat{\hat{\mathbf{\Lambda}}}$ (propagation) and $\hat{\hat{\mathbf{\Theta}}}$ (scattering). In this section we choose

$$\hat{\hat{\mathbf{\Lambda}}} = \hat{\mathbf{\Lambda}} \quad \text{(i.e. propagation in the actual medium)} \tag{20}$$

and

$$\hat{\hat{\mathbf{\Theta}}} = \mathbf{O} \quad \text{(i.e. no scattering)}. \tag{21}$$

As a consequence, the reference operator $\hat{\hat{\mathbf{B}}} = -j\omega\hat{\mathbf{\Lambda}}$ accounts for *primary* propagation in the *actual* medium (note that this is quite different from choosing a reference *medium*, as is usually done in two-way representations). Substituting this (block-) diagonal reference operator into eq. (11) yields

$$\partial_3\mathbf{G}_\mathrm{p}(\mathbf{x},\mathbf{x}') + j\omega\hat{\mathbf{\Lambda}}(\mathbf{x})\mathbf{G}_\mathrm{p}(\mathbf{x},\mathbf{x}') = \mathbf{I}\delta(\mathbf{x}-\mathbf{x}'), \tag{22}$$

where $\mathbf{G}_\mathrm{p}(\mathbf{x},\mathbf{x}')$ is the Green's matrix for primaries. By choosing the appropriate boundary conditions (i.e. outgoing waves for $x_3 \to -\infty$ and for $x_3 \to \infty$), it follows that this Green's matrix has the following structure:

$$\mathbf{G}_\mathrm{p}(\mathbf{x},\mathbf{x}') = \begin{pmatrix} H(x_3 - x_3')\mathbf{W}_\mathrm{p}^+(\mathbf{x},\mathbf{x}') & \mathbf{O} \\ \mathbf{O} & -H(x_3' - x_3)\mathbf{W}_\mathrm{p}^-(\mathbf{x},\mathbf{x}') \end{pmatrix}, \tag{23}$$

Table 4. States for the generalized Bremmer series.

	State A	State B
Wavefield	$\mathbf{G}_\mathrm{p}(\mathbf{x},\mathbf{x}')$	$\mathbf{P}(\mathbf{x})$
Operator	$-j\omega\hat{\mathbf{\Lambda}}(\mathbf{x})$	$\hat{\mathbf{B}}(\mathbf{x})$
Source	$\mathbf{I}\delta(\mathbf{x}-\mathbf{x}')$	$\mathbf{S}(\mathbf{x})$

where $H(x_3)$ is the Heaviside step function (the + and − signs are chosen for later convenience). $\mathbf{W}_\mathrm{p}^+(\mathbf{x},\mathbf{x}')$ and $\mathbf{W}_\mathrm{p}^-(\mathbf{x},\mathbf{x}')$ will be referred to as the propagators for the primary downgoing and upgoing waves. In the elastodynamic situation, $\mathbf{W}_\mathrm{p}^+(\mathbf{x},\mathbf{x}')$ and $\mathbf{W}_\mathrm{p}^-(\mathbf{x},\mathbf{x}')$ are 3×3 matrices; in the acoustic situation they are scalars (i.e. W_p^+ and W_p^-). From eqs (15) and (23) it follows that

$$\mathbf{W}_\mathrm{p}^+(\mathbf{x},\mathbf{x}') = \{\mathbf{W}_\mathrm{p}^-(\mathbf{x}',\mathbf{x})\}^\mathrm{T}. \tag{24}$$

This primary propagator is further discussed in Appendix A.

Generalized Bremmer series representation

We use the Green's primary matrix to derive a generalized Bremmer series representation. For the two states in Table 4 we find that the contrast operator $\hat{\mathbf{\Delta}}$ reads

$$\hat{\mathbf{\Delta}} = \hat{\mathbf{B}} - (-j\omega\hat{\mathbf{\Lambda}}) = \hat{\mathbf{\Theta}}. \tag{25}$$

We thus obtain, instead of eq. (17),

$$\chi(\mathbf{x}')\mathbf{P}(\mathbf{x}') = \int_{\mathscr{V}} \mathbf{G}_\mathrm{p}(\mathbf{x}',\mathbf{x})\mathbf{S}(\mathbf{x})\,d^3\mathbf{x} - \int_{\partial\mathscr{V}} \mathbf{G}_\mathrm{p}(\mathbf{x}',\mathbf{x})\mathbf{P}(\mathbf{x})n_3(\mathbf{x})\,d^2\mathbf{x}_\mathrm{H} + \int_{\mathscr{V}} \mathbf{G}_\mathrm{p}(\mathbf{x}',\mathbf{x})\hat{\mathbf{\Theta}}(\mathbf{x})\mathbf{P}(\mathbf{x})\,d^3\mathbf{x}. \tag{26}$$

For the special case that $\mathscr{V}$ is taken to be equal to $\mathbb{R}^3$, the boundary integral vanishes. The remaining expression was derived previously by Corones (1975), who proposed (in a different notation) the following series expansion:

$$\mathbf{P}^{(k)}(\mathbf{x}') = \mathbf{P}^{(0)}(\mathbf{x}') + \int_{\mathbb{R}^3} \mathbf{G}_\mathrm{p}(\mathbf{x}',\mathbf{x})\hat{\mathbf{\Theta}}(\mathbf{x})\mathbf{P}^{(k-1)}(\mathbf{x})\,d^3\mathbf{x} \tag{27}$$

for $k > 0$, with

$$\mathbf{P}^{(0)}(\mathbf{x}') = \int_{\mathbb{R}^3} \mathbf{G}_\mathrm{p}(\mathbf{x}',\mathbf{x})\mathbf{S}(\mathbf{x})\,d^3\mathbf{x}. \tag{28}$$

This result may be seen as a generalized Bremmer series (generalized geometrical optics, see Brekhovskikh 1960). Each term fully accounts for one order of multiple scattering (unlike in the Neumann series expansion associated with the two-way wave equation). Its convergence is shown by de Hoop (1992) for wavefields in fluids. A convergence analysis for wavefields in solids is beyond the scope of this paper.

Primary representation for seismic reflection data

For $k = 1$, eq. (27) describes primary scattered data. In this subsection we consider the reflection response of an inhomogeneous (anisotropic) lower half-space $x_3 > x_{3,0}$, probed by a one-way source at $\mathbf{x}_\mathrm{S}$ and a one-way detector at $\mathbf{x}_\mathrm{D}$, both in a homogeneous isotropic upper half-space $x_3 \le x_{3,0}$. Using the following partitioning of $\mathbf{P}^{(1)}$ and $\mathbf{S}$,

$$\mathbf{P}^{(1)} = \begin{pmatrix} \{\mathbf{P}^+\}^{(1)} \\ \{\mathbf{P}^-\}^{(1)} \end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} \mathbf{S}_0^+ \\ \mathbf{O} \end{pmatrix} \delta(\mathbf{x} - \mathbf{x}_S), \tag{29}$$

(where $\mathbf{O}$ is the null vector) and using the partitioning of $\hat{\mathbf{\Theta}}$ and $\mathbf{G}_p$ as described in eqs (6) and (23) we find the primary upgoing response $\{\mathbf{P}^-(\mathbf{x}_D)\}^{(1)}$ from eqs (27) and (28):

$$\{\mathbf{P}^-(\mathbf{x}_D)\}^{(1)} = \int_{\Omega} \mathbf{W}_p^-(\mathbf{x}_D, \mathbf{x}) \hat{\mathbf{R}}^+(\mathbf{x}) \mathbf{W}_p^+(\mathbf{x}, \mathbf{x}_S) \mathbf{S}_0^+(\mathbf{x}_S) \, d^3\mathbf{x}, \tag{30}$$

where Ω denotes the lower half-space $x_3 > x_{3,0}$. Eq. (30) is a straightforward one-way representation of primary reflection data. It was introduced in a discrete formulation by Berkhout (1982) for acoustic one-way wavefields in fluids, and modified by Wapenaar & Berkhout (1989) for elastodynamic one-way wavefields in solids.

An attractive feature of representation (30) is that the reflection operator $\hat{\mathbf{R}}^+$ is proportional to the vertical *variations* of the medium parameters (see eqs 5 and 6). Hence, for each 'reflector' this operator has a compact support in depth [opposed to the contrast function for the usual (two-way) Born approximation; see Fig. 2 for a very simple example]. This makes the one-way representation particularly suited to seismic applications, where the main cause of scattering is given by the contrasts between the different layers in the subsurface (rather than by the contrasts between the actual and the background medium).

In a later section we will see that this simple representation may account for internal multiple scattering as well, when the primary propagators $\mathbf{W}_p^+$ and $\mathbf{W}_p^-$ are replaced by generalized primary propagators.

Finally, note that the assumptions made at the beginning of this subsection are far from realistic for seismic applications: the Earth's surface ($x_3 = x_{3,0}$) is a strongly reflecting free boundary, the (multi-component) seismic vibrators induce tensile and shear stresses at this surface, and the (multi-component) geophones register the particle velocity of the two-way wavefield at the surface. Hence, for the validity of eq. (30) in practical situations the following two pre-processing steps are required:

(1) decomposition of the physical measurements into one-way wavefields;

(2) elimination of multiple reflections related to the free surface.

This so-called surface-related pre-processing procedure is described in detail in Berkhout (1982), Verschuur, Berkhout & Wapenaar (1992), Carvalho, Weglein & Stolt (1982) and Fokkema & van den Berg (1993) for acoustic waves in fluids and by Wapenaar & Berkhout (1989) for elastodynamic waves in solids.

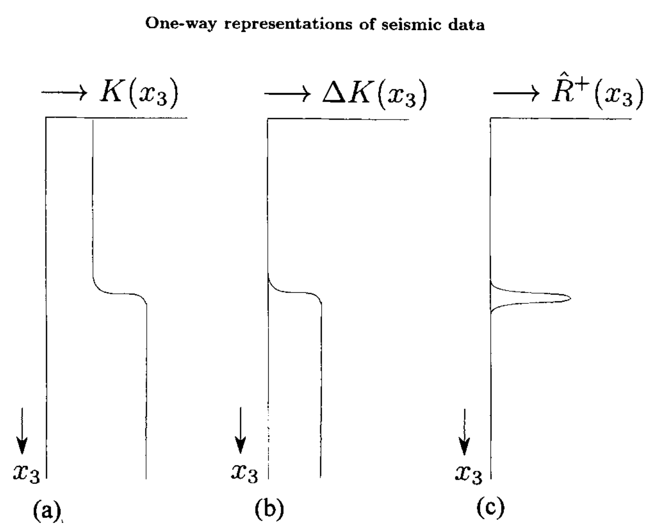

Figure 2. Illustration of the compactness in depth of the reflection operator in an acoustic medium. (a) Vertical cross-section of one medium parameter (compression modulus). (b) Contrast function for the usual (two-way) Born approximation. (c) Reflection operator for the one-way representation.

Inverse primary propagator

The primary representation (30) shows that primary seismic reflection data are proportional to the reflection operator $\hat{\mathbf{R}}^+$, modified by the propagators $\mathbf{W}_p^+$ and $\mathbf{W}_p^-$. Hence, inversion of the propagators $\mathbf{W}_p^+$ and $\mathbf{W}_p^-$ plays an essential role in reflection imaging. A full discussion of reflection imaging is beyond the scope of this paper. Here we restrict ourselves to the derivation of the inverse propagators $\mathbf{F}_p^+$ and $\mathbf{F}_p^-$. We introduce two horizontal reference levels Σ_0 and Σ_m, defined by $x_3 = x_{3,0}$ and $x_3 = x_{3,m} > x_{3,0}$, respectively. For any Σ_m below Σ_0 the inverse propagators $\mathbf{F}_p^+$ and $\mathbf{F}_p^-$ ideally satisfy the following relations:

$$\int_{\Sigma_m} \mathbf{F}_p^+(\mathbf{x}', \mathbf{x}) \mathbf{W}_p^+(\mathbf{x}, \mathbf{x}'') \, d^2\mathbf{x}_H = \mathbf{I}\delta(\mathbf{x}'_H - \mathbf{x}''_H) \tag{31}$$

and

$$\int_{\Sigma_m} \mathbf{W}_p^-(\mathbf{x}'', \mathbf{x}) \mathbf{F}_p^-(\mathbf{x}, \mathbf{x}') \, d^2\mathbf{x}_H = \mathbf{I}\delta(\mathbf{x}'_H - \mathbf{x}''_H) \tag{32}$$

for $\mathbf{x}'$ and $\mathbf{x}''$ at Σ_0 (and $\mathbf{x}$ at Σ_m), where $\delta(\mathbf{x}_H) = \delta(x_1)\delta(x_2)$. Note that

$$\mathbf{F}_p^+(\mathbf{x}', \mathbf{x}) = \{\mathbf{F}_p^-(\mathbf{x}, \mathbf{x}')\}^T \tag{33}$$

on account of eqs (24), (31) and (32).

In order to derive explicit expressions for the inverse propagators we apply reciprocity theorem (8) to the states in Table 5.

Since both one-way operators are the same, the first volume integral on the right-hand side of eq. (8) vanishes. Unlike in eq. (7), the boundary integral on the left-hand side of eq. (8) does not vanish when $\mathbf{x}'$ and $\mathbf{x}''$ are chosen in $\mathscr{V}$ (see also Fig. 6 in Paper A). Instead, we choose $\mathbf{x}'$ and $\mathbf{x}''$ both to be *outside* $\partial\mathscr{V}$. As a result, the second volume integral on the

Table 5. States for deriving the inverse primary propagators.

	State A	State B
Wavefield	$\mathbf{G}_p(\mathbf{x}, \mathbf{x}')$	$\mathbf{G}_p(\mathbf{x}, \mathbf{x}'')$
Operator	$-j\omega\hat{\mathbf{\Lambda}}(\mathbf{x})$	$-j\omega\hat{\mathbf{\Lambda}}(\mathbf{x})$
Source	$\mathbf{I}\delta(\mathbf{x} - \mathbf{x}')$	$\mathbf{I}\delta(\mathbf{x} - \mathbf{x}'')$

right-hand side of eq. (8) also vanishes. This leaves

$$\int_{\partial\mathcal{V}} \mathbf{G}_p^H(\mathbf{x},\mathbf{x}')\mathbf{J}\mathbf{G}_p(\mathbf{x},\mathbf{x}'')n_3(\mathbf{x})\,d^2\mathbf{x}_H = \mathbf{O}. \tag{34}$$

In the following we let $\partial\mathcal{V}$ consist of two horizontal reference levels Σ_0^+ and Σ_m, where Σ_0^+ lies just below Σ_0 (it is defined by $x_3 = x_{3,0} + \varepsilon$, with ε a vanishing positive constant). Hence, we obtain from eq. (34)

$$\int_{\Sigma_m} \mathbf{G}_p^H(\mathbf{x},\mathbf{x}')\mathbf{J}\mathbf{G}_p(\mathbf{x},\mathbf{x}'')\,d^2\mathbf{x}_H = \int_{\Sigma_0^+} \mathbf{G}_p^H(\mathbf{x},\mathbf{x}')\mathbf{J}\mathbf{G}_p(\mathbf{x},\mathbf{x}'')\,d^2\mathbf{x}_H \tag{35}$$

for $\mathbf{x}'$ and $\mathbf{x}''$ at Σ_0 (i.e. just outside $\partial\mathcal{V}$). In analogy with eqs (23) and (A2) we can write

$$\mathbf{G}_p(\mathbf{x},\mathbf{x}') = \begin{pmatrix} \mathbf{W}_p^+(\mathbf{x},\mathbf{x}') & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \tag{36}$$

for $\mathbf{x}'$ at Σ_0 and $\mathbf{x}$ at Σ_m, and

$$\mathbf{G}_p(\mathbf{x},\mathbf{x}') = \begin{pmatrix} \mathbf{I}\delta(\mathbf{x}_H - \mathbf{x}'_H) & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \tag{37}$$

for $\mathbf{x}'$ at Σ_0 and $\mathbf{x}$ at Σ_0^+. Using these expressions in eq. (35) yields

$$\int_{\Sigma_m} \{\mathbf{W}_p^+(\mathbf{x},\mathbf{x}')\}^H \mathbf{W}_p^+(\mathbf{x},\mathbf{x}'')\,d^2\mathbf{x}_H = \mathbf{I}\delta(\mathbf{x}'_H - \mathbf{x}''_H), \tag{38}$$

for $\mathbf{x}'$ and $\mathbf{x}''$ at Σ_0. Comparing this result with eq. (31) we find, for the inverse propagator for primary waves,

$$\mathbf{F}_p^+(\mathbf{x}',\mathbf{x}) = \{\mathbf{W}_p^+(\mathbf{x},\mathbf{x}')\}^H. \tag{39}$$

With eq. (33) we find, for the inverse propagator in eq. (32),

$$\mathbf{F}_p^-(\mathbf{x},\mathbf{x}') = \{\mathbf{W}_p^-(\mathbf{x}',\mathbf{x})\}^H. \tag{40}$$

Eqs (39) and (40) formulate very simple relations between the forward wavefield propagators that appear in the primary representation (30) and the inverse propagators that are essential for reflection imaging. This result can be seen as a generalization of the so-called 'matched filter' for inverse wavefield extrapolation (Berkhout 1982; Wapenaar & Berkhout 1989). The present expressions apply to flux-normalized waves in arbitrary inhomogeneous media (for comparison, in the cited references it was assumed that there are no lateral variations at Σ_0 and Σ_m).

Since eqs (39) and (40) are based on reciprocity theorem (8), evanescent wave modes are erroneously handled. Although this limits the maximum obtainable spatial resolution, it also ensures stability since the evanescent wave modes are suppressed instead of amplified (see also Appendix A).

GENERALIZED LINEAR REPRESENTATION, INCLUDING MULTIPLE SCATTERING

The aim of this section is to derive a representation of the form of eq. (30), with internal multiple scattering included in a consistent manner in the propagators. We will call this the 3-D generalized primary representation. The term generalized primary was introduced by Hubral, Treitel & Gutowski (1980) for 1-D acoustic wave propagation through 1-D discretely layered fluids and adopted by Resnick, Lerche & Shuey (1986) for 1-D acoustic wave propagation through 1-D continuously layered fluids.

We first introduce a Green's matrix for generalized primaries. Next we use this matrix in the derivation of a generalized linear representation from which the generalized primary representation follows immediately. Finally, we derive inverse generalized primary propagators, which form the basis for reflection imaging techniques, generalized for continuous 3-D finely layered media.

Green's matrix for generalized primaries

In this subsection we introduce a Green's matrix that accounts for multiple scattering in the half-space $x_3 < \zeta$, where, for the moment, ζ denotes an arbitrary depth level. Again we exploit the natural distinction between propagation and scattering in the one-way reference operator $\hat{\bar{\mathbf{B}}} = -j\omega\hat{\bar{\mathbf{\Lambda}}} + \hat{\bar{\mathbf{\Theta}}}$. This time we choose

$$\hat{\bar{\mathbf{\Lambda}}} = \hat{\mathbf{\Lambda}} \quad \text{(i.e. propagation in the actual medium)} \tag{41}$$

and

$$\hat{\bar{\mathbf{\Theta}}} = H(\zeta - x_3)\hat{\mathbf{\Theta}} \quad \text{(i.e. scattering only for } x_3 < \zeta). \tag{42}$$

For the reference operator we thus obtain

$$\hat{\bar{\mathbf{B}}}(\mathbf{x}|\zeta) = -j\omega\hat{\mathbf{\Lambda}}(\mathbf{x}) + H(\zeta - x_3)\hat{\mathbf{\Theta}}(\mathbf{x}). \tag{43}$$

For $x_3 < \zeta$ this operator is equal to $\hat{\mathbf{B}}(\mathbf{x})$, defined in the actual medium. For $x_3 > \zeta$ the coupling term $\hat{\mathbf{\Theta}}(\mathbf{x})$ is missing. Hence, $\hat{\bar{\mathbf{B}}}(\mathbf{x}|\zeta)$ applies to a configuration that is identical to the actual medium for the upper half-space $x_3 < \zeta$ and that is scatter-free for the lower half-space $x_3 > \zeta$. In analogy with eq. (11), we let this operator govern a Green's matrix $\mathbf{G}(\mathbf{x},\mathbf{x}''|\zeta)$. We can express this matrix in terms of the Green's primary matrix by applying reciprocity theorem (7) to the two states in Table 6, taking $\mathcal{V}$ equal to $\mathbb{R}^3$ and using reciprocity relation (14) for $\mathbf{G}_p$. We thus obtain

$$\mathbf{G}(\mathbf{x}',\mathbf{x}''|\zeta) = \mathbf{G}_p(\mathbf{x}',\mathbf{x}'') + \int_{\mathbb{R}^3} H(\zeta - x_3)\mathbf{G}_p(\mathbf{x}',\mathbf{x})\hat{\mathbf{\Theta}}(\mathbf{x})\mathbf{G}(\mathbf{x},\mathbf{x}''|\zeta)\,d^3\mathbf{x}. \tag{44}$$

We consider the special situation for which the half-space $x_3 \le x_{3,0}$ is homogeneous and isotropic. Moreover, we choose $\zeta = x'_3 > x''_3 = x_{3,0}$. With these choices, the half-spaces above x''_3 as well as below x'_3 are scatter-free. Hence, in analogy with eq. (23), $\mathbf{G}(\mathbf{x}',\mathbf{x}''|x'_3)$ now has the following structure:

$$\mathbf{G}(\mathbf{x}',\mathbf{x}''|x'_3) = \begin{pmatrix} \mathbf{W}_g^+(\mathbf{x}',\mathbf{x}'') & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}; \tag{45}$$

Table 6. States for deriving a representation for $\mathbf{G}(\mathbf{x},\mathbf{x}''|\zeta)$.

	State A	State B
Wavefield	$\mathbf{G}_p(\mathbf{x},\mathbf{x}')$	$\mathbf{G}(\mathbf{x},\mathbf{x}''\|\zeta)$
Operator	$\hat{\bar{\mathbf{B}}}(\mathbf{x}) = -j\omega\hat{\mathbf{\Lambda}}(\mathbf{x})$	$\hat{\bar{\mathbf{B}}}(\mathbf{x}\|\zeta) = -j\omega\hat{\mathbf{\Lambda}}(\mathbf{x}) + H(\zeta - x_3)\hat{\mathbf{\Theta}}(\mathbf{x})$
Source	$\mathbf{I}\delta(\mathbf{x}-\mathbf{x}')$	$\mathbf{I}\delta(\mathbf{x}-\mathbf{x}'')$

similarly,

$$\mathbf{G}(\mathbf{x}'', \mathbf{x}'|x_3') = \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & -\mathbf{W}_g^-(\mathbf{x}'', \mathbf{x}') \end{pmatrix}. \tag{46}$$

$\mathbf{W}_g^+(\mathbf{x}', \mathbf{x}'')$ and $\mathbf{W}_g^-(\mathbf{x}'', \mathbf{x}')$ will be referred to as the propagators for the *generalized primary* downgoing and upgoing waves in the actual medium between depth levels x_3'' and x_3'. In the elastodynamic situation, $\mathbf{W}_g^+(\mathbf{x}', \mathbf{x}'')$ and $\mathbf{W}_g^-(\mathbf{x}'', \mathbf{x}')$ are 3×3 matrices; in the acoustic situation they are scalars (i.e. W_g^+ and W_g^-). From eqs (15), (45) and (46) it follows that

$$\mathbf{W}_g^+(\mathbf{x}', \mathbf{x}'') = \{\mathbf{W}_g^-(\mathbf{x}'', \mathbf{x}')\}^{\mathrm{T}}. \tag{47}$$

This generalized primary propagator is further discussed in Appendix B.

Generalized linear representation

We use the Green's generalized primary matrix to derive a generalized linear representation. To this end we apply reciprocity theorem (7) to the two states in Table 7, taking $\mathcal{V}$ equal to $\mathbb{R}^3$. Using, in analogy with eq. (14), $\mathbf{G}(\mathbf{x}', \mathbf{x}''|\zeta) = -\mathbf{N}^{-1}\mathbf{G}^{\mathrm{T}}(\mathbf{x}'', \mathbf{x}'|\zeta)\mathbf{N}$, we thus obtain

$$\mathbf{G}(\mathbf{x}', \mathbf{x}''|\zeta') - \mathbf{G}(\mathbf{x}', \mathbf{x}''|\zeta) = \int_{\mathbb{R}^3} \mathbf{G}(\mathbf{x}', \mathbf{x}|\zeta)\{\hat{\hat{\mathbf{B}}}(\mathbf{x}|\zeta') - \hat{\hat{\mathbf{B}}}(\mathbf{x}|\zeta)\}\mathbf{G}(\mathbf{x}, \mathbf{x}''|\zeta')\, d^3\mathbf{x}. \tag{48}$$

Next we choose $\zeta' = \zeta + d\zeta$ and we take the limit for $d\zeta \to 0$. This yields

$$\partial_\zeta \mathbf{G}(\mathbf{x}', \mathbf{x}''|\zeta) = \int_{\mathbb{R}^3} \mathbf{G}(\mathbf{x}', \mathbf{x}|\zeta)\{\partial_\zeta \hat{\hat{\mathbf{B}}}(\mathbf{x}|\zeta)\}\mathbf{G}(\mathbf{x}, \mathbf{x}''|\zeta)\, d^3\mathbf{x}, \tag{49}$$

where ∂_ζ denotes differentiation with respect to ζ. Using eq. (43) we obtain

$$\partial_\zeta \mathbf{G}(\mathbf{x}', \mathbf{x}''|\zeta) = \int_{\mathbb{R}^2} \{\mathbf{G}(\mathbf{x}', \mathbf{x}|\zeta)\hat{\mathbf{\Theta}}(\mathbf{x})\mathbf{G}(\mathbf{x}, \mathbf{x}''|\zeta)\}_{x_3=\zeta}\, d^2\mathbf{x}_{\mathrm{H}}. \tag{50}$$

This equation quantifies the changes in the Green's matrix due to variations of the depth level ζ. In this sense, it resembles the method of invariant imbedding for initial-value problems (Bellman & Wing 1975; Fishman, McCoy & Wales 1987). To see this, we consider again the special situation for which the half-space $x_3 \leq x_{3,0}$ is homogeneous and isotropic. Moreover, this time we choose $x_3' = x_3'' = x_{3,0}$. For this situation, eq. (50) for the lower left 1×1 or 3×3 sub-matrix $\mathbf{G}^{-,+}(\mathbf{x}', \mathbf{x}''|\zeta)$ is identical to eq. (34) in Fishman *et al.* (1987) for the same sub-matrix (in a different notation). The latter authors use their eq. (34) as the starting point for a numerical modelling algorithm. In this paper we use our eq. (50) to derive a generalized linear representation (for arbitrary media and arbitrary x_3' and x_3'').

Replacing ζ by x_3, integrating both sides with respect to x_3 from $-\infty$ to ∞, and using $\mathbf{G}(\mathbf{x}', \mathbf{x}''|-\infty) = \mathbf{G}_p(\mathbf{x}', \mathbf{x}'')$ yields

$$\mathbf{G}(\mathbf{x}', \mathbf{x}''|\infty) = \mathbf{G}_p(\mathbf{x}', \mathbf{x}'') + \int_{\mathbb{R}^3} \mathbf{G}(\mathbf{x}', \mathbf{x}|x_3)\hat{\mathbf{\Theta}}(\mathbf{x})\mathbf{G}(\mathbf{x}, \mathbf{x}''|x_3)\, d^3\mathbf{x}. \tag{51}$$

Note that $\mathbf{G}(\mathbf{x}', \mathbf{x}''|\infty)$ represents the Green's matrix in the actual medium. Multiplying both sides from the right by $\mathbf{S}(\mathbf{x}'')$ and integrating over $\mathbf{x}''$ yields

$$\mathbf{P}(\mathbf{x}') = \int_{\mathbb{R}^3} \mathbf{G}_p(\mathbf{x}', \mathbf{x}'')\mathbf{S}(\mathbf{x}'')\, d^3\mathbf{x}'' + \int_{\mathbb{R}^3} \mathbf{G}(\mathbf{x}', \mathbf{x}|x_3)\hat{\mathbf{\Theta}}(\mathbf{x})\bar{\mathbf{P}}(\mathbf{x}|x_3)\, d^3\mathbf{x}, \tag{52}$$

where

$$\bar{\mathbf{P}}(\mathbf{x}|x_3) = \int_{\mathbb{R}^3} \mathbf{G}(\mathbf{x}, \mathbf{x}''|x_3)\mathbf{S}(\mathbf{x}'')\, d^3\mathbf{x}''. \tag{53}$$

Note that eqs (52) and (53) very much resemble eqs (27) and (28) for the generalized Bremmer series. However, multiple scattering is organized quite differently. In eqs (52) and (53) the multiple scattering effects are fully included in $\mathbf{G}(\mathbf{x}, \mathbf{x}''|x_3)$ and $\mathbf{G}(\mathbf{x}', \mathbf{x}|x_3)$: see eq. (44) and Appendix B. Hence, implicitly eqs (52) and (53) are non-linear in the scattering operator $\hat{\mathbf{\Theta}}(\mathbf{x})$. However, in its explicit form, the system of eqs (52) and (53) is *linear* in $\hat{\mathbf{\Theta}}(\mathbf{x})$. The consequences will be discussed in the next subsection.

Generalized primary representation for seismic reflection data

In this subsection we consider the reflection response of an inhomogeneous (anisotropic) lower half-space $x_3 > x_{3,0}$, probed by a one-way source at $\mathbf{x}_S$ and a one-way detector at $\mathbf{x}_D$, both in a homogeneous isotropic upper half-space $x_3 \leq x_{3,0}$. Using the partitioning of $\mathbf{P}$, $\hat{\mathbf{\Theta}}$, $\mathbf{S}$, and $\mathbf{G}$ as described in eqs (1), (6), (29), (45) and (46) we find, from eqs (52) and (53), the total upgoing response

$$\mathbf{P}^-(\mathbf{x}_D) = \int_\Omega \mathbf{W}_g^-(\mathbf{x}_D, \mathbf{x})\hat{\mathbf{R}}^+(\mathbf{x})\mathbf{W}_g^+(\mathbf{x}, \mathbf{x}_S)\mathbf{S}_0^+(\mathbf{x}_S)\, d^3\mathbf{x}, \tag{54}$$

see Fig. 3, where Ω denotes the lower half-space $x_3 > x_{3,0}$.

Note that this representation is almost identical to eq. (30), except that the primary upgoing wavefield $\{\mathbf{P}^-(\mathbf{x}_D)\}^{(1)}$ in eq. (30) has been replaced by the full upgoing wavefield $\mathbf{P}^-(\mathbf{x}_D)$ in eq. (54), hence the name generalized primary representation.

Following a similar argument to that used for the primary representation, it can be seen that the generalized primary representation (54) accurately describes the seismic reflection data *after* decomposition and surface-related multiple elimination.

Internal multiple scattering is included in the propagators

Table 7. States for the generalized linear representation.

	State A	State B
Wavefield	$\mathbf{G}(\mathbf{x}, \mathbf{x}'|\zeta)$	$\mathbf{G}(\mathbf{x}, \mathbf{x}''|\zeta')$
Operator	$\hat{\hat{\mathbf{B}}}(\mathbf{x}|\zeta) = -j\omega\hat{\mathbf{A}}(\mathbf{x}) + H(\zeta - x_3)\hat{\mathbf{\Theta}}(\mathbf{x})$	$\hat{\hat{\mathbf{B}}}(\mathbf{x}|\zeta') = -j\omega\hat{\mathbf{A}}(\mathbf{x}) + H(\zeta' - x_3)\hat{\mathbf{\Theta}}(\mathbf{x})$
Source	$\mathbf{I}\delta(\mathbf{x} - \mathbf{x}')$	$\mathbf{I}\delta(\mathbf{x} - \mathbf{x}'')$

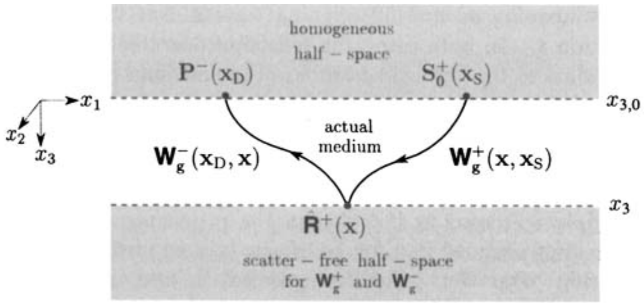

Figure 3. Configuration for the generalized primary representation. $\mathbf{W}_g^+(\mathbf{x}, \mathbf{x}_S)$ and $\mathbf{W}_g^-(\mathbf{x}_D, \mathbf{x})$ are defined in the actual medium between $x_{3,0}$ and x_3 and a scatter-free half-space below x_3. $\mathbf{P}^-(\mathbf{x}_D)$ is the response of the actual medium.

$\mathbf{W}_g^+$ and $\mathbf{W}_g^-$ (see also Appendix B). Hence, eq. (54) is implicitly non-linear in the scattering operators $\hat{\mathbf{R}}^\pm$ and $\hat{\mathbf{T}}^\pm$. However, eq. (54) has been organized such that it distinguishes clearly between generalized propagation ($\mathbf{W}_g^\pm$) and linear reflection ($\hat{\mathbf{R}}^+$).

We briefly discuss the consequences for 3-D finely layered media.

(1) The main effect of fine-layering on the propagators $\mathbf{W}_g^\pm$ is an angle-dependent dispersion. These dispersion effects can be approximately accounted for by parametrizing $\mathbf{W}_g^\pm$, or by defining $\mathbf{W}_g^\pm$ in a 3-D extended macro model with anisotropic anelastic losses.

(2) Since eq. (54) in its explicit form is linear in $\hat{\mathbf{R}}^+$, this operator can be spatially low-pass-filtered, without changing the response $\mathbf{P}^-$. The pass band is related to the maximum temporal frequency of the seismic source via the minimum macro propagation velocity.

Hence, in $\mathbf{W}_g^\pm$ and in $\hat{\mathbf{R}}^+$ the fine-layering parameters can be 'upscaled' in two independent ways. For a more elaborate discussion and the consequences for 3-D reflection imaging, see Wapenaar (1996).

Inverse generalized primary propagator

Inversion of the propagators $\mathbf{W}_g^+$ and $\mathbf{W}_g^-$ plays an essential role in reflection imaging in finely layered media. A full discussion is beyond the scope of this paper. Here we restrict ourselves to the derivation of the inverse propagators $\mathbf{F}_g^+$ and $\mathbf{F}_g^-$. We consider again two horizontal reference levels Σ_0 and Σ_m, defined by $x_3 = x_{3,0}$ and $x_3 = x_{3,m} > x_{3,0}$, respectively. For any Σ_m below Σ_0, the inverse propagators $\mathbf{F}_g^+$ and $\mathbf{F}_g^-$ ideally satisfy the following relations:

$$\int_{\Sigma_m} \mathbf{F}_g^+(\mathbf{x}', \mathbf{x})\mathbf{W}_g^+(\mathbf{x}, \mathbf{x}'')\, d^2\mathbf{x}_H = \mathbf{I}\delta(\mathbf{x}'_H - \mathbf{x}''_H) \tag{55}$$

and

$$\int_{\Sigma_m} \mathbf{W}_g^-(\mathbf{x}'', \mathbf{x})\mathbf{F}_g^-(\mathbf{x}, \mathbf{x}')\, d^2\mathbf{x}_H = \mathbf{I}\delta(\mathbf{x}'_H - \mathbf{x}''_H) \tag{56}$$

for $\mathbf{x}'$ and $\mathbf{x}''$ at Σ_0. Note that

$$\mathbf{F}_g^+(\mathbf{x}', \mathbf{x}) = \{\mathbf{F}_g^-(\mathbf{x}, \mathbf{x}')\}^{\mathrm{T}} \tag{57}$$

on account of eqs (47), (55) and (56). In order to derive explicit expressions for the inverse propagators, we apply reciprocity theorem (8) to the states in Table 7, with $\zeta = \zeta' = x_{3,m}$. We let $\partial\mathscr{V}$ again consist of two horizontal reference levels Σ_0^+ and Σ_m, where Σ_0^+ lies just below Σ_0 (it is defined by $x_3 = x_{3,0} + \varepsilon$, with ε a vanishing positive constant). This yields the following modified form of eq. (35):

$$\int_{\Sigma_m} \mathbf{G}^{\mathrm{H}}(\mathbf{x}, \mathbf{x}'|x_{3,m})\mathbf{J}\mathbf{G}(\mathbf{x}, \mathbf{x}''|x_{3,m})\, d^2\mathbf{x}_H$$
$$= \int_{\Sigma_0^+} \mathbf{G}^{\mathrm{H}}(\mathbf{x}, \mathbf{x}'|x_{3,m})\mathbf{J}\mathbf{G}(\mathbf{x}, \mathbf{x}''|x_{3,m})\, d^2\mathbf{x}_H \tag{58}$$

for $\mathbf{x}'$ and $\mathbf{x}''$ at Σ_0 (i.e. just outside $\partial\mathscr{V}$). With reference to Fig. 4 we write

$$\mathbf{G}(\mathbf{x}, \mathbf{x}'|x_{3,m}) = \begin{pmatrix} \mathbf{W}_g^+(\mathbf{x}, \mathbf{x}') & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \tag{59}$$

for $\mathbf{x}'$ at Σ_0 and $\mathbf{x}$ at Σ_m (compare with eq. 36) and

$$\mathbf{G}(\mathbf{x}, \mathbf{x}'|x_{3,m}) = \begin{pmatrix} \mathbf{I}\delta(\mathbf{x}_H - \mathbf{x}'_H) & \mathbf{O} \\ \mathbf{G}^{-,+}(\mathbf{x}, \mathbf{x}'|x_{3,m}) & \mathbf{O} \end{pmatrix} \tag{60}$$

for $\mathbf{x}'$ at Σ_0 and $\mathbf{x}$ at Σ_0^+ (compare with eq. 37). Using these expressions in eq. (58) yields

$$\int_{\Sigma_m} \{\mathbf{W}_g^+(\mathbf{x}, \mathbf{x}')\}^{\mathrm{H}}\mathbf{W}_g^+(\mathbf{x}, \mathbf{x}'')\, d^2\mathbf{x}_H$$
$$= \mathbf{I}\delta(\mathbf{x}'_H - \mathbf{x}''_H) - \mathbf{C}(\mathbf{x}', \mathbf{x}''|x_{3,m}), \tag{61}$$

where

$$\mathbf{C}(\mathbf{x}', \mathbf{x}''|x_{3,m}) = \int_{\Sigma_0^+} \{\mathbf{G}^{-,+}(\mathbf{x}, \mathbf{x}'|x_{3,m})\}^{\mathrm{H}}\mathbf{G}^{-,+}(\mathbf{x}, \mathbf{x}''|x_{3,m})\, d^2\mathbf{x}_H \tag{62}$$

for $\mathbf{x}'$ and $\mathbf{x}''$ at Σ_0. In comparison with eq. (38), the correlation function $\mathbf{C}$ on the right-hand side of eq. (61) is extra. Both terms $\mathbf{G}^{-,+}$ contributing to this correlation function represent scattered waves (see Fig. 4), hence this function is proportional to multiply scattered waves and is therefore two orders of magnitude lower than the integral on the left-hand side of eq. (61). Hence, for a weakly scattering medium this correlation function can be ignored, so that, by analogy with eqs (39) and (40) we obtain

$$\mathbf{F}_g^+(\mathbf{x}', \mathbf{x}) \approx \{\mathbf{W}_g^+(\mathbf{x}, \mathbf{x}')\}^{\mathrm{H}}, \tag{63}$$

and

$$\mathbf{F}_g^-(\mathbf{x}, \mathbf{x}') \approx \{\mathbf{W}_g^-(\mathbf{x}', \mathbf{x})\}^{\mathrm{H}}. \tag{64}$$

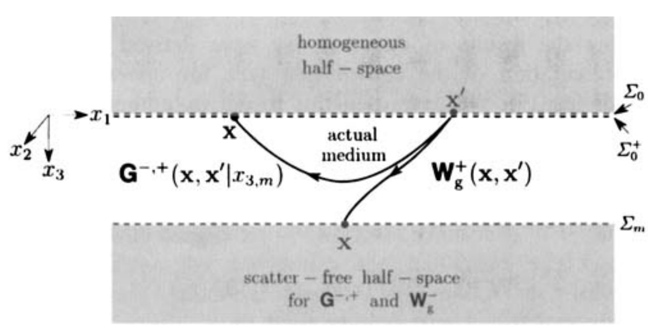

Figure 4. Schematic representation of the Green's sub-matrices $\mathbf{W}_g^+(\mathbf{x}, \mathbf{x}')$ and $\mathbf{G}^{-,+}(\mathbf{x}, \mathbf{x}'|x_{3,m})$.

For a finely layered medium, however, this approximation breaks down and another approach must be followed (see also Wapenaar & Berkhout 1989, Chapters 9 and 10). Substituting the left-hand side of eq. (55) for $\mathbf{I}\delta(\mathbf{x}'_H - \mathbf{x}''_H)$ in eq. (61) and reordering some terms yields

$$\int_{\Sigma_m} \mathbf{F}_g^+(\mathbf{x}', \mathbf{x})\mathbf{W}_g^+(\mathbf{x}, \mathbf{x}'')\, d^2\mathbf{x}_H$$
$$= \int_{\Sigma_m} \{\mathbf{W}_g^+(\mathbf{x}, \mathbf{x}')\}^H \mathbf{W}_g^+(\mathbf{x}, \mathbf{x}'')\, d^2\mathbf{x}_H + \mathbf{C}(\mathbf{x}', \mathbf{x}''|x_{3,m}) \tag{65}$$

for $\mathbf{x}'$ and $\mathbf{x}''$ at Σ_0. Applying the inverse propagator $\mathbf{F}_g^+(\mathbf{x}'', \mathbf{x}''')$, with $\mathbf{x}'''$ at Σ_m, to the right of each term and integrating along $\mathbf{x}''_H$ yields

$$\mathbf{F}_g^+(\mathbf{x}', \mathbf{x}''') = \{\mathbf{W}_g^+(\mathbf{x}''', \mathbf{x}')\}^H + \int_{\Sigma_0} \mathbf{C}(\mathbf{x}', \mathbf{x}''|x_{3,m})\mathbf{F}_g^+(\mathbf{x}'', \mathbf{x}''')\, d^2\mathbf{x}''_H \tag{66}$$

for $\mathbf{x}'$ at Σ_0 and $\mathbf{x}'''$ at Σ_m. This is an integral equation of the second kind for $\mathbf{F}_g^+$. In the following, we replace $\mathbf{x}'''$ by $\mathbf{x}$ for notational convenience. An iterative solution can thus be formulated as

$$\{\mathbf{F}_g^+(\mathbf{x}', \mathbf{x})\}^{(k)} = \{\mathbf{F}_g^+(\mathbf{x}', \mathbf{x})\}^{(0)} + \int_{\Sigma_0} \mathbf{C}(\mathbf{x}', \mathbf{x}''|x_{3,m})\{\mathbf{F}_g^+(\mathbf{x}'', \mathbf{x})\}^{(k-1)}\, d^2\mathbf{x}''_H \tag{67}$$

and

$$\{\mathbf{F}_g^+(\mathbf{x}', \mathbf{x})\}^{(0)} = \{\mathbf{W}_g^+(\mathbf{x}, \mathbf{x}')\}^H \tag{68}$$

for $\mathbf{x}'$ at Σ_0 and $\mathbf{x}$ at Σ_m. Hence, the inverse generalized primary propagator is given by the matched filter, modified by a correction operator containing the correlation function $\mathbf{C}(\mathbf{x}', \mathbf{x}''|x_{3,m})$. Note that in practice $\mathbf{C}$ can be estimated from the spatial cross-correlation of the reflection measurements—see eq. (62) and Fig. 4. An expression similar to eq. (67) could be formulated for $\mathbf{F}_g^-$. It is more straightforward, however, to relate $\mathbf{F}_g^-$ to $\mathbf{F}_g^+$ via reciprocity relation (57).

By applying the inverse propagators $\mathbf{F}_g^+$ and $\mathbf{F}_g^-$ to seismic reflection data (as represented by eq. 54), one eliminates the downward and upward propagation effects, including the complicated anisotropic dispersion effects due to fine-layering. This forms the nucleus of seismic reflection imaging in 3-D finely layered media (see also Wapenaar & Herrmann 1996).

CONCLUSIONS

Using the results of Paper A, we have derived a general representation of the convolution type for one-way wavefields (eq. 17). We have used this to derive, amongst others, a *primary* representation (eq. 30) and a *generalized primary* representation (eq. 54) of seismic reflection data (after surface-related pre-processing). These representations can be summarized as

$$\mathbf{P}^-(\mathbf{x}_D) = \int_{\Omega} \mathbf{W}^-(\mathbf{x}_D, \mathbf{x})\hat{\mathbf{R}}^+(\mathbf{x})\mathbf{W}^+(\mathbf{x}, \mathbf{x}_S)\mathbf{S}_0^+(\mathbf{x}_S)\, d^3\mathbf{x}, \tag{69}$$

where $\mathbf{W}^{\pm}$ may stand for the primary or for the generalized primary propagator. Accordingly, $\mathbf{P}^-(\mathbf{x}_D)$ represents the primary upgoing or the full upgoing wavefield at the detector location $\mathbf{x}_D$. In both cases, the reflection operator $\hat{\mathbf{R}}^+$ is proportional to the vertical *variations* of the medium parameters. This makes these one-way representations particularly suited for seismic applications, where the vertical variations (due to layering) are much more pronounced than the horizontal variations. In the generalized primary representation, internal multiple scattering is included in the propagators $\mathbf{W}_g^{\pm}$. We have briefly argued that for 3-D finely layered media the fine-layering parameters may be 'upscaled' in two independent ways in $\mathbf{W}_g^{\pm}$ and $\hat{\mathbf{R}}^+$.

Due to its simple form, the one-way representation (69) is an excellent starting point for the derivation of reflection imaging techniques. The nucleus of these techniques is the elimination of the one-way (generalized) propagation effects described by $\mathbf{W}^+$ and $\mathbf{W}^-$. To this end we derived inverse propagators $\mathbf{F}^+$ and $\mathbf{F}^-$. We showed that for the primary representation these propagators are simply defined as the complex conjugate transpose of the forward propagators (eqs 39 and 40). For the generalized primary representation, these inverse propagators have been modified (eqs 57 and 67) in order to account for the dispersion effects due to fine-layering. The required modification can be estimated from the spatial cross-correlation of the reflection measurements.

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APPENDIX A: THE PRIMARY PROPAGATOR

Upon substitution of eq. (23) into (22), using eq. (6), it follows that the primary propagator $\mathbf{W}_p^{\pm}(\mathbf{x},\mathbf{x}')$ satisfies the following one-way wave equation:

$$\partial_3 \mathbf{W}_p^{\pm}(\mathbf{x},\mathbf{x}') = \mp j\omega\hat{\mathbf{\Lambda}}^{\pm}(\mathbf{x})\mathbf{W}_p^{\pm}(\mathbf{x},\mathbf{x}'), \tag{A1}$$

where, for $x_3 = x_3'$,

$$\mathbf{W}_p^{\pm}(\mathbf{x},\mathbf{x}') = \mathbf{I}\delta(\mathbf{x}_H - \mathbf{x}_H'). \tag{A2}$$

Using a Taylor series expansion (see Wapenaar & Berkhout 1989, Chapter 3) we obtain

$$\mathbf{W}_p^{\pm}(\mathbf{x},\mathbf{x}') = \hat{\mathbf{W}}_p^{\pm}(\mathbf{x},x_3')\delta(\mathbf{x}_H - \mathbf{x}_H'), \tag{A3}$$

where

$$\hat{\mathbf{W}}_p^{\pm}(\mathbf{x},x_3') = \sum_{n=0}^{\infty}\frac{(x_3-x_3')^n}{n!}(\mp j\omega)^n\hat{\mathbf{H}}_n^{\pm}(\mathbf{x}_H,x_3'), \tag{A4}$$

with

$$\hat{\mathbf{H}}_n^{\pm}(\mathbf{x}) = \hat{\mathbf{H}}_{n-1}^{\pm}(\mathbf{x})\hat{\mathbf{\Lambda}}^{\pm}(\mathbf{x}) \mp (j\omega)^{-1}\partial_3\hat{\mathbf{H}}_{n-1}^{\pm}(\mathbf{x}) \tag{A5}$$

for $n > 0$ and

$$\hat{\mathbf{H}}_0^{\pm}(\mathbf{x}) = \mathbf{I}. \tag{A6}$$

According to eq. (A3), the *propagator* $\mathbf{W}_p^{\pm}(\mathbf{x},\mathbf{x}')$ may be seen as the kernel of an *operator* $\hat{\mathbf{W}}_p^{\pm}(\mathbf{x},x_3')$. Note that the propagator can be built up recursively, according to

$$\mathbf{W}_p^{\pm}(\mathbf{x},\mathbf{x}'') = \int_{\mathbb{R}^2}\mathbf{W}_p^{\pm}(\mathbf{x},\mathbf{x}')\mathbf{W}_p^{\pm}(\mathbf{x}',\mathbf{x}'')\,d^2\mathbf{x}_H', \tag{A7}$$

for $x_3 \gtrless x_3' \gtrless x_3''$. For sufficiently small recursion steps (i.e. for sufficiently small $|x_3 - x_3'|$) it is justified to ignore the vertical derivative of the medium parameters (bear in mind that we are considering flux-normalized primary waves). In that case we have $\hat{\mathbf{H}}_n^{\pm}(\mathbf{x}) = \{\hat{\mathbf{\Lambda}}^{\pm}(\mathbf{x})\}^n$, and eq. (A4) can be rewritten as

$$\hat{\mathbf{W}}_p^{\pm}(\mathbf{x},x_3') = \exp\{-j\omega|x_3-x_3'|\hat{\mathbf{\Lambda}}^{\pm}(\mathbf{x}_H,x_3')\}, \tag{A8}$$

for $x_3 \gtrless x_3'$. We evaluate this result for the special situation of an acoustic medium. In this case $\hat{\mathbf{\Lambda}}^{\pm}$ reduces to the scalar operator $\omega^{-1}\hat{\mathscr{H}}_1$, where $\hat{\mathscr{H}}_1$ is the square-root operator—see Paper A. Hence

$$\hat{W}_p^{\pm}(\mathbf{x},x_3') = \exp\{-j|x_3-x_3'|\hat{\mathscr{H}}_1(\mathbf{x}_H,x_3')\}, \tag{A9}$$

for $x_3 \gtrless x_3'$. Using the results of Paper A, we can express the kernel of $\hat{\mathscr{H}}_1$ in terms of eigenfunctions, according to

$$\begin{aligned}\mathscr{H}_1(\mathbf{x}_H,x_3';\mathbf{x}_H') &= \hat{\mathscr{H}}_1(\mathbf{x}_H,x_3')\delta(\mathbf{x}_H-\mathbf{x}_H')\\ &= \int_{\mathbb{R}^2}\phi(\mathbf{x}_H,\boldsymbol{\kappa})\lambda^{1/2}(\boldsymbol{\kappa})\phi^*(\mathbf{x}_H',\boldsymbol{\kappa})\,d^2\boldsymbol{\kappa}\\ &\quad+\sum_{\lambda_i\in\sigma_{\rm discr}}\phi^{(i)}(\mathbf{x}_H)\lambda_i^{1/2}\phi^{(i)}(\mathbf{x}_H'),\end{aligned} \tag{A10}$$

where the signs of the square roots have been chosen as

$$\mathscr{R}e(\lambda^{1/2}) \geq 0 \quad \text{for } \lambda \geq 0 \quad \text{(propagating wave modes)}, \tag{A11}$$

$$\mathscr{I}m(\lambda^{1/2}) < 0 \quad \text{for } \lambda < 0 \quad \text{(evanescent wave modes)}. \tag{A12}$$

Using the orthonormality property of the eigenfunctions, we obtain the following expression for the kernel $W_p^{\pm}(\mathbf{x},\mathbf{x}')$ from eqs (A9) and (A10):

$$\begin{aligned}W_p^{\pm}(\mathbf{x},\mathbf{x}') &= \hat{W}_p^{\pm}(\mathbf{x},x_3')\delta(\mathbf{x}_H-\mathbf{x}_H')\\ &= \int_{\mathbb{R}^2}\phi(\mathbf{x}_H,\boldsymbol{\kappa})\exp\{-j|x_3-x_3'|\lambda^{1/2}(\boldsymbol{\kappa})\}\phi^*(\mathbf{x}_H',\boldsymbol{\kappa})\,d^2\boldsymbol{\kappa}\\ &\quad+\sum_{\lambda_i\in\sigma_{\rm discr}}\phi^{(i)}(\mathbf{x}_H)\exp\{-j|x_3-x_3'|\lambda_i^{1/2}\}\phi^{(i)}(\mathbf{x}_H'),\end{aligned} \tag{A13}$$

for $x_3 \gtrless x_3'$; see Grimbergen, Wapenaar & Dessing (1996), also for numerical examples. Note that, given the signs of $\lambda^{1/2}$ as defined in eqs (A11) and (A12), both $W_p^{\pm}(\mathbf{x},\mathbf{x}')$ and $\{W_p^{\pm}(\mathbf{x},\mathbf{x}')\}^*$ are unconditionally stable.

APPENDIX B: THE GENERALIZED PRIMARY PROPAGATOR

Using a Bremmer series expansion analogous to eq. (27), we can write, instead of eq. (44),

$$\begin{aligned}\mathbf{G}(\mathbf{x}',\mathbf{x}''|\zeta) &= \mathbf{G}_p(\mathbf{x}',\mathbf{x}'')\\ &\quad+\int_{\mathbb{R}^3}H(\zeta-x_3)\mathbf{G}_p(\mathbf{x}',\mathbf{x})\hat{\mathbf{\Theta}}(\mathbf{x})\mathbf{G}_p(\mathbf{x},\mathbf{x}'')\,d^3\mathbf{x}\\ &\quad+\int_{\mathbb{R}^3}H(\zeta-x_3)\mathbf{G}_p(\mathbf{x}',\mathbf{x})\hat{\mathbf{\Theta}}(\mathbf{x})\,d^3\mathbf{x}\\ &\quad\times\int_{\mathbb{R}^3}H(\zeta-x_3''')\mathbf{G}_p(\mathbf{x},\mathbf{x}''')\hat{\mathbf{\Theta}}(\mathbf{x}''')\\ &\quad\times\mathbf{G}_p(\mathbf{x}''',\mathbf{x}'')\,d^3\mathbf{x}'''+\cdots.\end{aligned} \tag{B1}$$

Eq. (B1) contains reflection and transmission contributions. Here we will analyse eq. (B1) for the transmission response in a configuration for which the half-space $x_3 \leq x_{3,0}$ is homogeneous and isotropic. Moreover, we choose $x_3' = \zeta > x_3'' = x_{3,0}$. With these choices, the half-spaces above x_3'' as well as below x_3' are scatter-free. Upon substitution of eqs (6) and (23) into eq. (B1) we thus obtain the following expression for $\mathbf{W}_g^+(\mathbf{x}', \mathbf{x}'')$ as defined in eq. (45):

$$\begin{aligned}\mathbf{W}_g^+(\mathbf{x}', \mathbf{x}'') = \mathbf{W}_p^+(\mathbf{x}', \mathbf{x}'') &+ \int_{\Omega_{tot}} \mathbf{W}_p^+(\mathbf{x}', \mathbf{x})\hat{\mathbf{T}}^+(\mathbf{x})\mathbf{W}_p^+(\mathbf{x}, \mathbf{x}'')\, d^3\mathbf{x} \\ &+ \int_{\Omega_{tot}} \mathbf{W}_p^+(\mathbf{x}', \mathbf{x})\hat{\mathbf{R}}^-(\mathbf{x})\, d^3\mathbf{x} \\ &\times \int_{\Omega_1} \mathbf{W}_p^-(\mathbf{x}, \mathbf{x}''')\hat{\mathbf{R}}^+(\mathbf{x}''')\mathbf{W}_p^+(\mathbf{x}''', \mathbf{x}'')\, d^3\mathbf{x}''' \\ &+ \int_{\Omega_{tot}} \mathbf{W}_p^+(\mathbf{x}', \mathbf{x})\hat{\mathbf{T}}^+(\mathbf{x})\, d^3\mathbf{x} \\ &\times \int_{\Omega_2} \mathbf{W}_p^+(\mathbf{x}, \mathbf{x}''')\hat{\mathbf{T}}^+(\mathbf{x}''')\mathbf{W}_p^+(\mathbf{x}''', \mathbf{x}'')\, d^3\mathbf{x}''' + \cdots,\end{aligned} \tag{B2}$$

where Ω_{tot}, Ω_1 and Ω_2 denote 3-D volumes, according to

$$\begin{aligned}&\Omega_{tot}: \mathbf{x} \in \mathbb{R}^3 | x_{3,0} < x_3 < x_3', \\ &\Omega_1: \mathbf{x}''' \in \mathbb{R}^3 | x_3 < x_3''' < x_3', \\ &\Omega_2: \mathbf{x}''' \in \mathbb{R}^3 | x_{3,0} < x_3''' < x_3.\end{aligned} \tag{B3}$$

For 1-D finely layered media, eq. (B2) leads in a straightforward manner (via a Rytov expansion) to the continuous counterpart of the O'Doherty–Anstey expressions for the transmission response (O'Doherty & Anstey 1971). The further analysis of eq. (B2) for 3-D finely layered media is the subject of current research.