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WAVE FIELD EXTRAPOLATION TECHNIQUES FOR INHOMOGENEOUS MEDIA WHICH INCLUDE CRITICAL ANGLE EVENTS. PART II: METHODS USING THE TWO-WAY WAVE EQUATION*

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Abstract

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In one-way wave field extrapolation downgoing and upgoing waves are treated independently, which is allowed if propagation at small angles against the vertical in (weakly) inhomogeneous media is considered. In practical implementation the slow convergence of the square-root operator causes numerical deficiencies. On the other hand, in two-way wave field extrapolation no assumptions need to be made on the separability of downgoing and upgoing waves. Furthermore, in practical implementation the use of the square-root operator is avoided. To put the two-way techniques into perspective, it is shown that two-way wave field extrapolation could be described in terms of one-way processes, namely: (1) decomposition of the total wave field into downgoing and upgoing waves; (2) one-way wave field extrapolation; (3) composition of the total wave field from its downgoing and upgoing constituents. This alternative description of two-way wave field extrapolation is valid for media which are homogeneous along the z-coordinate as well as for small dip angles in arbitrarily inhomogeneous media. In addition, it is shown that this description is also valid for large dip angles in 1-D (vertically) inhomogeneous media, including critical-angle events, when the WKBJ one-way wave functions discussed in part I of this paper are considered.

For large dip angles in arbitrarily inhomogeneous media the two-way wave equation is solved by means of Taylor series expansion. For practical implementation a truncated operator is designed, assuming gentle horizontal variations of the medium properties. This operator is stable and converges already in the first order approximation, also for critical-angle events.

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1. Introduction

Both in modeling and migration schemes, wave field extrapolation operators play an important role. In depth extrapolation techniques these operators describe the propagation effects of the wave field from one depth level to another. For this purpose a horizontally layered *computational* model is often chosen, as shown in fig. 1. It should be stressed that the depth levels $z_0, z_1, z_2, \ldots, z_{i-1}, z_i, \ldots, z_I$

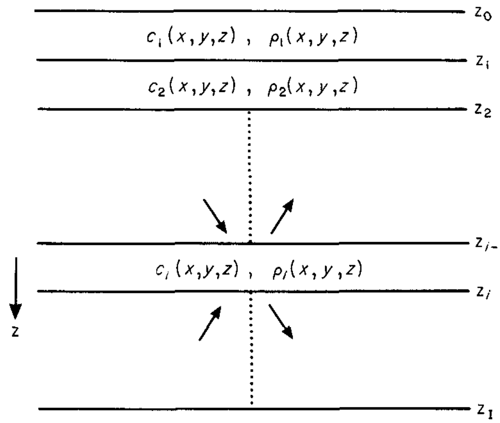

Fig. 1. Computationally convenient subsurface model for depth extrapolation techniques.

generally do not coincide with reflecting boundaries in the subsurface model. As a consequence, the medium properties c and ρ (propagation velocity and mass density) between two depth levels may be arbitrary functions of the spatial coordinates (x, y, z). Most approaches to wave field extrapolation are based on the assumption that the downgoing (source) wave field and the upgoing (reflected) wave field may be treated independently. This *one-way approach* is extensively discussed by Berkhout (1982) and in part I of this paper (Wapenaar and Berkhout 1985). In the frequency (ω) domain the one-way operations can be formulated in terms of spatial convolutions over the x- and y-coordinates. We consider four cases:

(i) *forward extrapolation of downgoing waves* P^+ (the positive z-axis is pointing downward) is symbolically described by

$$P^+(z_i) = W^+(z_i, z_{i-1}) * P^+(z_{i-1}), \tag{1.1a}$$

see also fig. 2a. (For notational convenience we abbreviate wave functions $P(x, y, z, \omega)$ as $P(z)$ or P; the symbol * denotes spatial convolutions);

(ii) *forward extrapolation of upgoing waves* P^- is symbolically described by

$$P^-(z_{i-1}) = W^-(z_{i-1}, z_i) * P^-(z_i), \tag{1.1b}$$

(see also fig. 2b);

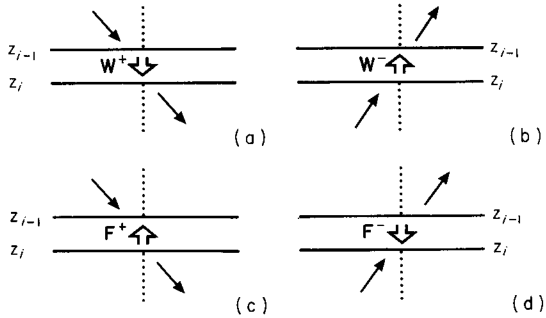

Fig. 2. In the one-way approach to wave field extrapolation downgoing and upgoing waves are treated independently: (a) forward extrapolation of downgoing waves; (b) forward extrapolation of upgoing waves; (c) inverse extrapolation of downgoing waves; (d) inverse extrapolation of upgoing waves.

(iii) *inverse extrapolation of downgoing waves* P^+ is symbolically described by

$$P^+(z_{i-1}) = F^+(z_{i-1}, z_i) * P^+(z_i), \tag{1.1c}$$

(see also fig. 2c);

(iv) *inverse extrapolation of upgoing waves* P^- is symbolically described by

$$P^-(z_i) = F^-(z_i, z_{i-1}) * P^-(z_{i-1}), \tag{1.1d}$$

(see also fig. 2d).

Relationships (1.1a) and (1.1b) provide the basis for modeling schemes based on the one-way wave equations, relationships (1.1c) and (1.1d) for migration schemes based on the one-way wave equations. As shown in part I, the one-way approach breaks down for strong vertical velocity gradients and large propagation angles (critical angle events). Interestingly, lateral gradients do not introduce any theoretical complications. In part I we discussed an extension to the one-way approach suitable for critical angle events as well. However, for this extension we had to assume that the medium properties are a function of depth only.

Here we discuss a more fundamental approach to wave field extrapolation which evades many of the problems typical for the one-way approach. We consider extrapolation of the *total* wave field, described in terms of P and $\rho^{-1}\,\partial_z P$. Because the total wave field is a superposition of downgoing and upgoing waves we may also speak of *two-way* wave field extrapolation. It is important to realize that no assumptions need be made on the separability of downgoing and upgoing waves (as for one-way techniques). In the frequency domain the two-way operations can be formulated in terms of spatial convolutions. We consider two cases:

(i) *upward extrapolation of the total wave field* $(P, \rho^{-1}\,\partial_z P)^T$ is symbolically described in matrix notation by

$$\begin{bmatrix} P(z_{i-1}) \\ \dfrac{1}{\rho}\dfrac{\partial P}{\partial z}\bigg|_{z_{i-1}} \end{bmatrix} = \begin{bmatrix} W_{\mathrm{I}}(z_{i-1}, z_i) * & W_{\mathrm{II}}(z_{i-1}, z_i) * \\ W_{\mathrm{III}}(z_{i-1}, z_i) * & W_{\mathrm{IV}}(z_{i-1}, z_i) * \end{bmatrix} \begin{bmatrix} P(z_i) \\ \dfrac{1}{\rho}\dfrac{\partial P}{\partial z}\bigg|_{z_i} \end{bmatrix}, \tag{1.2a}$$

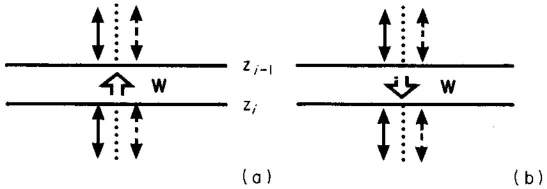

Fig. 3. In the two-way approach to wave field extrapolation downgoing and upgoing waves are treated simultaneously: (a) upward extrapolation of the total wave field; (b) downward extrapolation of the total wave field.

or, in abbreviated form, by

$$\mathbf{Q}(z_{i-1}) = \mathbf{W}(z_{i-1}, z_i)\mathbf{Q}(z_i), \tag{1.2b}$$

(see also fig. 3a);

(ii) *downward extrapolation of the total wave field* $[P, \rho^{-1}\, \partial_z P]^T$ is symbolically described in matrix notation by

$$\begin{bmatrix} P(z_i) \\ \left.\dfrac{1}{\rho}\dfrac{\partial P}{\partial z}\right|_{z_i} \end{bmatrix} = \begin{bmatrix} W_{\mathrm{I}}(z_i, z_{i-1}) * & W_{\mathrm{II}}(z_i, z_{i-1}) * \\ W_{\mathrm{III}}(z_i, z_{i-1}) * & W_{\mathrm{IV}}(z_i, z_{i-1}) * \end{bmatrix} \begin{bmatrix} P(z_{i-1}) \\ \left.\dfrac{1}{\rho}\dfrac{\partial P}{\partial z}\right|_{z_{i-1}} \end{bmatrix}, \tag{1.2c}$$

or, in abbreviated form, by

$$\mathbf{Q}(z_i) = \mathbf{W}(z_i, z_{i-1})\mathbf{Q}(z_{i-1}) \tag{1.2d}$$

(see also fig. 3b).

Various expressions for the two-way operator $\mathbf{W}$ are presented further on in this paper. Notice that upward and downward two-way wave field extrapolation are fundamentally equivalent: in both cases downgoing and upgoing waves are extrapolated simultaneously. It will be shown in part III that upward extrapolation algorithm (1.2a) or (1.2b) provides the basis for two-way modeling schemes and that downward extrapolation algorithm (1.2c) or (1.2d) provides the basis for two-way migration schemes.

2. A Comparison of the One-Way and the Two-Way Approach

In this section we start with the derivation of the matrix formulation of the two-way wave equation. Next, we decompose this formulation and show the close relationship with the one-way wave equations. Finally, we discuss the physical interpretation of both approaches.

In the frequency domain, the wave equation for inhomogeneous media is given by

$$\nabla^2 P + k^2 P = \nabla \ln \rho \cdot \nabla P, \tag{2.1}$$

where

$P = P(x, y, z, \omega)$ is the acoustic pressure,
$k = \omega/c$ is the wave number,
$c = c(x, y, z)$ is the propagation velocity,
$\rho = \rho(x, y, z)$ is the mass density, and
ω is the circular frequency.

In order to describe wave field propagation along the depth coordinate we separate z-derivatives from x- and y-derivatives according to

$$\frac{\partial^2 P}{\partial z^2} - \frac{\partial \ln \rho}{\partial z}\frac{\partial P}{\partial z} = -k^2 P - \frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial y^2} + \frac{\partial \ln \rho}{\partial x}\frac{\partial P}{\partial x} + \frac{\partial \ln \rho}{\partial y}\frac{\partial P}{\partial y}. \tag{2.2}$$

The z-derivatives can be expressed as a lateral operator H_2 working on P as follows:

$$\rho \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial P}{\partial z}\right) = -H_2 * P, \tag{2.3a}$$

where

$$H_2 = k^2\, d_0(x, y) + d_2(x) + d_2(y) - \frac{\partial \ln \rho}{\partial x}\, d_1(x) - \frac{\partial \ln \rho}{\partial y}\, d_1(y), \tag{2.3b}$$

with

$$d_0(x, y) = \delta(x)\delta(y). \tag{2.3c}$$

Notice that (2.3) represents a space variant spatial convolution along the x- and y-coordinates. $d_m(x)$ and $d_m(y)$ are space invariant one-dimensional spatial differentiation operators with respect to x and y, respectively, where m is the order of differentiation; the quantities k^2, $\partial_x \ln \rho$ and $\partial_y \ln \rho$ are space variant weighting factors. Note that for practical implementation spatially band-limited operators $d_m(x)$ and $d_m(y)$ should be used (Berkhout 1982).

Relation (2.3) is not yet suitable for wave field extrapolation because it is a second order differential equation. An easily manageable first order differential equation is obtained if we rewrite (2.3) as a matrix differential equation, according to

$$\frac{\partial \mathbf{Q}}{\partial z} = \mathbf{AQ}, \tag{2.4a}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & \rho\, d_0\, * \\ -\frac{1}{\rho} H_2 * & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} P \\ \frac{1}{\rho}\frac{\partial P}{\partial z} \end{bmatrix}. \tag{2.4b, c}$$

Notice that *two-way wave equation* (2.4) requires one boundary condition for the wave vector $\mathbf{Q}$ only.

If we define operators H_1 and H_1^{-1} such that

$$H_1 * H_1 = H_2, \tag{2.5a}$$

$$H_1^{-1} * H_1 = \delta(x)\,\delta(y), \tag{2.5b}$$

then operator **A** can be expressed as

$$\mathbf{A} = \mathbf{L}\boldsymbol{\Lambda}\mathbf{L}^{-1}, \tag{2.6a}$$

with

$$\mathbf{L} = \begin{bmatrix} d_0 * & d_0 * \\ \dfrac{-j}{\rho} H_1 * & \dfrac{j}{\rho} H_1 * \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} -jH_1 * & 0 \\ 0 & jH_1 * \end{bmatrix},$$

$$\mathbf{L}^{-1} = \frac{1}{2}\begin{bmatrix} d_0 * & jH_1^{-1} * \rho\, d_0 * \\ d_0 * & -jH_1^{-1} * \rho\, d_0 * \end{bmatrix}. \tag{2.6b, c, d}$$

In addition, if we define P^+ and P^- such that

$$P = P^+ + P^-, \tag{2.7a}$$

$$\frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{j}{\rho} H_1 * (P^+ - P^-), \tag{2.7b}$$

or, in matrix notation,

$$\mathbf{Q} = \mathbf{L}\mathbf{P} \tag{2.7c}$$

or, equivalently,

$$\mathbf{P} = \mathbf{L}^{-1}\mathbf{Q}, \tag{2.7d}$$

with

$$\mathbf{P} = \begin{bmatrix} P^+ \\ P^- \end{bmatrix}, \tag{2.7e}$$

we find by substituting (2.6a) and (2.7c) into two-way wave equation (2.4a)

$$\frac{\partial \mathbf{P}}{\partial z} = \left[\boldsymbol{\Lambda} - \mathbf{L}^{-1}\frac{\partial \mathbf{L}}{\partial z}\right]\mathbf{P}, \tag{2.8a}$$

or

$$\frac{\partial P^+}{\partial z} = -jH_1 * P^+ - \tfrac{1}{2}H_1^{-1} * \left[\rho \frac{\partial}{\partial z}\left(\frac{1}{\rho} H_1\right)\right] * (P^+ - P^-), \tag{2.8b}$$

and

$$\frac{\partial P^-}{\partial z} = +jH_1 * P^- + \tfrac{1}{2}H_1^{-1} * \left[\rho \frac{\partial}{\partial z}\left(\frac{1}{\rho} H_1\right)\right] * (P^+ - P^-). \tag{2.8c}$$

Apparently P^+ and P^- satisfy the coupled one-way wave equations for downgoing and upgoing waves, which were derived on physical grounds in part I. Note that in the conventional one-way approach it is assumed that $|P^-| \ll |P^+|$ in (2.8b) and $|P^+| \ll |P^-|$ in (2.8c), which means that critical angle events are not considered. Hence

$$\frac{\partial P^+}{\partial z} \approx -\mathrm{j}\mathrm{H}_1^+ * P^+, \tag{2.9a}$$

$$\frac{\partial P^-}{\partial z} \approx +\mathrm{j}\mathrm{H}_1^- * P^-, \tag{2.9b}$$

where the new operators H_1^+ and H_1^- are defined as

$$\mathrm{j}\mathrm{H}_1^+ = \mathrm{j}\mathrm{H}_1 + \tfrac{1}{2}\mathrm{H}_1^{-1} * \left[\rho \frac{\partial}{\partial z}\left(\frac{1}{\rho}\mathrm{H}_1\right)\right], \tag{2.9c}$$

$$\mathrm{j}\mathrm{H}_1^- = \mathrm{j}\mathrm{H}_1 - \tfrac{1}{2}\mathrm{H}_1^{-1} * \left[\rho \frac{\partial}{\partial z}\left(\frac{1}{\rho}\mathrm{H}_1\right)\right]. \tag{2.9d}$$

Finally, note that (2.8b) and (2.8c) *fully* decouple for media which are homogeneous along the z-coordinate, that is, when $\partial_z(\rho^{-1}\mathrm{H}_1) = 0$. Summarizing, by decomposing operator $\mathbf{A}$ we showed that two-way wave equation (2.4a)

$$\frac{\partial \mathbf{Q}}{\partial z} = \mathbf{AQ} \tag{2.10}$$

can be transformed into coupled one-way wave equations (2.8b, c), or, assuming that critical angle events may be neglected, into decoupled one-way wave equations (2.9a, b), in matrix notation given by

$$\frac{\partial \mathbf{P}}{\partial z} \approx \mathrm{j}\mathbf{H}_1\mathbf{P}, \tag{2.11a}$$

with

$$\mathbf{H}_1 = \begin{bmatrix} -\mathrm{H}_1^+ * & 0 \\ 0 & \mathrm{H}_1^- * \end{bmatrix}. \tag{2.11b}$$

For downward extrapolation, a solution of one-way wave equations (2.11) is symbolically described by [see also (1.1a) and (1.1d)]

$$\begin{bmatrix} P^+(z_i) \\ P^-(z_i) \end{bmatrix} = \begin{bmatrix} W^+(z_i, z_{i-1}) * & 0 \\ 0 & F^-(z_i, z_{i-1}) * \end{bmatrix} \begin{bmatrix} P^+(z_{i-1}) \\ P^-(z_{i-1}) \end{bmatrix}, \tag{2.12a}$$

or, in abbreviated form, by

$$\mathbf{P}(z_i) = \mathbf{V}(z_i, z_{i-1})\mathbf{P}(z_{i-1}). \tag{2.12b}$$

A similar relation holds for upward extrapolation.

With respect to these independent *one-way* solutions, notice the following:

(i) critical angle events are not included because the underlying wave equation (2.11) is a decoupled approximated version of (2.8);

(ii) in practical implementations for inhomogeneous media the solution is highly affected by numerical inaccuracy (limited dip angle performance) because the underlying wave equation is based on the implicitly defined *one-way square root operator* H_1 [see (2.5a)];

(iii) in recursive applications additional effort is required with respect to the boundary conditions between the consecutive extrapolation steps, because downgoing and upgoing waves are not continuous at layer interfaces. In practical implementations these boundary conditions are often neglected, which means that transmission effects and multiple reflections are not incorporated.

On the other hand, a solution of two-way wave (2.10) is symbolically described by [see also (1.2d)]

$$\mathbf{Q}(z_i) = \mathbf{W}(z_i, z_{i-1})\mathbf{Q}(z_{i-1}). \tag{2.13}$$

Notice that for media which are homogeneous along the z-coordinate, as well as for *small dip angle* applications in arbitrary inhomogeneous media, two-way wave field extrapolation, as described by (2.13), could be replaced by three sub-processes as follows:

(i) *decomposition* of the total wave field $\mathbf{Q} = [P, \rho^{-1}\,\partial_z P]^{\mathrm{T}}$ into downgoing and upgoing waves $\mathbf{P} = [P^+, P^-]^{\mathrm{T}}$, according to (2.7d):

$$\mathbf{P}(z_{i-1}) = \mathbf{L}^{-1}(z_{i-1})\mathbf{Q}(z_{i-1}); \tag{2.14a}$$

(ii) *independent one-way wave field extrapolation* of downgoing and upgoing waves, according to (2.12b):

$$\mathbf{P}(z_i) = \mathbf{V}(z_i, z_{i-1})\mathbf{P}(z_{i-1}); \tag{2.14b}$$

(iii) *composition* of the total wave field from its downgoing and upgoing constituents, according to (2.7c):

$$\mathbf{Q}(z_i) = \mathbf{L}(z_i)\mathbf{P}(z_i). \tag{2.14c}$$

Combination of these three steps yields

$$\mathbf{Q}(z_i) = [\mathbf{L}(z_i)\mathbf{V}(z_i, z_{i-1})\mathbf{L}^{-1}(z_{i-1})]\mathbf{Q}(z_{i-1}). \tag{2.15}$$

This total process is indicated symbolically in fig. 4. It is important to realize, however, that direct *two-way* wave field extrapolation, as described by (2.13), is to be preferred to the above forementioned one-way processes for the following reasons:

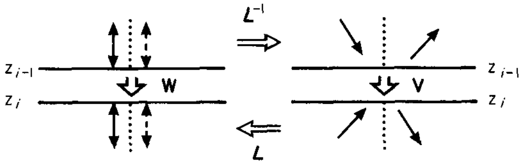

Fig. 4. Diagram showing the relationship between two-way and one-way wave field extrapolation.

(i) critical angle events are included because the underlying wave equation (2.10) is exact (in linear acoustics);

(ii) in practical implementations for inhomogeneous media the solution may be very accurate (90° dip angle performance, see section 6) because the underlying wave equation is based on the explicitly defined *two-way operator* H_2 [see (2.3b)];

(iii) in recursive applications *no* additional effort is required with respect to the boundary conditions, because the total wave field $\mathbf{Q} = [P, \rho^{-1}\, \partial_z P]^T$ is continuous at layer interfaces. Hence, transmission effects and multiple reflections are automatically incorporated; it is shown by Wapenaar, Kinneging and Berkhout (1985) that wave conversion may also be incorporated when the method is based on the full elastic two-way wave equation.

3. Two-way Solution for 1-D Inhomogeneous Media Including Critical Angle Events

We start with a review of the two-way wave equation in the wave number-frequency domain, following Ursin (1983). Next, we give the solution for a homogeneous layer. Finally, we present two solutions for piecewise continuously-layered media.

Consider the two-way wave equation in the wave number-frequency domain

$$\frac{\partial \tilde{\mathbf{Q}}}{\partial z} = \tilde{\mathbf{A}}\tilde{\mathbf{Q}}, \tag{3.1a}$$

where

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & \rho \\ -\tilde{H}_2/\rho & 0 \end{bmatrix}, \quad \tilde{\mathbf{Q}} = \begin{bmatrix} \tilde{P} \\ \dfrac{1}{\rho}\dfrac{\partial \tilde{P}}{\partial z} \end{bmatrix} \tag{3.1b, c}$$

and

$$\tilde{H}_2 = k_z^2 = \omega^2/c^2 - k_x^2 - k_y^2. \tag{3.1d}$$

The symbol $\tilde{}$ refers to a double spatial Fourier transformation from x to k_x and y to k_y, where k_x and k_y represent the horizontal components of the wave vector $\mathbf{k}$. The medium properties c and ρ are functions of the depth coordinate only: $c = c(z)$, $\rho = \rho(z)$. If we define $\tilde{H}_1$ such that

$$\tilde{H}_1^2 = \tilde{H}_2, \tag{3.2}$$

then *eigenvalue decomposition* applied to operator $\tilde{\mathbf{A}}$ yields

$$\tilde{\mathbf{A}} = \tilde{\mathbf{L}}\tilde{\mathbf{\Lambda}}\tilde{\mathbf{L}}^{-1}, \tag{3.3a}$$

with

$$\tilde{\mathbf{L}} = \begin{bmatrix} 1 & 1 \\ -j\tilde{H}_1/\rho & j\tilde{H}_1/\rho \end{bmatrix}, \quad \tilde{\mathbf{\Lambda}} = \begin{bmatrix} -j\tilde{H}_1 & 0 \\ 0 & j\tilde{H}_1 \end{bmatrix}, \quad \tilde{\mathbf{L}}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & j\rho/\tilde{H}_1 \\ 1 & -j\rho/\tilde{H}_1 \end{bmatrix}. \tag{3.3b, c, d}$$

Notice that this equation represents the wave number domain equivalent of (2.6). Similar as in (2.7), we define downgoing waves $\tilde{P}^+$ and upgoing waves $\tilde{P}^-$ according to

$$\tilde{\mathbf{Q}} = \tilde{\mathbf{L}}\tilde{\mathbf{P}} \tag{3.4a}$$

or

$$\tilde{\mathbf{P}} = \tilde{\mathbf{L}}^{-1}\tilde{\mathbf{Q}}, \tag{3.4b}$$

with

$$\tilde{\mathbf{P}} = \begin{bmatrix} \tilde{P}^+ \\ \tilde{P}^- \end{bmatrix}. \tag{3.4c}$$

Notice that decomposition (3.4b) breaks down for critical angle events, that is, for $\tilde{\mathrm{H}}_1 \to 0$. This phenomenon was already discussed in part I. Later in this section we present an alternative decomposition which is valid for critical angle events also.

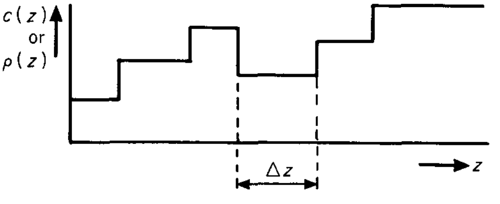

Fig. 5. Subsurface model with homogeneous layers.

We now consider the case that both c and ρ are constant within a given depth interval (see fig. 5). Then the solution of (3.1) is given by

$$\tilde{\mathbf{Q}}(z) = \tilde{\mathbf{W}}(z, z_0)\tilde{\mathbf{Q}}(z_0), \tag{3.5a}$$

where, symbolically,

$$\tilde{\mathbf{W}}(z, z_0) = \exp\,[\tilde{\mathbf{A}}\;\Delta z], \tag{3.5b}$$

with $\Delta z = z - z_0$. Using (3.3), (3.5b) can be written as

$$\tilde{\mathbf{W}}(z, z_0) = \mathbf{I} + \tilde{\mathbf{L}}(\tilde{\mathbf{\Lambda}}\;\Delta z)\tilde{\mathbf{L}}^{-1} + \tilde{\mathbf{L}}(\tilde{\mathbf{\Lambda}}\;\Delta z)\tilde{\mathbf{L}}^{-1}\tilde{\mathbf{L}}(\tilde{\mathbf{\Lambda}}\;\Delta z)\tilde{\mathbf{L}}^{-1} + \cdots \tag{3.6a}$$

or

$$\tilde{\mathbf{W}}(z, z_0) = \tilde{\mathbf{L}}[\mathbf{I} + (\tilde{\mathbf{\Lambda}}\;\Delta z) + (\tilde{\mathbf{\Lambda}}\;\Delta z)^2 + \cdots]\tilde{\mathbf{L}}^{-1} \tag{3.6b}$$

or

$$\tilde{\mathbf{W}}(z, z_0) = \tilde{\mathbf{L}}(z)\tilde{\mathbf{V}}(z, z_0)\tilde{\mathbf{L}}^{-1}(z_0), \tag{3.7a}$$

with

$$\tilde{\mathbf{V}}(z, z_0) = \exp\,[\tilde{\mathbf{\Lambda}}\;\Delta z], \tag{3.7b}$$

or

$$\tilde{\mathbf{V}}(z, z_0) = \begin{bmatrix} \exp(-j\tilde{H}_1 \Delta z) & 0 \\ 0 & \exp(j\tilde{H}_1 \Delta z) \end{bmatrix}. \tag{3.7c}$$

(3.7) shows that for this special case of a homogeneous medium, two-way operator $\tilde{\mathbf{W}}(z, z_0)$ can be written in terms of one-way sub-processes. This phenomenon was already discussed in the previous section [see (2.14) and (2.15)] and is visualized in fig. 4. On the other hand, if we define two-way operator $\tilde{\mathbf{W}}(z, z_0)$ as

$$\tilde{\mathbf{W}}(z, z_0) = \begin{bmatrix} \tilde{W}_{\mathrm{I}}(z, z_0) & \tilde{W}_{\mathrm{II}}(z, z_0) \\ \tilde{W}_{\mathrm{III}}(z, z_0) & \tilde{W}_{\mathrm{IV}}(z, z_0) \end{bmatrix}, \tag{3.8a}$$

then expressions for the sub-operators $\tilde{W}_{\mathrm{I}} \ldots \tilde{W}_{\mathrm{IV}}$ follow directly from relation (3.7):

$$\tilde{W}_{\mathrm{I}}(z, z_0) = \cos[\tilde{H}_1 \Delta z], \qquad \tilde{W}_{\mathrm{II}}(z, z_0) = \frac{\rho}{\tilde{H}_1} \sin[\tilde{H}_1 \Delta z], \tag{3.8b, c}$$

$$\tilde{W}_{\mathrm{III}}(z, z_0) = \tilde{Z}_2 \tilde{W}_{\mathrm{II}}(z, z_0), \qquad \tilde{W}_{\mathrm{IV}}(z, z_0) = \tilde{W}_{\mathrm{I}}(z, z_0), \tag{3.8d, e}$$

with

$$\tilde{Z}_2 = -\frac{1}{\rho^2} \tilde{H}_2. \tag{3.8f}$$

Notice that the limit for $\tilde{H}_1 \rightarrow 0$ exists. For evanescent waves ($k_x^2 + k_y^2 > \omega^2/c^2$) the operator $\tilde{H}_1$ becomes imaginary. The goniometric functions should then preferably be replaced by hyperbolic functions of the real argument $j\tilde{H}_1 \Delta z$.

For propagating waves ($k_x^2 + k_y^2 < \omega^2/c^2$) sub-operator $\tilde{W}_{\mathrm{I}}$ describes the real part of the phase shift operator $\exp(-j\tilde{H}_1 \Delta z)$, which represents the spatial Fourier transform of the Rayleigh II operator. In a similar way, operators $\tilde{W}_{\mathrm{II}}$ and $\tilde{W}_{\mathrm{III}}$ are related to the imaginary part of the transformed Rayleigh I and Rayleigh III operators. Hence, the spatially *band-limited* inverse Fourier transform of (3.5a) is given by the following relation in the space-frequency domain:

$$\mathbf{Q}(z) = \mathbf{W}(z, z_0)\mathbf{Q}(z_0), \tag{3.9a}$$

where

$$\mathbf{W}(z, z_0) = \begin{bmatrix} W_{\mathrm{I}}(z, z_0) * & W_{\mathrm{II}}(z, z_0) * \\ W_{\mathrm{III}}(z, z_0) * & W_{\mathrm{IV}}(z, z_0) * \end{bmatrix}, \tag{3.9b}$$

with

$$W_{\mathrm{I}}(z, z_0) = \mathrm{Re}\,(\text{Rayleigh II}), \qquad W_{\mathrm{II}}(z, z_0) = \frac{1}{\omega}\,\mathrm{Im}\,(\text{Rayleigh I}), \tag{3.9c, d}$$

$$W_{\mathrm{III}}(z, z_0) = -\omega\,\mathrm{Im}\,(\text{Rayleigh III}), \qquad W_{\mathrm{IV}}(z, z_0) = W_{\mathrm{I}}(z, z_0). \tag{3.9e, f}$$

The three Rayleigh operators are extensively discussed by Berkhout (1982). Notice that (3.9) describes stable two-way wave field extrapolation in the z-direction (the convolutions are carried out in the x- and y-directions).

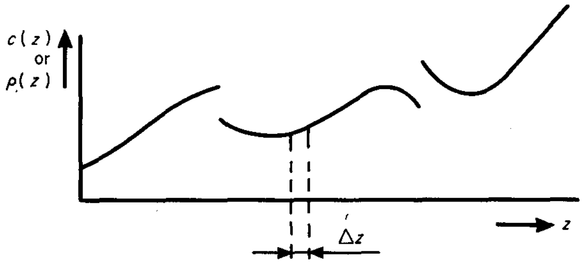

Fig. 6. Piecewise continuously layered subsurface model.

When $c(z)$ and $\rho(z)$ are arbitrary continuous functions within a given depth interval (see fig. 6), then two-way wave equation (3.1) cannot be solved as above. Suppose a solution is given by

$$\tilde{\mathbf{Q}}(z) = \tilde{\mathbf{W}}(z, z_0)\tilde{\mathbf{Q}}(z_0), \tag{3.10a}$$

then operator $\tilde{\mathbf{W}}(z, z_0)$ should satisfy the wave equation

$$\frac{\partial \tilde{\mathbf{W}}(z, z_0)}{\partial z} = \tilde{\mathbf{A}}(z)\tilde{\mathbf{W}}(z, z_0). \tag{3.10b}$$

Furthermore, when the gradients of c and ρ vanish, then $\tilde{\mathbf{W}}(z, z_0)$ should be equal to operator (3.8). Assuming that the medium properties may be linearized within a sufficiently thin layer, according to

$$c(z) = c_0[1 + q\,\Delta z] \tag{3.11a}$$

and

$$\rho(z) = \rho_0[1 + r\,\Delta z], \tag{3.11b}$$

with

$$\Delta z = z - z_0,$$

while

$$|q\,\Delta z| \ll 1 \tag{3.11c}$$

and

$$|r\,\Delta z| \ll 1, \tag{3.11d}$$

then it can be verified by substitution that operator $\tilde{\mathbf{W}}(z, z_0)$ is given by

$$\tilde{\mathbf{W}}(z, z_0) = \begin{bmatrix} \tilde{W}_{\mathrm{I}}(z, z_0) & \tilde{W}_{\mathrm{II}}(z, z_0) \\ \tilde{W}_{\mathrm{III}}(z, z_0) & \tilde{W}_{\mathrm{IV}}(z, z_0) \end{bmatrix}, \tag{3.12a}$$

with

$$\tilde{W}_{\mathrm{I}}(z, z_0) = (1 + R)\tilde{\psi}_1 + (S - R)\tilde{\psi}_2 + S\tilde{\psi}_3, \tag{3.12b}$$

$$\tilde{W}_{\mathrm{II}}(z, z_0) = \rho_0\,\Delta z[(1 + R)\tilde{\psi}_2 - S\tilde{\psi}_3], \tag{3.12c}$$

$$\tilde{W}_{\mathrm{III}}(z, z_0) = \tilde{Z}_2 \tilde{W}_{\mathrm{II}}(z, z_0), \tag{3.12d}$$

$$\tilde{W}_{\mathrm{IV}}(z, z_0) = (1 - R)\tilde{\psi}_1 + (S + R)\tilde{\psi}_2 - S\tilde{\psi}_3, \tag{3.12e}$$

$$\tilde{Z}_2 = -\frac{1}{\rho_0^2}\left[(1 - r\,\Delta z)\tilde{\mathrm{H}}_2(z_0) - q\,\Delta z\,\frac{\omega^2}{c_0^2}\right], \tag{3.12f}$$

where

$$S = \frac{q\omega^2}{2c_0^2}\Delta z^3, \qquad R = \frac{r}{2}\Delta z. \tag{3.12g, h}$$

For propagating waves, i.e., for $\tilde{\mathrm{H}}_2 > 0$, the operators $\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3$ are given by

$$\tilde{\psi}_1 = \cos\tilde{\Phi}, \tag{3.12i}$$

$$\tilde{\psi}_2 = (\sin\tilde{\Phi})/\tilde{\Phi}, \tag{3.12j}$$

$$\tilde{\psi}_3 = (\cos\tilde{\Phi})/\tilde{\Phi}^2 - (\sin\tilde{\Phi})/\tilde{\Phi}^3, \tag{3.12k}$$

with

$$\tilde{\Phi} = \tilde{\mathrm{H}}_1(z_0)\,\Delta z. \tag{3.12l}$$

Notice that the limit for $\tilde{\mathrm{H}}_1(z_0) \to 0$ exists. For evanescent waves the operator $\tilde{\mathrm{H}}_1(z_0)$ is imaginary. The goniometric functions should then preferably be replaced by hyperbolic functions of the real argument $\mathrm{j}\tilde{\mathrm{H}}_1(z_0)\,\Delta z$. Notice that the only approximation is the thin layer assumption (3.11). This means that *critical angle events* are properly incorporated in operator $\tilde{\mathbf{W}}(z, z_0)$, given by (3.12). For large extrapolation distances this operator must be applied recursively with steps such that in each recursion step thin layer assumption (3.11) is satisfied.

In part I we have shown that decomposition of the total wave field into downgoing and upgoing waves is not uniquely defined. Based on the WKBJ approach for 1-D inhomogeneous media, we discussed an alternative choice of *decoupled* downgoing and upgoing propagating waves $\tilde{\mathscr{P}}^+$ and $\tilde{\mathscr{P}}^-$ in the vicinity of a turning point, which include critical angle events. In the following, curled symbols refer to the incorporation of critical angle events in the one-way approach. The superposition of $\tilde{\mathscr{P}}^+$ and $\tilde{\mathscr{P}}^-$, given by

$$\tilde{P} = \tilde{\mathscr{P}}^+ + \tilde{\mathscr{P}}^- \tag{3.13a}$$

satisfies the wave equation

$$\frac{\partial^2}{\partial z^2}\left(\frac{\tilde{P}}{\sqrt{\rho}}\right) = -\tilde{\mathrm{H}}_2\left(\frac{\tilde{P}}{\sqrt{\rho}}\right), \tag{3.13b}$$

whereas $\tilde{\mathscr{P}}^+$ and $\tilde{\mathscr{P}}^-$ satisfy the following decoupled one-way wave equations:

$$\frac{\partial\tilde{\mathscr{P}}^+}{\partial z} = -\mathrm{j}\tilde{\mathscr{H}}_1^+\tilde{\mathscr{P}}^+, \tag{3.13c}$$

$$\frac{\partial\tilde{\mathscr{P}}^-}{\partial z} = +\mathrm{j}\tilde{\mathscr{H}}_1^-\tilde{\mathscr{P}}^-. \tag{3.13d}$$

Wave equation (3.13b) represents the spatial Fourier transform of wave equation (2.3), assuming $|E_\rho| = |(1/2\rho)\, \partial_z^2 \rho - (3/4)(\rho^{-1}\, \partial_z \rho)^2| \ll k^2$. Furthermore, it was assumed in part I that in the vicinity of the turning point z_t the operator $\tilde{\mathrm{H}}_2$ can be linearized in depth according to

$$\tilde{\mathrm{H}}_2(z) = (z - z_t)\chi. \tag{3.13e}$$

Hence, it is assumed that in the vicinity of z_t the propagation velocity $c(z)$ satisfies

$$c^{-2}(z) = c^{-2}(z_t) + (z - z_t)\chi\omega^{-2}, \tag{3.13f}$$

or, if z_0 is close to z_t,

$$c^{-2}(z) = c_0^{-2}(1 + c_0^2\,\omega^{-2}\chi\, \Delta z), \tag{3.13g}$$

with $c_0 = c(z_0)$ and $\Delta z = z - z_0$. In wave equations (3.13c) and (3.13d) the operators $\tilde{\mathscr{H}}_1^+$ and $\tilde{\mathscr{H}}_1^-$ are based on Airy functions, according to (7.4c, d; part I). We may construct a matrix formalism, based on (3.13a–d). The total wave field $\tilde{\mathbf{Q}} = [\tilde{P}, \rho^{-1}\, \partial_z \tilde{P}]^{\mathrm{T}}$ can be composed from the wave functions $\tilde{\mathscr{P}}^+$ and $\tilde{\mathscr{P}}^-$ according to

$$\tilde{\mathbf{Q}} = \tilde{\mathscr{L}}\tilde{\boldsymbol{\mathscr{P}}}, \tag{3.14a}$$

where

$$\tilde{\mathscr{L}} = \begin{bmatrix} 1 & 1 \\ -\mathrm{j}\tilde{\mathscr{H}}_1^+/\rho & \mathrm{j}\tilde{\mathscr{H}}_1^-/\rho \end{bmatrix}, \qquad \tilde{\boldsymbol{\mathscr{P}}} = \begin{bmatrix} \tilde{\mathscr{P}}^+ \\ \tilde{\mathscr{P}}^- \end{bmatrix}, \tag{3.14b, c}$$

and where $\tilde{\mathbf{Q}}$ satisfies two-way wave equation (3.1a), assuming $|E_\rho| \ll k^2$. Similarly, decomposition is described by

$$\tilde{\boldsymbol{\mathscr{P}}} = \tilde{\mathscr{L}}^{-1}\tilde{\mathbf{Q}}, \tag{3.14d}$$

where

$$\tilde{\mathscr{L}}^{-1} = \frac{1}{\tilde{\mathscr{H}}_1^+ + \tilde{\mathscr{H}}_1^-}\begin{bmatrix} \tilde{\mathscr{H}}_1^- & \mathrm{j}\rho \\ \tilde{\mathscr{H}}_1^+ & -\mathrm{j}\rho \end{bmatrix}. \tag{3.14e}$$

Notice that $\tilde{\mathscr{L}}^{-1}$ defines a decomposition operator which is valid for sub-critical as well as critical angle events. Finally, the one-way wave equations (3.13c) and (3.13d) can be combined into the following matrix equation:

$$\frac{\partial \tilde{\boldsymbol{\mathscr{P}}}}{\partial z} = \mathrm{j}\tilde{\boldsymbol{\mathscr{H}}}_1\tilde{\boldsymbol{\mathscr{P}}}, \tag{3.15a}$$

with

$$\tilde{\boldsymbol{\mathscr{H}}}_1 = \begin{bmatrix} -\tilde{\mathscr{H}}_1^+ & 0 \\ 0 & \tilde{\mathscr{H}}_1^- \end{bmatrix}. \tag{3.15b}$$

For downward extrapolation, a solution of one-way wave equation (3.15) is described by [see also (7.6) and (7.7) in part I]

$$\begin{bmatrix} \tilde{\mathscr{P}}^+(z) \\ \tilde{\mathscr{P}}^-(z) \end{bmatrix} = \begin{bmatrix} \tilde{\mathscr{W}}^+(z, z_0) & 0 \\ 0 & \tilde{\mathscr{F}}^-(z, z_0) \end{bmatrix}\begin{bmatrix} \tilde{\mathscr{P}}^+(z_0) \\ \tilde{\mathscr{P}}^-(z_0) \end{bmatrix}, \tag{3.16a}$$

or, in abbreviated form, by

$$\tilde{\mathscr{P}}(z) = \tilde{\mathscr{V}}(z, z_0)\tilde{\mathscr{P}}(z_0). \tag{3.16b}$$

A similar relationship holds for upward extrapolation. Notice that (3.14) and (3.16) can be elegantly combined into one relation for downward extrapolation of the total wave field $\tilde{\mathbf{Q}}$:

$$\tilde{\mathbf{Q}}(z) = [\tilde{\mathscr{L}}(z)\tilde{\mathscr{V}}(z, z_0)\tilde{\mathscr{L}}^{-1}(z_0)]\tilde{\mathbf{Q}}(z_0). \tag{3.17}$$

Under thin-layer assumption (3.11), with $q = -\frac{1}{2}c_0^2\omega^{-2}\chi$, this solution is equivalent to

$$\tilde{\mathbf{Q}}(z) = \tilde{\mathbf{W}}(z, z_0)\tilde{\mathbf{Q}}(z_0), \tag{3.18}$$

with two-way operator $\tilde{\mathbf{W}}(z, z_0)$ given by (3.12).

Hence, (3.17) shows that also in the special case of a 1-D inhomogeneous medium, two-way wave field extrapolation can be written in terms of one-way sub-processes, including *critical angle events.* For practical implementations, however, two-way algorithm (3.18) is preferred because it avoids the use of Airy functions. Two-way operator $\tilde{\mathbf{W}}(z, z_0)$, as defined by (3.12), is fully based on simple goniometric functions, which provide the basis for recursive finite difference schemes that include critical angle events (see section 5).

We have derived two-way wave field extrapolation operators for media with depth-dependent properties $c(z)$ and $\rho(z)$, as shown in figs 5 and 6. Both in modeling and migration schemes, the operators should be applied recursively, which is allowed because the extrapolated total wave field $\tilde{\mathbf{Q}}$ is continuous for all depths. This means that transmission effects as well as multiple reflections are automatically incorporated when applying operator (3.8) or (3.9). In addition, critical angle events are incorporated when applying operator (3.12). Modeling and migration schemes, based on the extrapolation operators (3.8) and (3.12), are discussed in part III. In addition, it will be shown in part III that the composition and decomposition algorithms (3.4) and (3.14) play an important role in two-way modeling and migration schemes.

4. Two-Way Solution for Arbitrary Inhomogeneous Media Including Critical Angle Events

In this section we discuss two-way wave field extrapolation in arbitrary inhomogeneous media. It is shown that in principle lateral derivatives of the medium properties can be incorporated. In addition, the operator may include all propagation angles as well as evanescent waves.

Our starting point is two-way wave equation (2.4a)

$$\frac{\partial \mathbf{Q}}{\partial z} = \mathbf{A}\mathbf{Q}, \tag{4.1}$$

where $\mathbf{A}$ and $\mathbf{Q}$ are defined by (2.4b, c). Assuming that the derivatives $\partial^m\mathbf{Q}/\partial z^m$ exist and are continuous between z_0 and z, we can define two-way wave field extrapolation by means of the following Taylor series summation:

$$\mathbf{Q}(z) = \sum_{m=0}^{\infty} \frac{\Delta z^m}{m!} \left[\frac{\partial^m \mathbf{Q}}{\partial z^m} \right]_{z_0}, \tag{4.2}$$

with $\Delta z = z - z_0$. Notice that in practice the scheme should be applied recursively for small $|\Delta z|$. Berkhout (1982) discussed a similar approach for one-way wave field extrapolation. He concluded that the total error per extrapolation step depends on two different sub-errors:

1. the error in the estimates of the derivatives with respect to z;
2. the error due to truncation of the Taylor series.

In one-way wave field extrapolation, the z-derivatives are based on the *implicit* square-root operator H_1, defined by (2.5a). On the other hand, in two-way wave field extrapolation, the z-derivatives are based on the *explicit* operator H_2, defined by (2.3b).

They can be calculated exactly within the seismic band width, which means that sub-error 1 vanishes in case of two-way wave field extrapolation. In the next section we show that for gradual horizontal variations of the medium properties also sub-error 2 remains small.

From (4.1), the z-derivatives follow directly by recursively applying

$$\frac{\partial^m \mathbf{Q}}{\partial z^m} = \frac{\partial}{\partial z} \left[\frac{\partial^{m-1} \mathbf{Q}}{\partial z^{m-1}} \right], \tag{4.3a}$$

so

$$\frac{\partial \mathbf{Q}}{\partial z} = \mathbf{A}\mathbf{Q}, \tag{4.3b}$$

$$\frac{\partial^2 \mathbf{Q}}{\partial z^2} = \left[\mathbf{A}\mathbf{A} + \frac{\partial \mathbf{A}}{\partial z} \right] \mathbf{Q}, \tag{4.3c}$$

$$\frac{\partial^3 \mathbf{Q}}{\partial z^3} = \left[\mathbf{A}\mathbf{A}\mathbf{A} + 2\frac{\partial \mathbf{A}}{\partial z}\mathbf{A} + \mathbf{A}\frac{\partial \mathbf{A}}{\partial z} + \frac{\partial^2 \mathbf{A}}{\partial z^2} \right] \mathbf{Q}, \tag{4.3d}$$

etc.

As in the previous section, we assume linearized medium properties in the z-direction. This means that sufficient thin-layers should be taken (see fig. 7b). For this situation the scheme is worked out in appendix A. In this section we only show the principle, assuming the medium properties are constant in depth, that is, $c = c(x, y)$, $\rho = \rho(x, y)$ within one layer (see fig. 7a). In this case all derivatives of operator $\mathbf{A}$ with respect to z vanish, so relation (4.2) can be written as

$$\mathbf{Q}(z) = \sum_{m=0}^{\infty} \frac{\Delta z^m}{m!} \mathbf{A}^m \mathbf{Q}(z_0). \tag{4.4}$$

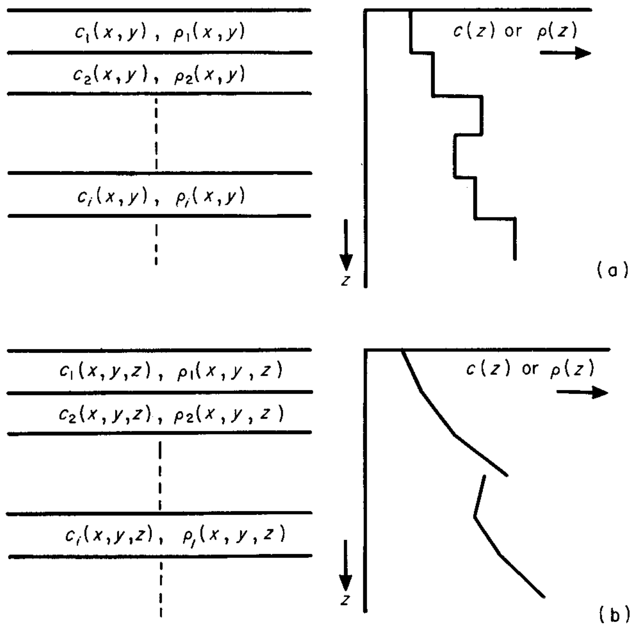

Fig. 7. Computationally convenient subsurface models with thin inhomogeneous layers: (a) c and ρ constant in depth per layer; (b) c and ρ linearized in depth per layer.

This relation can be rewritten as

$$\mathbf{Q}(z) = \sum_{n=0}^{\infty} \left[\frac{\Delta z^{2n}}{(2n)!} \mathbf{A}^{2n} + \frac{\Delta z^{2n+1}}{(2n+1)!} \mathbf{A}^{2n+1} \right] \mathbf{Q}(z_0), \tag{4.5a}$$

where

$$\mathbf{A}^{2n} = (-1)^n \begin{bmatrix} \mathrm{H}_{2n} * & 0 \\ 0 & \dfrac{1}{\rho} \mathrm{H}_{2n} * \rho \mathrm{H}_0 * \end{bmatrix}, \tag{4.5b}$$

$$\mathbf{A}^{2n+1} = (-1)^n \begin{bmatrix} 0 & \mathrm{H}_{2n} * \rho \mathrm{H}_0 * \\ -\dfrac{1}{\rho} \mathrm{H}_{2n+2} * & 0 \end{bmatrix}, \tag{4.5c}$$

with H_{2n+2} recursively defined by

$$\mathrm{H}_{2n+2} = \mathrm{H}_2 * \mathrm{H}_{2n}, \tag{4.5d}$$

$$\mathrm{H}_{2n} = \mathrm{H}_2 * \mathrm{H}_{2n-2}, \tag{4.5e}$$

etc., and

$$\mathrm{H}_0 = d_0(x, y) = \delta(x)\,\delta(y). \tag{4.5f}$$

Notice that (4.5) can be written as

$$\mathbf{Q}(z) = \mathbf{W}(z, z_0)\mathbf{Q}(z_0), \tag{4.6a}$$

where

$$\mathbf{W}(z, z_0) = \begin{bmatrix} W_{\rm I}(z, z_0) * & W_{\rm II}(z, z_0) * \\ W_{\rm III}(z, z_0) * & W_{\rm IV}(z, z_0) * \end{bmatrix}, \tag{4.6b}$$

$$W_{\rm I}(z, z_0) = \sum_{n=0}^{\infty} a_n \mathrm{H}_{2n}, \qquad a_n = \frac{\Delta z^{2n}}{(2n)!}(-1)^n, \tag{4.6c, d}$$

$$W_{\rm II}(z, z_0) = \sum_{n=0}^{\infty} b_n \mathrm{H}_{2n} * \rho \mathrm{H}_0, \qquad b_n = \frac{\Delta z^{2n+1}}{(2n+1)!}(-1)^n, \tag{4.6e, f}$$

$$W_{\rm III}(z, z_0) = \sum_{n=0}^{\infty} -b_n \frac{1}{\rho} \mathrm{H}_{2n+2}, \tag{4.6g}$$

$$W_{\rm IV}(z, z_0) = \sum_{n=0}^{\infty} a_n \frac{1}{\rho} \mathrm{H}_{2n} * \rho \mathrm{H}_0. \tag{4.6h}$$

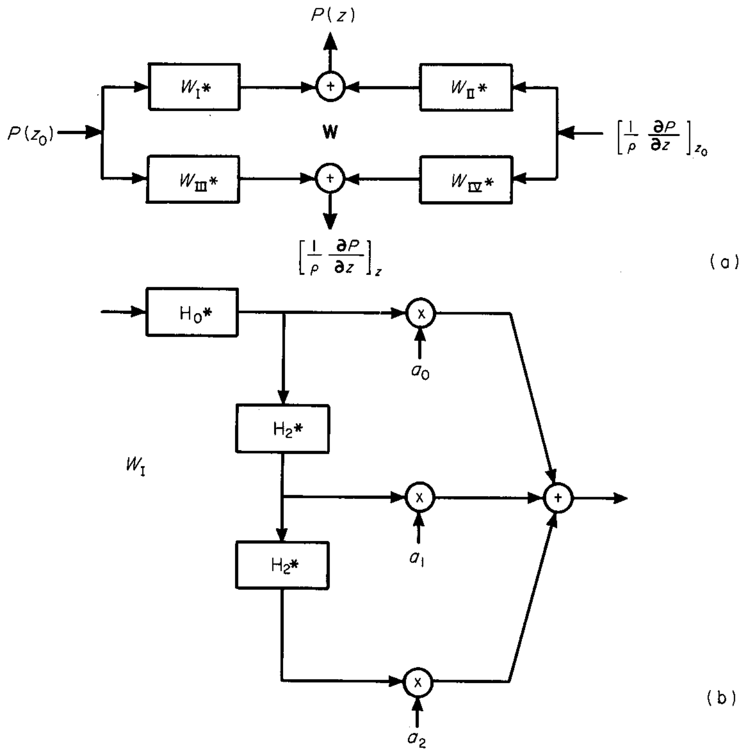

Fig. 8. (a) Two-way wave field extrapolation scheme. (b) Detailed diagram of the second order approximation of sub-operator $W_{\rm I}$. For operators $W_{\rm II}$, $W_{\rm III}$, $W_{\rm IV}$, operator H_0 should be replaced by $\rho\mathrm{H}_0$, H_2, $\rho\mathrm{H}_0$, respectively, while coefficients a_n should be replaced by b_n, $-b_n/\rho$, a_n/ρ, respectively.

Notice that for two-dimensional as well as three-dimensional applications (4.6) describes an explicit finite difference two-way wave field extrapolation operator in the space-frequency domain, based on one-dimensional convolutions. The extrapolation scheme is shown in diagram in fig. 8.

Note that in the special situation that c and ρ are constant in one layer, extrapolation may be carried out in the wave number-frequency domain, so H_{2n} may be replaced by $\tilde{H}_2^n$, with $\tilde{H}_2$ given by (3.1d). Now the infinite series in (4.6) can be summed to closed expressions, yielding operators (3.8b–e).

We have derived a two-way wave field extrapolation operator for inhomogeneous media, assuming that in each layer the medium properties are constant in depth (fig. 7a). Since no approximation were made, the operator (4.6b) is exact. Of course, for practical applications this formal operator should be truncated. This is discussed in the following sections. In appendix A we generalized the operator assuming that in each layer the medium properties may be linearized in depth (fig. 7b) as described by (A.2). Furthermore, we neglected mixed derivatives of the medium properties containing the first derivative with respect to z. Operator (A.5b) takes properly into account sub-critical as well as critical angle events.

5. A Fast Converging Two-Way Extrapolation Scheme for Inhomogeneous Media

In the previous section we have seen that one of the sub-errors in wave field extrapolation by means of Taylor series summation vanishes when the scheme is based on the two-way wave equation, since the z-derivatives of the total wave field can be calculated exactly within the seismic band width. However, significant errors may arise in practice due to the truncation of the Taylor series, particularly for horizontal plane waves. For one-way wave field extrapolation, Claerbout (1976) introduced a floating time reference in order to improve the convergence speed. This means that the horizontal plane wave phase shift operator $\exp(-jk\,\Delta z)$ is kept out of the Taylor series expansion (Berkhout 1982). In two-way wave field extrapolation the floating time reference concept cannot be followed, because downgoing and upgoing waves are considered simultaneously (see fig. 3). Instead, we rearrange the Taylor series expansion such that the two-way horizontal plane wave extrapolation operator can be kept out of the Taylor series expansion. In this section we only show the principle, assuming that c and ρ are constant in depth within each layer (fig. 7a). In appendix B we consider the case that c and ρ are linear functions of depth within each layer (fig. 7b).

Assuming slow horizontal variations of the medium properties such that the lateral derivatives may be neglected, operator (4.6b) can be written as

$$\mathbf{W}(z, z_0) = \sum_{n=0}^{\infty} [(\mathbf{E}_n + \mathbf{A}\mathbf{F}_n)\mathbf{H}_2^n], \tag{5.1a}$$

where

$$\mathbf{E}_n = \begin{bmatrix} a_n & 0 \\ 0 & a_n \end{bmatrix}, \qquad \mathbf{F}_n = \begin{bmatrix} b_n & 0 \\ 0 & b_n \end{bmatrix}, \qquad \mathbf{H}_2^n = \begin{bmatrix} \mathrm{H}_{2n} * & 0 \\ 0 & \mathrm{H}_{2n} * \end{bmatrix}, \tag{5.1b, c, d}$$

and $\mathbf{A}$ defined by (2.4b). Applying a binomial expansion for $\mathbf{H}_2^n$ we may write

$$\mathbf{H}_2^n = (\mathbf{K} + \mathbf{D}_2)^n = \sum_{m=0}^{n} \left[\frac{1}{m!} \left(\frac{\partial^m}{\partial \kappa^m} \mathbf{K}^n \right) \mathbf{D}_2^m \right], \tag{5.2a}$$

where

$$\mathbf{K} = \begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix}, \qquad \mathbf{D}_2 = \begin{bmatrix} \mathrm{D}_2 * & 0 \\ 0 & \mathrm{D}_2 * \end{bmatrix}, \tag{5.2b, c}$$

$$\kappa = k^2, \qquad \mathrm{D}_2 = \mathrm{d}_2(x) + \mathrm{d}_2(y). \tag{5.2d, e}$$

Substituting (5.2a) in (5.1a), changing the order of summations, and using the property $\partial_\kappa^m \kappa^n = 0$ for $m > n$, yields

$$\mathbf{W}(z, z_0) = \sum_{m=0}^{\infty} \left[\frac{1}{m!} \left\{ \frac{\partial^m}{\partial \kappa^m} \mathbf{M} + \mathbf{A} \frac{\partial^m}{\partial \kappa^m} \mathbf{N} \right\} \mathbf{D}_2^m \right], \tag{5.3a}$$

where

$$\mathbf{M} = \sum_{n=0}^{\infty} [\mathbf{E}_n \mathbf{K}^n], \qquad \mathbf{N} = \sum_{n=0}^{\infty} [\mathbf{F}_n \mathbf{K}^n]. \tag{5.3b, c}$$

So far, we did nothing but rewriting operator $\mathbf{W}(z, z_0)$, assuming the lateral derivatives of the medium properties may be neglected. In our next step, however, we replace the infinite series (5.3b, c) by closed expressions, according to

$$\mathbf{M} = \begin{bmatrix} W_{\mathrm{I},0} & 0 \\ 0 & W_{\mathrm{I},0} \end{bmatrix}, \qquad \mathbf{N} = \frac{1}{\rho} \begin{bmatrix} W_{\mathrm{II},0} & 0 \\ 0 & W_{\mathrm{II},0} \end{bmatrix}, \tag{5.4a, b}$$

where

$$W_{\mathrm{I},0} = \cos(k\,\Delta z), \qquad W_{\mathrm{II},0} = \frac{\rho}{k} \sin(k\,\Delta z). \tag{5.4c, d}$$

Notice that $W_{\mathrm{I},0}$ and $W_{\mathrm{II},0}$ are equal to the operators $\tilde{W}_{\mathrm{I}}$ and $\tilde{W}_{\mathrm{II}}$, respectively, given by relations (3.8b, c), for a horizontal plane wave (i.e., for $k_x^2 = k_y^2 = 0$).

By substituting (5.4) in (5.3), it follows that operator $\mathbf{W}(z, z_0)$ is given by

$$\mathbf{W}(z, z_0) = \begin{bmatrix} W_{\mathrm{I}}(z, z_0) * & W_{\mathrm{II}}(z, z_0) * \\ W_{\mathrm{III}}(z, z_0) * & W_{\mathrm{IV}}(z, z_0) * \end{bmatrix}, \tag{5.5a}$$

where

$$W_{\mathrm{I}}(z, z_0) = \sum_{m=0}^{\infty} \alpha_m \mathrm{D}_{2m}, \qquad W_{\mathrm{II}}(z, z_0) = \sum_{m=0}^{\infty} \beta_m \mathrm{D}_{2m}, \tag{5.5b, c}$$

$$W_{\mathrm{III}}(z, z_0) = \mathrm{Z}_2 * W_{\mathrm{II}}(z, z_0), \qquad W_{\mathrm{IV}}(z, z_0) = \sum_{m=0}^{\infty} \gamma_m \mathrm{D}_{2m}, \tag{5.5d, e}$$

with

$$\alpha_m = \frac{1}{m!}\left(\frac{\partial^m}{\partial \kappa^m} W_{\mathrm{I},0}\right), \qquad \beta_m = \frac{1}{m!}\left(\frac{\partial^m}{\partial \kappa^m} W_{\mathrm{II},0}\right), \qquad \gamma_m = \alpha_m, \tag{5.5f, g, h}$$

$$\mathrm{Z}_2 = -\frac{1}{\rho^2}\,\mathrm{H}_2, \tag{5.5i}$$

and D_{2m} defined recursively by

$$\mathrm{D}_{2m} = \mathrm{D}_2 * \mathrm{D}_{2m-2}, \tag{5.5j}$$

$$\mathrm{D}_{2m-2} = \mathrm{D}_2 * \mathrm{D}_{2m-4}, \tag{5.5k}$$

etc., and

$$\mathrm{D}_0 = \mathrm{d}_0(x, y) = \delta(x)\,\delta(y). \tag{5.5l}$$

Notice that for two-dimensional as well as three-dimensional applications (5.5) describes an explicit finite difference two-way wave field extrapolation scheme based on one-dimensional convolutions. Operator D_{2m} represents space invariant spatial convolutions for all m, while α_m, β_m and γ_m represent space-dependent coefficients:

$$\alpha_0 = \gamma_0 = \cos k\,\Delta z, \qquad \alpha_1 = \gamma_1 = -\frac{\Delta z}{2k}\sin k\,\Delta z, \text{ etc.,} \tag{5.6a, b}$$

$$\beta_0 = \frac{\rho}{k}\sin k\,\Delta z, \qquad \beta_1 = -\frac{\rho}{2k^3}[\sin k\,\Delta z - k\,\Delta z\cos k\,\Delta z], \text{ etc.,} \tag{5.6c, d}$$

with

$$k = \omega/c(x, y), \qquad \rho = \rho(x, y). \tag{5.6e, f}$$

In practice only a finite number of terms can be used. We define the Mth order approximation of $\mathbf{W}(z, z_0)$ by

$$\mathbf{W}_M(z, z_0) = \begin{bmatrix} W_{\mathrm{I},M}(z, z_0) * & W_{\mathrm{II},M}(z, z_0) * \\ W_{\mathrm{III},M}(z, z_0) * & W_{\mathrm{IV},M}(z, z_0) * \end{bmatrix}, \tag{5.7a}$$

where

$$W_{\mathrm{I},M}(z, z_0) = \sum_{m=0}^{M} \alpha_m \mathrm{D}_{2m}, \qquad W_{\mathrm{II},M}(z, z_0) = \sum_{m=0}^{M} \beta_m \mathrm{D}_{2m}, \tag{5.7b, c}$$

$$W_{\mathrm{III},M}(z, z_0) = \mathrm{Z}_2 * W_{\mathrm{II},M}(z, z_0), \qquad W_{\mathrm{IV},M}(z, z_0) = \sum_{m=0}^{M} \gamma_m \mathrm{D}_{2m}. \tag{5.7d, e}$$

Notice that for a horizontal plane wave all lateral derivatives are zero, which means that the zeroth-order scheme ($M = 0$) already converges for this situation. We may conclude that the zeroth-order terms represent the "floating time reference for the two-way wave equation".

Accuracy and stability properties for various order are studied in the next section. The first order extrapolation scheme is shown in diagram in fig. 9. Notice that the operator D_2 is used efficiently in two suboperators.

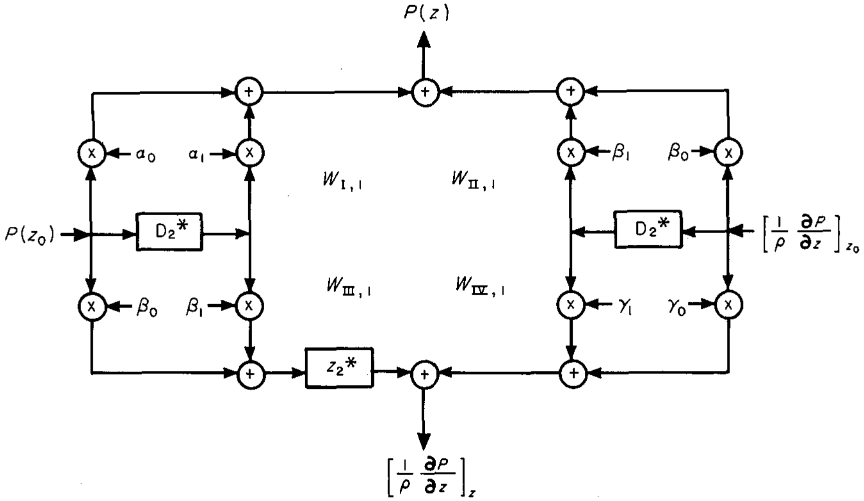

Fig. 9. Rapidly converging two-way wave field extrapolation scheme for sub-critical as well as critical angle events.

In this section we have derived a quickly converging two-way wave field extrapolation operator for inhomogeneous media, assuming that the medium properties are constant in depth for each layer (fig. 7a). In the derivation we assumed that the lateral *derivatives* of the medium properties may be neglected. In appendix B we generalize the operator, assuming that the medium properties for each layer may be linearized in depth (fig. 7b), as described by (A.2). Operator (B.5) is comparable to (5.5); only Z_2 and the coefficients α_m, β_m, γ_m are defined differently. These coefficients are based on operators $\tilde{\psi}_1$, $\tilde{\psi}_2$, $\tilde{\psi}_3$, given by (3.12i–k) for $k_x^2 = k_y^2 = 0$, and their derivatives with respect to κ. Operator (B.5) takes properly into account sub-critical as well as critical angle events. Hence, the diagram in fig. 9 represents the first order two-way wave field extrapolation scheme for sub-critical as well as critical angle events.

Finally, we present a two-term operator $\mathbf{W}_M^{(2)}(z, z_0)$, which is defined as follows:

$$\mathbf{W}_M^{(2)}(z, z_0) = \begin{bmatrix} W_{\mathrm{I},M}^{(2)}(z, z_0) * & W_{\mathrm{II},M}^{(2)}(z, z_0) * \\ W_{\mathrm{III},M}^{(2)}(z, z_0) * & W_{\mathrm{IV},M}^{(2)}(z, z_0) * \end{bmatrix}, \tag{5.8a}$$

where

$$W_{\mathrm{I},M}^{(2)}(z, z_0) = \bar{W}_{\mathrm{I}}(z, z_0) + \Delta W_{\mathrm{I},M}(z, z_0), \tag{5.8b}$$

$$W_{\mathrm{II},M}^{(2)}(z, z_0) = \bar{W}_{\mathrm{II}}(z, z_0) + \Delta W_{\mathrm{II},M}(z, z_0), \tag{5.8c}$$

$$W_{\mathrm{III},M}^{(2)}(z, z_0) = Z_2 * W_{\mathrm{II},M}^{(2)}(z, z_0), \tag{5.8d}$$

$$W_{\mathrm{IV},M}^{(2)}(z, z_0) = \bar{W}_{\mathrm{IV}}(z, z_0) + \Delta W_{\mathrm{IV},M}(z, z_0). \tag{5.8e}$$

Operator Z_2 is defined by (B.11). The operators $\bar{W}_{\mathrm{I}}$, $\bar{W}_{\mathrm{II}}$, $\bar{W}_{\mathrm{IV}}$ describe two-way wave field extrapolation in a homogeneous *reference* layer, with propagation velocity $\bar{c}$

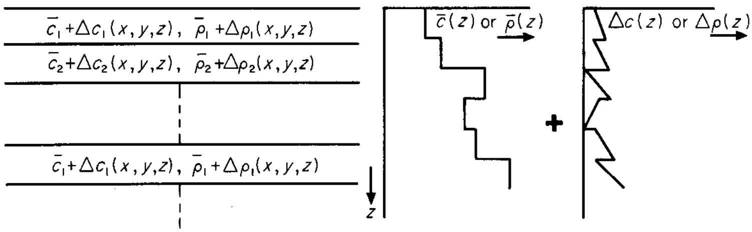

Fig. 10. Computationally-convenient subsurface model with inhomogeneous layers, used for the two-term operator.

and mass density $\bar{\rho}$ (fig. 10). They should preferably be applied in the wave number-frequency domain. The double spatial Fourier transforms of these operators are given by (3.8b, c, e), where c and ρ should be replaced by $\bar{c}$ and $\bar{\rho}$. The operators $\Delta W_{\mathrm{I},M}$, $\Delta W_{\mathrm{II},M}$ and $\Delta W_{\mathrm{IV},M}$ take into account the two-way propagation effects due to the *deviation* $\Delta c(x, y, z)$, $\Delta\rho(x, y, z)$ of the true medium properties $c(x, y, z)$, $\rho(x, y, z)$ from the reference medium properties $\bar{c}$, $\bar{\rho}$. They are defined as follows:

$$\Delta W_{\mathrm{I},M}(z, z_0) = \sum_{m=0}^{M} \Delta\alpha_m \mathbf{D}_{2m}, \qquad \Delta\alpha_m = \alpha_m - \bar{\alpha}_m, \tag{5.9a, b}$$

$$\Delta W_{\mathrm{II},M}(z, z_0) = \sum_{m=0}^{M} \Delta\beta_m \mathbf{D}_{2m}, \qquad \Delta\beta_m = \beta_m - \bar{\beta}_m, \tag{5.9c, d}$$

$$\Delta W_{\mathrm{IV},M}(z, z_0) = \sum_{m=0}^{M} \Delta\gamma_m \mathbf{D}_{2m}, \qquad \Delta\gamma_m = \gamma_m - \bar{\gamma}_m, \tag{5.9e, f}$$

where α_m, β_m, γ_m are defined by (B.5f, g, h) and where $\bar{\alpha}_m$, $\bar{\beta}_m$, $\bar{\gamma}_m$ are defined by (5.5f, g, h), with c, ρ replaced by $\bar{c}$, $\bar{\rho}$. A Born-type two-term wave field extrapolation operator was discussed by Kennett (1972). In his approach the deviation term describes the effects of a moderate inhomogeneity (10% contrast with the surrounding medium) of small lateral and vertical extent. For comparison, in two-term operator (5.8) no assumptions are made with respect to the dimensions of the contrast. It is shown in the next section that the first order scheme ($M = 1$) converges for sub-critical as well as critical angle events, even when the contrast $\Delta c/\bar{c}$ is in the order of 25% in the whole layer.

6. Accuracy and Stability of Two-Way Extrapolation

In this section we discuss the accuracy and stability properties of the Mth order finite difference operators $\mathbf{W}_M(z, z_0)$ and $\mathbf{W}_M^{(2)}(z, z_0)$, given by (5.7) and (5.8), respectively, as a function of the propagation angle. Therefore, it is convenient if we assume that c and ρ are functions of z only. First we consider operator $\mathbf{W}_M(z, z_0)$, in the wave number-frequency domain given by

$$\tilde{\mathbf{W}}_M(z, z_0) = \begin{bmatrix} \tilde{W}_{\mathrm{I},M}(z, z_0) & \tilde{W}_{\mathrm{II},M}(z, z_0) \\ \tilde{W}_{\mathrm{III},M}(z, z_0) & \tilde{W}_{\mathrm{IV},M}(z, z_0) \end{bmatrix}, \tag{6.1a}$$

where

$$\tilde{W}_{\mathrm{I},M}(z, z_0) = \sum_{m=0}^{M} \alpha_m \tilde{\mathrm{D}}_2^m, \qquad \tilde{W}_{\mathrm{II},M}(z, z_0) = \sum_{m=0}^{M} \beta_m \tilde{\mathrm{D}}_2^m, \tag{6.1b, c}$$

$$\tilde{W}_{\mathrm{III},M}(z, z_0) = \tilde{\mathrm{Z}}_2 \tilde{W}_{\mathrm{II},M}(z, z_0), \qquad \tilde{W}_{\mathrm{IV},M}(z, z_0) = \sum_{m=0}^{M} \gamma_m \tilde{\mathrm{D}}_2^m, \tag{6.1d, e}$$

$$\tilde{\mathrm{D}}_2^m = (-k_x^2 - k_y^2)^m, \tag{6.1f}$$

with α_m, β_m, γ_m given by (B.5f, g, h) and $\tilde{\mathrm{Z}}_2$ given by (3.12f).

Notice that

$$\lim_{M \to \infty} \tilde{\mathbf{W}}_M(z, z_0) = \tilde{\mathbf{W}}(z, z_0), \tag{6.2}$$

with $\tilde{\mathbf{W}}(z, z_0)$ given by (3.12). In the following analysis, we compare the eigenvalues $\tilde{\mu}_M$ of $\tilde{\mathbf{W}}_M(z, z_0)$ with the eigenvalues $\tilde{\mu}$ of operator $\tilde{\mathbf{W}}(z, z_0)$:

$$\tilde{\mu}_M^{\pm} = \tfrac{1}{2}(\tilde{W}_{\mathrm{I},M} + \tilde{W}_{\mathrm{IV},M}) \mp \mathrm{j}\sqrt{\det(\tilde{\mathbf{W}}_M) - \tfrac{1}{4}(\tilde{W}_{\mathrm{I},M} + \tilde{W}_{\mathrm{IV},M})^2}, \tag{6.3a}$$

$$\tilde{\mu}^{\pm} = \tfrac{1}{2}(\tilde{W}_{\mathrm{I}} + \tilde{W}_{\mathrm{IV}}) \mp \mathrm{j}\sqrt{1 - \tfrac{1}{4}(\tilde{W}_{\mathrm{I}} + \tilde{W}_{\mathrm{IV}})^2}. \tag{6.3b}$$

Notice that $\tilde{\mu}_M^+ \tilde{\mu}_M^- = \det(\tilde{\mathbf{W}}_M)$ and $\tilde{\mu}^+ \tilde{\mu}^- = 1$.

The angle-dependent amplitude and phase errors we define as

$$\Delta\tilde{A} = \sqrt{\det(\tilde{\mathbf{W}}_M)} - 1, \tag{6.4a}$$

$$\Delta\tilde{\Phi} = \pm[\arg(\tilde{\mu}_M^{\pm}) - \arg(\tilde{\mu}^{\pm})]. \tag{6.4b}$$

In order to specify the threshold values for these errors, we consider a homogeneous medium. Notice that in this case the eigenvalues of the exact operator, given by (6.3b), simplify to

$$\tilde{\mu}^{\pm} = \exp(\mp \mathrm{j}\tilde{\mathrm{H}}_1 \Delta z), \tag{6.5}$$

which is equivalent to the *phase-shift operator* for one-way wave field extrapolation. In recursive extrapolation, the total amplitude and phase errors after N extrapolation steps in a homogeneous medium read

$$\Delta\tilde{A}_{\mathrm{tot}} = (1 + \Delta\tilde{A})^N - 1 \approx N\,\Delta\tilde{A}, \tag{6.6a}$$

$$\Delta\tilde{\Phi}_{\mathrm{tot}} = N\,\Delta\tilde{\Phi}. \tag{6.6b}$$

We define the following (arbitrary) accuracy criteria:

> at an extrapolation depth of $N\,\Delta z = 50\lambda$ the absolute amplitude error $N\,\Delta\tilde{A}$ should be smaller than 3 dB and the absolute phase error $N\,\Delta\tilde{\Phi}$ should be smaller than $\pi/10$.

Here $\Delta z = z - z_0 > 0$, while λ represents the wave length. Since $N = 100\pi/(k\,\Delta z)$, the *accuracy* criteria read

$$|\Delta\tilde{A}|/(k\,\Delta z) \leq 0.001, \tag{6.7a}$$

$$|\Delta\tilde{\Phi}|/(k\,\Delta z) \leq 0.001. \tag{6.7b}$$

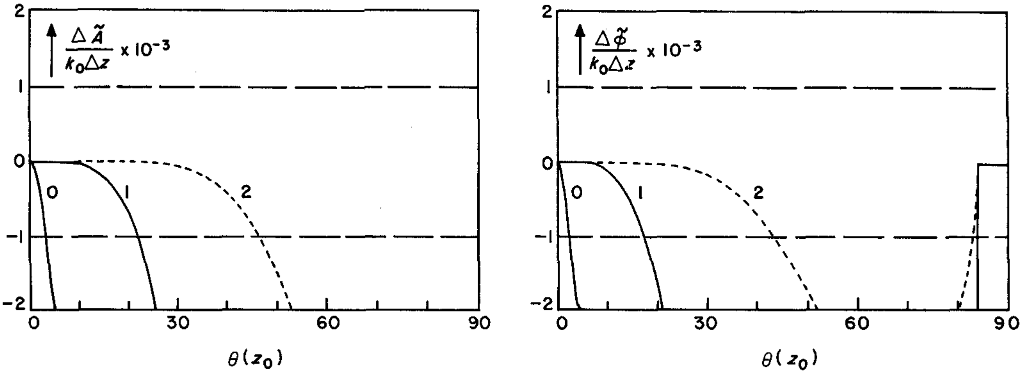

Fig. 11. Error curves for the zero-, first-, and second-order finite difference two-way wave field extrapolation operator, with $k_0\ \Delta z = \pi/2$, $q\ \Delta z = 0.01$, $r\ \Delta z = 0$.

These requirements cannot always be met for all propagation angles. To avoid that the solution grows out of bounds, our *stability* criterion reads

$$\Delta\tilde{A}/(k\ \Delta z) \leq 0.001 \text{ for all propagation angles.} \tag{6.7c}$$

We adopt these criteria for the inhomogeneous situation.

In this case k should be replaced by $k_0 = k(z_0)$. In fig. 11 the scaled amplitude and phase errors are shown as a function of θ, with

$$\sin^2\theta = (k_x^2 + k_y^2)/k^2(z_0). \tag{6.8}$$

Notice that θ represents the propagation angle at depth z_0: $\theta = \theta(z_0)$. The extrapolation step size is $\Delta z = \lambda/4$, so $k\ \Delta z = \pi/2$. The parameter q (3.11) is chosen such that $q\ \Delta z = 0.01$, which means that the velocity at extrapolation depth $z_0 + 25\lambda$ equals twice the velocity at depth z_0 (assuming the velocity function to be linear also outside the considered depth interval Δz). The density is assumed to be constant, i.e., $r = 0$. Notice that thin-layer condition (3.11c, d) is satisfied. From fig. 11 we observe that the first-order operator $\tilde{\mathbf{W}}_1$ is accurate up to 20° and that the second-order operator $\tilde{\mathbf{W}}_2$ is accurate up to 45°. Notice that all operators are stable. For a proper incorporation of critical angle events ($\theta \to 90°$), higher order schemes are required, which is not very attractive from a computational point of view. Therefore we consider also the two-term operator $\mathbf{W}_M^{(2)}$. Similarly as above we can define the eigenvalues $\tilde{\mu}_M^{(2)}$ of the two-term operator in the wave number-frequency domain. The amplitude and phase errors for one extrapolation step we define as

$$\Delta\tilde{A}^{(2)} = \sqrt{\det(\tilde{\mathbf{W}}_M^{(2)})} - 1, \tag{6.9a}$$

$$\Delta\tilde{\Phi}^{(2)} = \pm[\arg(\tilde{\mu}_M^{(2)\pm}) - \arg(\tilde{\mu}^{\pm})]. \tag{6.9b}$$

In fig. 12 the scaled amplitude and phase errors are shown as a function of $\theta(z_0)$. In all examples Δz is chosen such that $\bar{k}\ \Delta z = \pi/2$, where $\bar{k} = \omega/\bar{c}$. Furthermore, $\rho(z_0) = \bar{\rho}$, $r = 0$. In fig. 12a the true velocity is chosen close to the reference velocity, according to $[c(z_0) - \bar{c}]/\bar{c} = 0.05$, $q\ \Delta z = 0.01$. Notice that the first order two-term

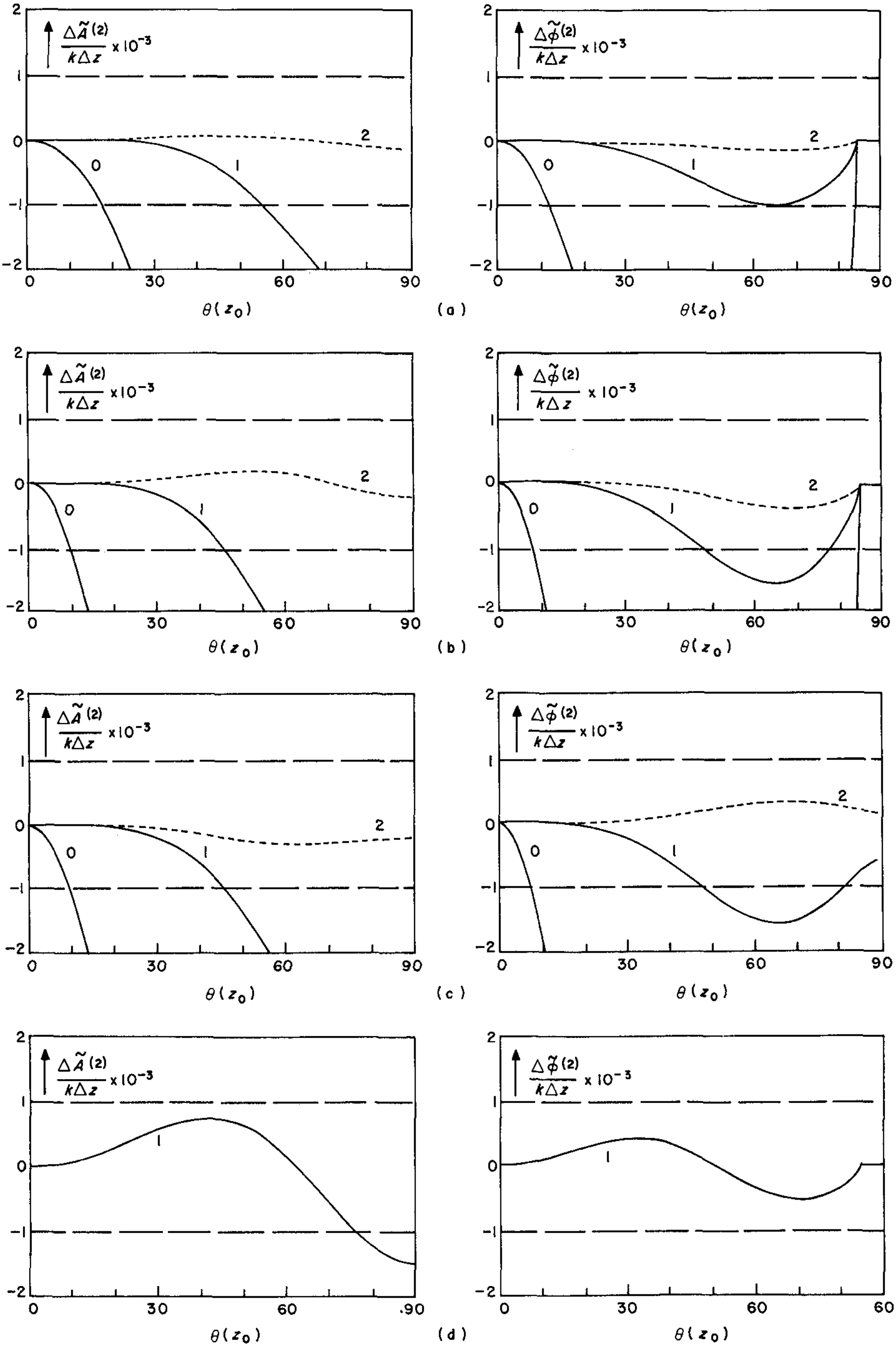

Fig. 12. Error curves for the zero-, first-, and second-order finite difference *two-term* two-way wave field extrapolation operator, with $\bar{k}\,\Delta z = \pi/2$, $\rho(z_0) = \bar{\rho}$, $r\,\Delta z = 0$. (a) $[c(z_0) - \bar{c}]/\bar{c} = 0.05$, $q\,\Delta z = 0.01$; (b) $[c(z_0) - \bar{c}]/\bar{c} = 0.25$, $q\,\Delta z = 0.01$; (c) $[c(z_0) - \bar{c}]/\bar{c} = 0.25$, $q\,\Delta z = -0.01$; (d) As in (b); improved first order scheme, $\theta_1(z_0) = 60°$.

operator is accurate up to 55° and that the second order operator is accurate up to 90°. All operators are stable. For angles higher than 82° a turning point occurs within the considered depth interval Δz. This accounts for the phase behaviour at high angles ($\theta \to 90°$), where the eigenvalues become purely real (for comparison, the phase-shift operator for one-way wave field extrapolation becomes real for evanescent waves). In fig. 12b the true velocity differs significantly from the reference velocity, according to $[c(z_0) - \bar{c}]/\bar{c} = 0.25$, $q\ \Delta z = 0.01$. Notice that the first order operator is accurate up to 45° and that the second order operator is accurate up to 90°. All operators are stable. In fig. 12c, a decreasing velocity is chosen, according to $[c(z_0) - \bar{c}]/\bar{c} = 0.25$, $q\ \Delta z = -0.01$. The error curves are comparable with fig. 12b. Notice that no turning point is present within the considered depth interval Δz. In fig. 12d the same example is chosen as in fig. 12b, however, the operator D_4 is approximated by $-[k^2(z_0)\sin^2\theta_1(z_0)]D_2$ with $\theta_1(z_0) = 60°$. This means that the second order scheme has been simplified to a first order scheme without loss of accuracy at 60°. Notice that this improved first order scheme is accurate and stable up to 90°.

We have formulated accuracy and stability conditions for the finite difference approximation of the eigenvalues of the two-way wave field extrapolation operator for inhomogeneous media. In the examples we studied the error curves for various orders, assuming a depth-dependent velocity function. Since density variations do not account for critical angle events, the density was kept constant in all examples. From the examples, it may be concluded that the finite difference approximations are stable and that critical angle events are properly taken into account. Particularly the improved first order two-term operator is very attractive from a computational point of view. Notice that the analysis has been performed in the wave number-frequency domain. It is assumed that the investigated accuracy and stability properties apply locally in a laterally varying medium.

7. Conclusions

In principle there are two approaches to modify the wave equation such that wave field depth extrapolation operators can be derived:

(i) decomposition into two first order one-way wave equations for P^+ and P^-, respectively;

(ii) reformulation into a first order two-way matrix wave equation for $(P, \rho^{-1}\ \partial P/\partial z)^{\mathrm{T}}$.

In this part we discussed methods using the two-way wave equation. In sections 1 and 2 we have shown the close relationship with the one-way wave equations which were derived in part I. We have discussed several solutions for increasing complexity of the medium:

(1) the medium consists of a sequence of homogeneous layers. *Exact* solutions for each layer have been formulated in both the space-frequency domain and the wave number-frequency domain (section 3). The recursive scheme is very simple since the total field is continuous for all depths. This means that the boundary

conditions (reflection, transmission) for downgoing and upgoing waves are automatically fulfilled at the layer interfaces. This is advantageous with respect to a proper treatment of multiple reflections in modeling as well as migration schemes, as will be discussed in part III;

(2) the medium consists of a sequence of layers where in each layer the medium properties are functions of the depth coordinate only. A solution has been formulated in the wave number-frequency domain, assuming the medium properties may be linearized in depth within each layer (section 3). Critical angle events are properly incorporated. We discussed the relationship with the decoupled WKBJ one-way wave equations for downgoing and upgoing waves which include critical angle events (as discussed in part I). Again the recursive scheme is very simple. Applications will be discussed in part III;

(3) the medium is arbitrary inhomogeneous. A formal solution has been formulated in the space-frequency domain, assuming that within each layer the medium properties may be linearized in depth. Critical angle events are incorporated (section 4 and appendix A). A quickly converging explicit finite difference scheme has been derived, assuming slow horizontal variations of the medium properties (section 5 and appendix B). It has been shown that the scheme is stable and converges already in the first order approximation, also for critical angle events (section 6).

ACKNOWLEDGMENTS

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APPENDIX A

We derive a two-way wave field extrapolation operator for arbitrarily inhomogeneous media. We follow the same procedure as in section 4; however, here we take into account the first derivative of the medium properties with respect to z.

Our starting point is relation (4.2):

$$\mathbf{Q}(z) = \sum_{m=0}^{\infty} \frac{(z - z_0)^m}{m!} \left[\frac{\partial^m \mathbf{Q}}{\partial z^m}\right]_{z0}, \tag{A.1a}$$

where

$$\frac{\partial^m \mathbf{Q}}{\partial z^m} = \frac{\partial}{\partial z}\left[\frac{\partial^{m-1}\mathbf{Q}}{\partial z^{m-1}}\right], \tag{A.1b}$$

with

$$\frac{\partial \mathbf{Q}}{\partial z} = \mathbf{A}\mathbf{Q}, \tag{A.1c}$$

A and **Q** being defined by (2.4b, c). In the following we assume linearized medium properties, according to

$$c(x, y, z) = c_0(x, y)[1 + q(x, y)\,\Delta z] \tag{A.2a}$$

and

$$\rho(x, y, z) = \rho_0(x, y)[1 + r(x, y)\,\Delta z], \tag{A.2b}$$

with

$$\Delta z = z - z_0,$$

while

$$|q(x, y)\,\Delta z| \ll 1 \tag{A.2c}$$

and

$$|r(x, y)\,\Delta z| \ll 1. \tag{A.2d}$$

Now operator **A** can be linearized, according to

$$\mathbf{A} = \mathbf{A}_0 + \partial_z \mathbf{A}_0\,\Delta z, \tag{A.3a}$$

with

$$\mathbf{A}_0 = \begin{bmatrix} 0 & \rho_0\,\mathrm{d}_0\,* \\ -\dfrac{1}{\rho_0} L_2\,* & 0 \end{bmatrix}, \qquad \partial_z \mathbf{A}_0 = \begin{bmatrix} 0 & \rho_0 r\,\mathrm{d}_0\,* \\ \dfrac{1}{\rho_0} G_2\,* & 0 \end{bmatrix}, \tag{A.3b, c}$$

$$L_2 = \mathrm{H}_2(z_0), \qquad G_2 = \rho_0\left[\frac{\partial}{\partial z}\left(-\frac{1}{\rho}\,\mathrm{H}_2\right)\right]_{z0}. \tag{A.3d, e}$$

We assume that lateral derivatives of $q(x, y)$ and $r(x, y)$ may be neglected and we obtain

$$G_2 = rL_2 + (2q\omega^2/c_0^2)\,\mathrm{d}_0, \tag{A.3f}$$

while $[\partial_z^m \mathbf{Q}]_{z0}$ can be approximated for even m ($m = 2n$) by

$$[\partial_z^{2n}\mathbf{Q}]_{z0} = [\mathbf{A}_0^{2n} + n^2\mathbf{A}_0^{2n-2}(\partial_z \mathbf{A}_0) + n(n-1)\mathbf{A}_0^{2n-3}(\partial_z \mathbf{A}_0)\mathbf{A}_0]\mathbf{Q}(z_0), \tag{A.4a}$$

and for odd m ($m = 2n + 1$) by

$$[\partial_z^{2n+1}\mathbf{Q}]_{z0} = [\mathbf{A}_0^{2n+1} + n^2\mathbf{A}_0^{2n-1}(\partial_z \mathbf{A}_0) + n(n+1)\mathbf{A}_0^{2n-2}(\partial_z \mathbf{A}_0)\mathbf{A}_0]\mathbf{Q}(z_0), \tag{A.4b}$$

where

$$\mathbf{A}_0^{2n} = (-1)^n \begin{bmatrix} L_{2n}\,* & 0 \\ 0 & \dfrac{1}{\rho_0} L_{2n} * \rho_0 L_0\,* \end{bmatrix}, \tag{A.4c}$$

$$\mathbf{A}_0^{2n+1} = (-1)^n \begin{bmatrix} 0 & L_{2n} * \rho_0 L_0\,* \\ -\dfrac{1}{\rho_0} L_{2n+2}\,* & 0 \end{bmatrix}, \tag{A.4d}$$

$$\mathbf{A}_0^{2n-2}(\partial_z \mathbf{A}_0) = -(-1)^n \begin{bmatrix} 0 & \rho_0 r L_{2n-2} * \\ \dfrac{1}{\rho_0} G_2 * L_{2n-2} * & 0 \end{bmatrix}, \tag{A.4e}$$

$$\mathbf{A}_0^{2n-1}(\partial_z \mathbf{A}_0) = -(-1)^n \begin{bmatrix} G_2 * L_{2n-2} * & 0 \\ 0 & -rL_{2n} * \end{bmatrix}, \tag{A.4f}$$

$$\mathbf{A}_0^{2n-3}(\partial_z \mathbf{A}_0)\mathbf{A}_0 = (-1)^n \begin{bmatrix} 0 & \rho_0 G_2 * L_{2n-4} * \\ \dfrac{r}{\rho_0} L_{2n} * & 0 \end{bmatrix}, \tag{A.4g}$$

$$\mathbf{A}_0^{2n-2}(\partial_z \mathbf{A}_0)\mathbf{A}_0 = -(-1)^n \begin{bmatrix} -rL_{2n} * & 0 \\ 0 & G_2 * L_{2n-2} * \end{bmatrix}, \tag{A.4h}$$

with L_{2n+2} defined recursively, according to

$$L_{2n+2} = L_2 * L_{2n}, \qquad L_0 = \mathrm{d}_0(x, y) = \delta(x)\,\delta(y). \tag{A.4i, j}$$

With these equations we finally find

$$\mathbf{Q}(z) = \mathbf{W}(z, z_0)\mathbf{Q}(z_0), \tag{A.5a}$$

where

$$\mathbf{W}(z, z_0) = \begin{bmatrix} W_{\mathrm{I}}(z, z_0) * & W_{\mathrm{II}}(z, z_0) * \\ W_{\mathrm{III}}(z, z_0) * & W_{\mathrm{IV}}(z, z_0) * \end{bmatrix}, \tag{A.5b}$$

$$W_{\mathrm{I}}(z, z_0) = \sum_{n=0}^{\infty} \left[a_n L_{2n} + 2nb_n \left(-nq \frac{\omega^2}{c_0^2} + \frac{r}{2} L_2 * \right) L_{2n-2} \right], \tag{A.5c}$$

$$W_{\mathrm{II}}(z, z_0) = \sum_{n=0}^{\infty} \left\{ b_n L_{2n} * \rho_0 L_0 + 2\rho_0 n a_n \left[(n-1)q \frac{\omega^2}{c_0^2} - \frac{r}{2} L_2 * \right] L_{2n-4} \right\}, \tag{A.5d}$$

$$W_{\mathrm{III}}(z, z_0) = \sum_{n=0}^{\infty} \left[-b_n \frac{1}{\rho_0} L_{2n+2} + \frac{2}{\rho_0} n a_n \left(-nq \frac{\omega^2}{c_0^2} - \frac{r}{2} L_2 * \right) L_{2n-2} \right], \tag{A.5e}$$

$$W_{\mathrm{IV}}(z, z_0) = \sum_{n=0}^{\infty} \left\{ a_n \frac{1}{\rho_0} L_{2n} * \rho_0 L_0 + 2nb_n \left[-(n+1)q \frac{\omega^2}{c_0^2} - \frac{r}{2} L_2 * \right] L_{2n-2} \right\}, \tag{A.5f}$$

with a_n and b_n defined by (4.6d, f). Notice that for $q = r = 0$ (4.6) is obtained. On the other hand, if the medium properties c and ρ are functions of z only, then extrapolation may be carried out in the wave number-frequency domain, so operator L_{2n} may be replaced by $\tilde{L}_2^n = \tilde{\mathrm{H}}_2^n(z_0)$. Now the infinite series in (A.5) can be summed to closed expressions, yielding operators (3.12b–e).

Appendix B

We derive a fast converging two-way wave field extrapolation operator for arbitrary inhomogeneous media. We follow the same procedure as in section 5, however, here we take into account the first derivative of the medium properties with respect to z.

Our starting point is operator (A.5b). Neglecting the lateral derivatives of the medium properties, this operator can be rewritten as

$$\mathbf{W}(z, z_0) = \sum_{n=0}^{\infty} [\{\mathbf{R}\mathbf{E}_n + (\mathbf{S} + \mathbf{J}\mathbf{U})\mathbf{F}_n + (\mathbf{T} + \mathbf{J}\mathbf{V})\mathbf{G}_n\}\mathbf{L}_2^n], \tag{B.1a}$$

where

$$\mathbf{R} = \begin{bmatrix} 1+R & 0 \\ 0 & 1-R \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} S-R & 0 \\ 0 & S+R \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} S & 0 \\ 0 & -S \end{bmatrix}, \tag{B.1b, c, d}$$

$$\mathbf{J} = \rho_0\, \Delta z \begin{bmatrix} 0 & d_0{}^* \\ Z_2{}^* & 0 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1+R & 0 \\ 0 & 1+R \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} -S & 0 \\ 0 & -S \end{bmatrix}, \tag{B.1e, f, g}$$

$$\mathbf{E}_n = \begin{bmatrix} a_n & 0 \\ 0 & a_n \end{bmatrix}, \qquad \mathbf{F}_n = \begin{bmatrix} b_n/\Delta z & 0 \\ 0 & b_n/\Delta z \end{bmatrix}, \tag{B.1h, i}$$

$$\mathbf{G}_n = \frac{1}{\Delta z^2}\,[\mathbf{E}_{n+1} - \mathbf{F}_{n+1}], \qquad \mathbf{L}_2^n = \begin{bmatrix} L_{2n}{}^* & 0 \\ 0 & L_{2n}{}^* \end{bmatrix}, \tag{B.1j, k}$$

$$Z_2 = -\frac{1}{\rho_0^2}\left[(1 - r\,\Delta z)L_2 - q\,\Delta z\,\frac{\omega^2}{c_0^2}\,d_0\right], \tag{B.1l}$$

$$S = \frac{q\omega^2}{2c_0^2}\,\Delta z^3, \qquad R = \frac{r}{2}\,\Delta z, \tag{B.1m, n}$$

with $c_0(x, y)$, $\rho_0(x, y)$, $q(x, y)$, $r(x, y)$ given by (A.2), L_{2n} given by (A.4i) and a_n, b_n given by (4.6d, f). In (B.1) we made use of the property $\Delta z a_n = (2n+1)b_n$.

As in section 5, we may write

$$\mathbf{L}_2^n = \sum_{m=0}^{n} \left[\frac{1}{m!}\left(\frac{\partial^m}{\partial\kappa^m}\,\mathbf{K}^n\right)\mathbf{D}_2^m\right], \qquad \kappa = \omega^2/c_0^2, \tag{B.2}$$

with $\mathbf{K}$ and $\mathbf{D}_2$ given by (5.2b, c). Substitution in (B.1), changing the order of summations, and using the property $\partial_\kappa^m \kappa^n = 0$ for $m > n$, yields

$$\mathbf{W}(z, z_0) = \sum_{m=0}^{\infty} \left\{\frac{1}{m!}\left[\mathbf{R}\left(\frac{\partial^m}{\partial\kappa^m}\,\mathbf{M}\right) + (\mathbf{S} + \mathbf{J}\mathbf{U})\left(\frac{\partial^m}{\partial\kappa^m}\,\mathbf{N}\right) + (\mathbf{T} + \mathbf{J}\mathbf{V})\left(\frac{\partial^m}{\partial\kappa^m}\,\mathbf{P}\right)\right]\mathbf{D}_2^m\right\}, \tag{B.3a}$$

where

$$\mathbf{M} = \sum_{n=0}^{\infty} [\mathbf{E}_n \mathbf{K}^n], \qquad \mathbf{N} = \sum_{n=0}^{\infty} [\mathbf{F}_n \mathbf{K}^n], \qquad \mathbf{P} = \sum_{n=0}^{\infty} [\mathbf{G}_n \mathbf{K}^n]. \tag{B.3b, c, d}$$

The infinite expansions for **M**, **N** and **P** can be replaced by closed expressions, according to

$$\mathbf{M} = \begin{bmatrix} \psi_{1,0} & 0 \\ 0 & \psi_{1,0} \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} \psi_{2,0} & 0 \\ 0 & \psi_{2,0} \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} \psi_{3,0} & 0 \\ 0 & \psi_{3,0} \end{bmatrix}, \quad \text{(B.4a, b, c)}$$

where

$$\psi_{1,0} = \cos \Phi_0, \qquad \psi_{2,0} = \frac{\sin \Phi_0}{\Phi_0}, \qquad \psi_{3,0} = \frac{\cos \Phi_0}{\Phi_0^2} - \frac{\sin \Phi_0}{\Phi_0^3}, \quad \text{(B.4d, e, f)}$$

$$\Phi_0 = \sqrt{\kappa}\, \Delta z = \frac{\omega}{c_0}\, \Delta z. \quad \text{(B.4g)}$$

Notice that $\psi_{1,0}$, $\psi_{2,0}$ and $\psi_{3,0}$ are identical to the operators $\tilde{\psi}_1$, $\tilde{\psi}_2$ and $\tilde{\psi}_3$, respectively, given by (3.12i, j, k) for horizontal plane waves, that is, for $k_x^2 = k_y^2 = 0$.

By substituting (B.4) in (B.3), it follows that operator $\mathbf{W}(z, z_0)$ is given by

$$\mathbf{W}(z, z_0) = \begin{bmatrix} W_{\mathrm{I}}(z, z_0) * & W_{\mathrm{II}}(z, z_0) * \\ W_{\mathrm{III}}(z, z_0) * & W_{\mathrm{IV}}(z, z_0) * \end{bmatrix}, \quad \text{(B.5a)}$$

where

$$W_{\mathrm{I}}(z, z_0) = \sum_{m=0}^{\infty} \alpha_m \mathbf{D}_{2m}, \qquad W_{\mathrm{II}}(z, z_0) = \sum_{m=0}^{\infty} \beta_m \mathbf{D}_{2m}, \quad \text{(B.5b, c)}$$

$$W_{\mathrm{III}}(z, z_0) = \mathbf{Z}_2 * W_{\mathrm{II}}(z, z_0), \qquad W_{\mathrm{IV}}(z, z_0) = \sum_{m=0}^{\infty} \gamma_m \mathbf{D}_{2m}, \quad \text{(B.5d, e)}$$

with

$$\alpha_m = (1 + R)\zeta_m + \frac{(S - R)}{\rho_0\, \Delta z} \eta_m + S\vartheta_m, \quad \text{(B.5f)}$$

$$\beta_m = (1 + R)\eta_m - \rho_0\, \Delta z S \vartheta_m, \quad \text{(B.5g)}$$

$$\gamma_m = (1 - R)\zeta_m + \frac{(S + R)}{\rho_0\, \Delta z} \eta_m - S\vartheta_m, \quad \text{(B.5h)}$$

$$\zeta_m = \frac{1}{m!}\left(\frac{\partial^m}{\partial \kappa^m} \psi_{1,0}\right), \qquad \eta_m = \rho_0\, \Delta z \frac{1}{m!}\left(\frac{\partial^m}{\partial \kappa^m} \psi_{2,0}\right), \qquad \vartheta_m = \frac{1}{m!}\left(\frac{\partial^m}{\partial \kappa^m} \psi_{3,0}\right).$$

(B.5i, j, k)

For $m = 0, 1, \ldots$, we find

$$\zeta_0 = \cos \Phi_0, \qquad \zeta_1 = -\frac{1}{2\kappa} \Phi_0 \sin \Phi_0, \ldots \quad \text{(B.6a, b)}$$

$$\eta_0 = \rho_0\, \Delta z \frac{\sin \Phi_0}{\Phi_0}, \qquad \eta_1 = -\frac{\rho_0\, \Delta z}{2\kappa}\left[\frac{\sin \Phi_0}{\Phi_0} - \cos \Phi_0\right], \ldots \quad \text{(B.6c, d)}$$

$$\vartheta_0 = \frac{\cos \Phi_0}{\Phi_0^2} - \frac{\sin \Phi_0}{\Phi_0^3}, \qquad \vartheta_1 = -\frac{1}{2\kappa}\left[\frac{\sin \Phi_0}{\Phi_0} + \frac{3 \cos \Phi_0}{\Phi_0^2} - \frac{3 \sin \Phi_0}{\Phi_0^3}\right], \ldots.$$

(B.6e, f)

Notice that for $q = r = 0$ (B.5) is identical to operator (5.5). On the other hand, if the medium properties c and ρ are functions of z only, then extrapolation may be carried out in the wave number-frequency domain, so D_{2m} may be replaced by $\tilde{D}_2^m = (-k_x^2 - k_y^2)^m$. Now the infinite series in (B.5) can be summed to closed expressions, yielding operators (3.12b–e).

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ERRATA

WAVE FIELD EXTRAPOLATION TECHNIQUES FOR INHOMOGENEOUS MEDIA WHICH INCLUDE CRITICAL ANGLE EVENTS, PARTS I–III

by C.P.A. WAPENAAR and A.J. BERKHOUT

Part I: Methods Using the One-Way Wave Equations, vol. 33, no. 8, December 1985, pp. 1138–1159.
Page 1153, relation (7.4b): $\tilde{\mathbf{P}}^-$ should read as $\tilde{\mathscr{P}}^-$.

Part II: Methods Using the Two-Way Equation, vol. 34, no. 2, April 1986, pp. 147–179.
Page 156: relations (3.6a) and (3.6b) should read as

$$\tilde{\mathbf{W}}(z, z_0) = \mathbf{I} + \tilde{\mathbf{L}}(\tilde{\boldsymbol{\Lambda}}\ \Delta z)\tilde{\mathbf{L}}^{-1} + \tfrac{1}{2}\,\tilde{\mathbf{L}}(\tilde{\boldsymbol{\Lambda}}\ \Delta z)\tilde{\mathbf{L}}^{-1}\tilde{\mathbf{L}}(\tilde{\boldsymbol{\Lambda}}\ \Delta z)\tilde{\mathbf{L}}^{-1} + \cdots, \tag{3.6a}$$

$$\tilde{\mathbf{W}}(z, z_0) = \tilde{\mathbf{L}}[\mathbf{I} + (\tilde{\boldsymbol{\Lambda}}\ \Delta z) + \tfrac{1}{2}\,(\tilde{\boldsymbol{\Lambda}}\ \Delta z)^2 + \cdots]\tilde{\mathbf{L}}^{-1}. \tag{3.6b}$$

Part III: Applications in Modeling, Migration and Inversion, vol. 34, no. 2, April 1986, pp. 180–207.
Page 184, last line: (3.9) should read as (3.12).
Page 190–197: the symbol * should read as a superscript (denoting complex conjugation) in the following relations: (5.4), (5.6), (6.7), (6.8) and (6.9).