GEOPHYSICS, VOL. 52, NO. 2 (FEBRUARY 1987); P. 151-173, 12 FIGS.

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Principle of prestack migration based on the full elastic two-way wave equation

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ABSTRACT

The acoustic approximation in seismic migration is not allowed when the effects of wave conversion cannot be neglected, as is often the case in data with large offsets. Hence, seismic migration should ideally be founded on the full elastic wave equation, which describes compressional as well as shear waves in solid media (such as rock layers, in which shear stresses may play an important role). In order to cope with conversions between those wave types, the full elastic wave equation should be expressed in terms of the particle velocity and the traction, because these field quantities are continuous across layer boundaries where the main interaction takes place. Therefore, the full elastic wave equation should be expressed as a matrix differential equation, in which a matrix operator acts on a full wave vector which contains both the particle velocity and the traction. The solution of this equation yields another matrix operator. This full elastic two-way wave field extrapolation operator describes the relation between the total (two-way) wave fields (in terms of the particle velocity and the traction) at two different depth levels. Therefore it can be used in prestack migration to perform recursive downward extrapolation of the surface data into the subsurface (at a "traction-free" surface, the total wave field can be described in terms of the detected particle velocity and the source traction). Results from synthetic data for a simplified subsurface configuration show that a multiple-free image of the subsurface can be obtained, from which the angle-dependent *P-P* and *P-SV* reflection functions can be recovered independently.

For more complicated subsurface configurations, full elastic migration is possible in principle, but it becomes computationally complex. Nevertheless, particularly for the 3-D case, our proposal has improved the feasibility of full elastic migration significantly compared with other proposed full elastic migration or inversion schemes, because our method is carried out per shot record and per frequency component.

INTRODUCTION

Although the Earth's subsurface contains solid rock layers, in which both compressional (*P*) and shear (*S*) waves may exist, current seismic migration schemes are founded on the acoustic wave equation, which describes *P*-waves only. In spite of this simplification, seismic migration has in many cases been a successful tool for determining the internal structures of the subsurface. This success can be explained easily. Most seismic sources generate mainly *P*-waves (apart from surface waves, which do not propagate into the earth). Wave conversion takes place at the interfaces between major layers, particularly for steep incidence angles. In many seismic data acquisition configurations, however, waves with steep propagation angles are not recorded due to the short source-receiver offsets, or they are removed by editing. Therefore, the contribution of *S*-waves may often be neglected, making the acoustic approximation in seismic migration acceptable. Of course, this conclusion is even more true for poststack applications, where the stacking process will favor primary compressional energy.

In the 1970s and the early 1980s, there has been an increasing interest in large-offset data acquisition techniques because large-offset data may contain wide-angle information (angle-dependent reflection amplitudes, refraction arrivals, etc.) which is not recorded by small-offset data acquisition techniques (of course, wide-angle information is not available in poststack data). In large-offset data, the effects of wave conversion can no longer be neglected (Figures 1a, 1b and 1c); seismic migration should be based on the full elastic wave equation.

The seismic literature gives few examples of full elastic migration schemes. The reason is that practically all operational seismic migration techniques are based on the common-

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midpoint (CMP) principle, in which converted waves, if any, are suppressed by CMP stacking. Hence, full elastic migration should be carried out *before* stack, preferably by single-shot record inversion (SSRI) followed by genuine common-depth-point (CDP) stacking. The scheme should be based on the *full elastic two-way* wave equation, because it properly describes wave conversion between *P*-waves and *S*-waves (Figure 1d).

BASIC EQUATIONS

We briefly review the basic equations which govern elastic wave motion in solid media. For a rigorous discussion, refer to Achenbach (1973), Pilant (1979), and Aki and Richards (1980).

Equation of motion

For lossless, inhomogeneous solids the linearized equation of motion (generalized Newton's law) reads

$$\rho \frac{\partial^2 u_q}{\partial t^2} = \sum_{r=1}^{3} \frac{\partial \tau_{qr}}{\partial r}, \tag{1}$$

where q (or r) = 1, 2, 3 stands for x, y, z, respectively. In this equation, u_q for $q = 1, 2, 3$ represents the three components of the particle displacement vector $\mathbf{u}$, all as a function of the space coordinates (x, y, z) and time t. Furthermore, τ_{qr} for $q = 1, 2, 3$ and $r = 1, 2, 3$ represents the nine components of the symmetric stress tensor $\underline{\tau}$, also a function of space and time. The rows of this tensor represent the tractions; that is, stress τ_{qr} is the rth component of the traction τ_q acting across the plane normal to the qth axis. Finally, $\rho = \rho(x, y, z)$ describes the space-dependent mass density in equilibrium.

Stress-displacement relation

For most seismic applications anisotropy is a second-order effect compared with inhomogeneity. Assuming isotropy, the linearized stress-displacement equation (generalized Hooke's law) reads

$$\tau_{qr} = \tau_{rq} = \lambda \delta_{qr} \nabla \cdot \mathbf{u} + \mu \left(\frac{\partial u_q}{\partial r} + \frac{\partial u_r}{\partial q} \right). \tag{2}$$

In this equation δ_{qr} is the Kronecker symbol and $\lambda = \lambda(x, y, z)$ and $\mu = \mu(x, y, z)$ represent the space-dependent Lamé coefficients. They are related to the bulk compression modulus $K = K(x, y, z)$ and the shear modulus $G = G(x, y, z)$ according to $K = \lambda + 2\mu/3$ and $G = \mu$.

Acoustic approximation

The basic equations given below govern acoustic wave motion in fluid-like media. Consider a fluid as a special case of a solid ($K = \lambda$, $G = \mu = 0$) in which shear stresses cannot exist; hence $\tau_{qr} = 0$ if $q \neq r$. Furthermore, define pressures $p_{qq} = -\tau_{qq}$ and note that $p = p_{11} = p_{22} = p_{33}$ (Pascal's law). Hence, the equation of motion (1) simplifies to

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = -\nabla p, \tag{3a}$$

and the stress-displacement equation (2) simplifies to

$$p = -K \nabla \cdot \mathbf{u}. \tag{3b}$$

ACOUSTIC TWO-WAY WAVE EQUATION

For acoustic two-way wave-field extrapolation along the depth coordinate, consider a horizontally layered computational model, as shown in Figure 2. It should be emphasized that the depth levels $z_0, z_1, z_2 \cdots z_{i-1}, z_i, \cdots z_I$ generally do not coincide with the major reflecting boundaries in the subsurface. As a consequence, the medium parameters K and ρ between two depth levels may be arbitrary functions of the space coordinates (x, y, z). Our aim is to derive operators

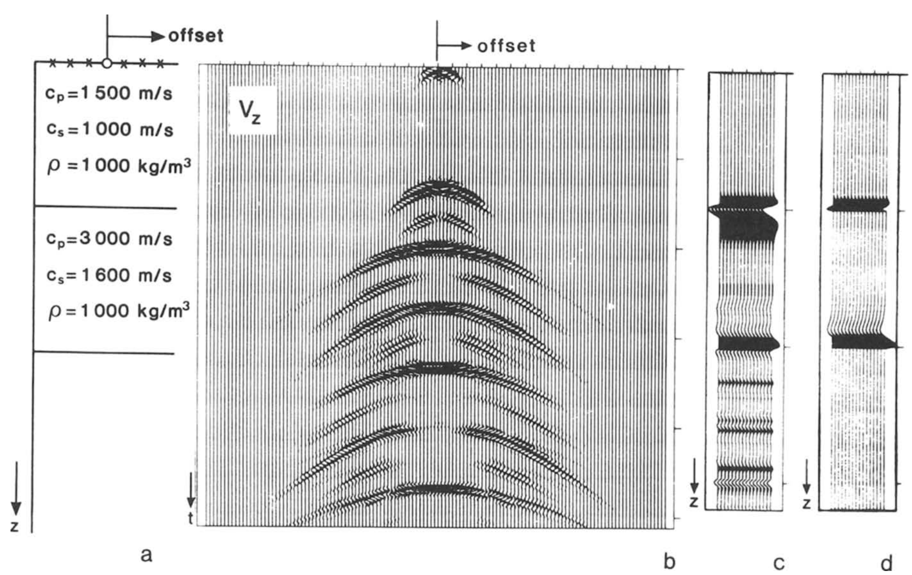

FIG. 1. In large-offset acquisition techniques the effects of wave conversion cannot be neglected. (a) Subsurface configuration. (b) Synthetic shot record, containing primary, multiply reflected, and converted waves. (c) Acoustic one-way migration result, containing various ghost-images. (d) Full elastic two-way migration result, without ghost-images.

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which extrapolate the total acoustic wave field from one depth to another. Because the total wave field is a superposition of downgoing and upgoing waves, we may also speak of *two-way wave-field* extrapolation. We choose a formulation in the frequency (ω) domain because it allows independent extrapolation of monochromatic wave fields (Berkhout, 1982). Furthermore, we define the total wave field in terms of the (monochromatic) pressure P and the (monochromatic) vertical component of the particle velocity V_z, because both quantities are continuous across the computational layer interfaces. Hence, to derive the desired two-way wave-field extrapolation operator, we first need to formulate the acoustic two-way wave equation for the wave vector $\mathbf{Q}_\ell$, defined as

$$\mathbf{Q}_\ell(x, y, z_i, \omega) = \begin{bmatrix} -P(x, y, z_i, \omega) \\ V_z(x, y, z_i, \omega) \end{bmatrix}.$$

The subscript ℓ refers to liquids (opposed to s for solids); the minus sign on P is introduced because, in the full elastic case, we consider the traction rather than the pressure. For convenience, we often denote wave fields $P(x, y, z_i, \omega)$ as $P(z_i)$ or P. A similar convention is used for operators.

In the frequency domain, differentiations with respect to time are replaced by multiplicative factors, e.g., $\mathbf{v} = \partial \mathbf{u}/\partial t$ transforms to $\mathbf{V} = i\omega \mathbf{U}$, with $i = \sqrt{-1}$. Hence, the equation of motion (3a) transforms to

$$i\omega\rho \mathbf{V} = -\nabla P, \tag{4a}$$

and the stress-displacement equation (3b) transforms to

$$i\omega P = -K \nabla \cdot \mathbf{V}. \tag{4b}$$

V_x and V_y must be eliminated from these equations because they are not in the wave vector $\mathbf{Q}_\ell$. Furthermore, we wish to express z-derivatives in terms of x- and y-derivatives, because the extrapolation takes place along the depth coordinate z. We thus obtain the following set of coupled equations:

$$\frac{\partial P}{\partial z} = -i\omega\rho V_z \tag{5a}$$

and

$$\frac{\partial V_z}{\partial z} = -\frac{i\omega}{K} P + \frac{\partial}{\partial x}\left(\frac{1}{i\omega\rho}\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{i\omega\rho}\frac{\partial P}{\partial y}\right). \tag{5b}$$

Because seismic data are always band-limited, equations (5a) and (5b) can be written in terms of lateral convolutions, according to

$$\frac{\partial P}{\partial z} = -i\omega\rho d_0 * V_z, \tag{6a}$$

and

$$\frac{\partial V_z}{\partial z} = \frac{1}{i\omega\rho} H_2 * P, \tag{6b}$$

where

$$H_2(x, y, z, \omega) = \left[k^2 d_0(x, y) + d_2(x) + d_2(y) - \frac{\partial \ln \rho}{\partial x} d_1(x) - \frac{\partial \ln \rho}{\partial y} d_1(y)\right]_z, \tag{6c}$$

$$k^2 = \omega^2/c^2, \tag{6d}$$

and

$$c^2 = K/\rho, \tag{6e}$$

where $c = c(x, y, z)$ represents the space-dependent propagation velocity. The asterisk denotes *spatial* convolutions along the x and y coordinates. The operators $d_m(x)$ and $d_m(y)$ represent *spatially invariant* band-limited spatial differentiation operators with respect to x and y, respectively, where m represents the order of differentiation; operator $d_0 = d_0(x, y)$ represents a spatial delta function:

$$d_0(x, y) = \delta(x)\delta(y). \tag{6f}$$

Parameters k^2, $\partial \ln \rho/\partial x$ and $\partial \ln \rho/\partial y$ represent *spatially variant* weighting factors which should be applied after the differentiations have been carried out.

Equations (6a) and (6b) can be rewritten as one second-order two-way wave equation, according to

$$\rho \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial P}{\partial z}\right) = -H_2 * P \tag{6g}$$

(which explains the subscript in H_2). Notice that for the homogeneous situation this relation simplifies to first-order one-way wave equations for downgoing or upgoing waves P^+ or P^-, respectively, according to

$$\frac{\partial P^\pm}{\partial z} = \mp i H_1 * P^\pm, \tag{6h}$$

with H_1 defined implicitly by $H_1 * H_1 = H_2$.

Alternatively, for the inhomogeneous situation, equations (6a) and (6b) can be rewritten as one first-order two-way wave equation for the wave vector $\mathbf{Q}_\ell = [-P, V_z]^T$, according to

$$\frac{\partial \mathbf{Q}_\ell}{\partial z} = \underline{\mathbf{A}}_\ell \mathbf{Q}_\ell, \tag{7a}$$

with

$$\underline{\mathbf{A}}_\ell = \begin{bmatrix} 0 & i\omega\rho d_0 * \\ \frac{-1}{i\omega\rho} H_2 * & 0 \end{bmatrix}. \tag{7b}$$

Equations (6g) and (7a) are equivalent. However, to derive two-way wave-field extrapolation operators, we prefer the more manageable first-order matrix equation (7a). If we define

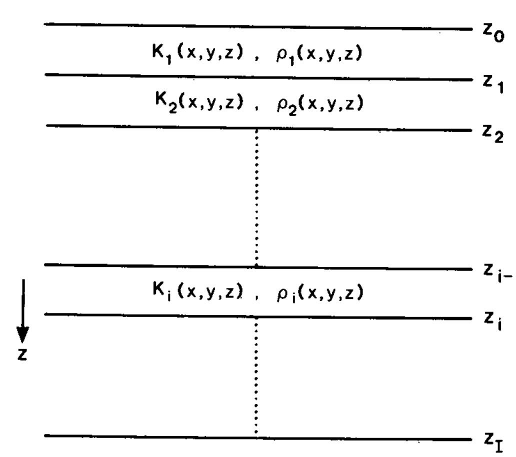

FIG. 2. A computationally convenient subsurface model for recursive acoustic two-way wave-field extrapolation in liquids.

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(for the inhomogeneous situation) operators H_1 and H_{-1} such that

$$H_1 * H_1 = H_2, \tag{8a}$$

$$H_{-1} * H_1 = \delta(x)\delta(y), \tag{8b}$$

then operator $\underset{\sim}{\mathbf{A}}_\ell$ can be expressed as

$$\underset{\sim}{\mathbf{A}}_\ell = \underset{\sim}{\mathbf{L}}_\ell \underset{\sim}{\mathbf{\Lambda}}_\ell \underset{\sim}{\mathbf{L}}_\ell^{-1}, \tag{9a}$$

with

$$\underset{\sim}{\mathbf{L}}_\ell = \begin{bmatrix} d_0 * & d_0 * \\ \dfrac{-1}{\omega\rho} H_1 * & \dfrac{1}{\omega\rho} H_1 * \end{bmatrix}, \tag{9b}$$

$$\underset{\sim}{\mathbf{\Lambda}}_\ell = \begin{bmatrix} -iH_1 * & 0 \\ 0 & iH_1 * \end{bmatrix}, \tag{9c}$$

and

$$\underset{\sim}{\mathbf{L}}_\ell^{-1} = \frac{1}{2}\begin{bmatrix} d_0 * & -\omega H_{-1} * \rho d_0 * \\ d_0 * & \omega H_{-1} * \rho d_0 * \end{bmatrix}. \tag{9d}$$

The physical meaning of these operators becomes clear if we define (for the inhomogeneous situation) a choice of downgoing and upgoing waves P^+ and P^-, respectively, such that [in accordance with the homogeneous situation; see also equation (6h)]

$$P = P^+ + P^-, \tag{10a}$$

$$\frac{\partial P}{\partial z} = \frac{\partial P^+}{\partial z} + \frac{\partial P^-}{\partial z} = -iH_1 * (P^+ - P^-). \tag{10b}$$

With these definitions and equation (5a), the following equations can be found.

$$\mathbf{Q}_\ell = \underset{\sim}{\mathbf{L}}_\ell \mathbf{P}_\ell, \tag{10c}$$

or

$$\mathbf{P}_\ell = \underset{\sim}{\mathbf{L}}_\ell^{-1} \mathbf{Q}_\ell, \tag{10d}$$

with

$$\mathbf{P}_\ell = \begin{bmatrix} -P^+ \\ -P^- \end{bmatrix}. \tag{10e}$$

Hence, operator $\underset{\sim}{\mathbf{L}}_\ell$ is a *composition* operator and $\underset{\sim}{\mathbf{L}}_\ell^{-1}$ is a *decomposition* operator. Substitution of equation (10c) into wave equation (7), using property (9), yields a set of coupled one-way wave equations, according to

$$\frac{\partial \mathbf{P}_\ell}{\partial z} = \underset{\sim}{\mathbf{B}}_\ell \mathbf{P}_\ell, \tag{11a}$$

with

$$\underset{\sim}{\mathbf{B}}_\ell = \underset{\sim}{\mathbf{\Lambda}}_\ell - \underset{\sim}{\mathbf{L}}_\ell^{-1} \frac{\partial \underset{\sim}{\mathbf{L}}_\ell}{\partial z}, \tag{11b}$$

or

$$\frac{\partial P^+}{\partial z} = -iH_1 * P^+ - \frac{1}{2} H_{-1} * \left[\rho \frac{\partial}{\partial z}\left(\frac{1}{\rho} H_1\right)\right] * (P^+ - P^-) \tag{11c}$$

and

$$\frac{\partial P^-}{\partial z} = +iH_1 * P^- + \frac{1}{2} H_{-1} * \left[\rho \frac{\partial}{\partial z}\left(\frac{1}{\rho} H_1\right)\right] * (P^+ - P^-). \tag{11d}$$

Notice that these equations fully decouple for media which are homogeneous along the z coordinate.

Thus, the acoustic two-way wave equation for inhomogeneous liquids can be described by the first-order differential matrix equation (7). By decomposing operator $\underset{\sim}{\mathbf{A}}_\ell$, given by equation (7b), we showed the close relationship between the acoustic two-way wave equation (7) and the coupled acoustic one-way wave equations (11). The physical aspects of this relationship are further discussed by Wapenaar and Berkhout (1986a). According to equation (10d), the total wave field $\mathbf{Q}_\ell$ can be decomposed into coupled downgoing waves P^+ and upgoing waves P^- by means of the decomposition operator $\underset{\sim}{\mathbf{L}}_\ell^{-1}$, given by equation (9d). This operator plays an important role in prestack modeling and migration schemes.

ACOUSTIC TWO-WAY WAVE-FIELD EXTRAPOLATION

Here we discuss the solution of the acoustic two-way wave equation. For convenience, first we consider a homogeneous medium.

For homogeneous media, the two-way wave equation in the wavenumber-frequency domain is

$$\frac{\partial \tilde{\mathbf{Q}}_\ell}{\partial z} = \tilde{\underset{\sim}{\mathbf{A}}}_\ell \tilde{\mathbf{Q}}_\ell, \tag{12a}$$

where

$$\tilde{\underset{\sim}{\mathbf{A}}}_\ell = \begin{bmatrix} 0 & i\omega\rho \\ -\tilde{H}_2/(i\omega\rho) & 0 \end{bmatrix}, \tag{12b}$$

$$\tilde{\mathbf{Q}}_\ell = \begin{bmatrix} -\tilde{P} \\ \tilde{V}_z \end{bmatrix}, \tag{12c}$$

and

$$\tilde{H}_2 = k_z^2 = k^2 - k_x^2 - k_y^2. \tag{12d}$$

The tilde above a variable refers to a spatial Fourier transformation from x to k_x and from y to k_y, where k_x and k_y represent the horizontal components of the wave vector $\mathbf{k}$. In the wavenumber-frequency domain, the differentiation operators $d_m(x)$ and $d_m(y)$ are replaced by the multiplicative factors $(-ik_x)^m$ and $(-ik_y)^m$, respectively (Berkhout, 1982). Eigenvalue decomposition applied to operator $\tilde{\underset{\sim}{\mathbf{A}}}_\ell$ yields

$$\tilde{\underset{\sim}{\mathbf{A}}}_\ell = \tilde{\underset{\sim}{\mathbf{L}}}_\ell \tilde{\underset{\sim}{\mathbf{\Lambda}}}_\ell \tilde{\underset{\sim}{\mathbf{L}}}_\ell^{-1}, \tag{13}$$

where operators $\tilde{\underset{\sim}{\mathbf{L}}}_\ell$, $\tilde{\underset{\sim}{\mathbf{\Lambda}}}_\ell$, and $\tilde{\underset{\sim}{\mathbf{L}}}_\ell^{-1}$ represent the spatial Fourier transforms of operators $\underset{\sim}{\mathbf{L}}_\ell$, $\underset{\sim}{\mathbf{\Lambda}}_\ell$, and $\underset{\sim}{\mathbf{L}}_\ell^{-1}$, given by equations (9b), (9c), and (9d). Notice that this decomposition breaks down for $\tilde{H}_1 = k_z = k \cos \theta \rightarrow 0$, that is, for waves which propagate in the horizontal direction ($\theta \rightarrow 90$ degrees). Following Ursin (1983), a solution of equation (12) can be given by

$$\tilde{\mathbf{Q}}_\ell(z) = \tilde{\underset{\sim}{\mathbf{W}}}_\ell(z, z_0)\tilde{\mathbf{Q}}_\ell(z_0), \tag{14a}$$

where (symbolically)

$$\tilde{\underset{\sim}{\mathbf{W}}}_\ell(z, z_0) = \exp(\tilde{\underset{\sim}{\mathbf{A}}}_\ell \Delta z), \tag{14b}$$

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or

$$\tilde{\mathbf{W}}_\ell(z, z_0) = \mathbf{I} + (\tilde{\mathbf{A}}_\ell\, \Delta z) + \tfrac{1}{2}(\tilde{\mathbf{A}}_\ell\, \Delta z)^2 + \cdots, \tag{14c}$$

with $\Delta z = z - z_0$. Using property (13), equation (14c) can be written as

$$\tilde{\mathbf{W}}_\ell(z, z_0) = \mathbf{I} + \tilde{\mathbf{L}}_\ell(\tilde{\mathbf{\Lambda}}_\ell\, \Delta z)\tilde{\mathbf{L}}_\ell^{-1} + \tfrac{1}{2}\tilde{\mathbf{L}}_\ell(\tilde{\mathbf{\Lambda}}_\ell\, \Delta z)\tilde{\mathbf{L}}_\ell^{-1}\tilde{\mathbf{L}}_\ell(\tilde{\mathbf{\Lambda}}_\ell\, \Delta z)\tilde{\mathbf{L}}_\ell^{-1} + \cdots, \tag{15a}$$

or

$$\tilde{\mathbf{W}}_\ell(z, z_0) = \tilde{\mathbf{L}}_\ell\left[\mathbf{I} + (\tilde{\mathbf{\Lambda}}_\ell\, \Delta z) + \tfrac{1}{2}(\tilde{\mathbf{\Lambda}}_\ell\, \Delta z)^2 + \cdots\right]\tilde{\mathbf{L}}_\ell^{-1}, \tag{15b}$$

or

$$\tilde{\mathbf{W}}_\ell(z, z_0) = \tilde{\mathbf{L}}_\ell(z)\tilde{\mathbf{V}}_\ell(z, z_0)\tilde{\mathbf{L}}_\ell^{-1}(z_0), \tag{15c}$$

with

$$\tilde{\mathbf{V}}_\ell(z, z_0) = \exp(\tilde{\mathbf{\Lambda}}_\ell\, \Delta z), \tag{15d}$$

or

$$\tilde{\mathbf{V}}_\ell(z, z_0) = \begin{bmatrix} \exp(-i\tilde{H}_1\, \Delta z) & 0 \\ 0 & \exp(i\tilde{H}_1\, \Delta z) \end{bmatrix}. \tag{15e}$$

If we define the two-way operator $\tilde{\mathbf{W}}_\ell(z, z_0)$ as

$$\tilde{\mathbf{W}}_\ell(z, z_0) = \begin{bmatrix} \tilde{W}_I(z, z_0) & \tilde{W}_{II}(z, z_0) \\ \tilde{W}_{III}(z, z_0) & \tilde{W}_{IV}(z, z_0) \end{bmatrix}, \tag{16a}$$

then expressions for the suboperators $\tilde{W}_I \ldots \tilde{W}_{IV}$ follow directly from equation (15):

$$\tilde{W}_I(z, z_0) = \cos(\tilde{H}_1\, \Delta z), \tag{16b}$$

$$\tilde{W}_{II}(z, z_0) = \frac{i\omega\rho}{\tilde{H}_1}\sin(\tilde{H}_1\, \Delta z), \tag{16c}$$

$$\tilde{W}_{III}(z, z_0) = \frac{1}{(\omega\rho)^2}\tilde{H}_2\,\tilde{W}_{II}(z, z_0), \tag{16d}$$

$$\tilde{W}_{IV}(z, z_0) = \tilde{W}_I(z, z_0), \tag{16e}$$

$$\tilde{H}_1 = \sqrt{\tilde{H}_2} = \sqrt{k^2 - k_x^2 - k_y^2}, \tag{16f}$$

and

$$\Delta z = z - z_0. \tag{16g}$$

Notice that these operators do *not* break down for $\tilde{H}_1 \to 0$.

We may conclude that, for the special case of a homogeneous layer, wave-field extrapolation can be carried out in two ways. Direct two-way wave-field extrapolation is described according to equation (14a) by

$$\tilde{\mathbf{Q}}_\ell(z_i) = \tilde{\mathbf{W}}_\ell(z_i, z_{i-1})\tilde{\mathbf{Q}}_\ell(z_{i-1}), \tag{17a}$$

with operator $\tilde{\mathbf{W}}_\ell$ defined by equation (16). Alternatively, according to equation (15c), this algorithm can be replaced by three subprocesses:

$$\tilde{\mathbf{Q}}_\ell(z_i) = \tilde{\mathbf{L}}_\ell(z_i)\tilde{\mathbf{V}}_\ell(z_i, z_{i-1})\tilde{\mathbf{L}}_\ell^{-1}(z_{i-1})\tilde{\mathbf{Q}}_\ell(z_{i-1}), \tag{17b}$$

where, from right to left, $\tilde{\mathbf{L}}_\ell^{-1}$ describes decomposition of the total wave field into downgoing and upgoing waves, $\tilde{\mathbf{V}}_\ell$ describes independent one-way wave-field extrapolation of downgoing and upgoing waves, and $\tilde{\mathbf{L}}_\ell$ describes composition of the total wave field from its downgoing and upgoing constituents. Both algorithms (17a) and (17b) are illustrated in Figure 3.

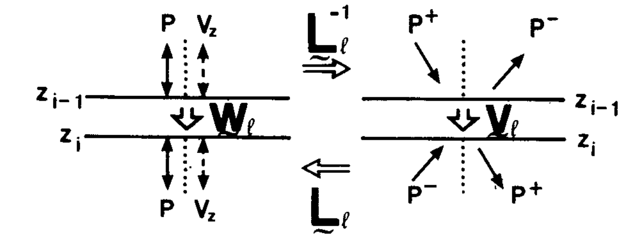

FIG. 3. Diagram showing the relationship between acoustic two-way and acoustic one-way wave-field extrapolation.

In order to derive a finite-difference formulation for two-way wave-field extrapolation in the space-frequency domain, we may expand either equation (17a) or equation (17b) as a Taylor series and compute the inverse spatial Fourier transform. Although the closed expressions for equations (17a) and (17b) are identical, the convergence speeds of their respective series expansions differ significantly. The direct two-way formulation [equation (17a)] is preferred because its series expansion contains integer powers of operator $\tilde{H}_2$ only; hence, expansion of the square-root operator $\tilde{H}_1 = \sqrt{\tilde{H}_2}$ (typical for the one-way approach) is avoided. Appendix A shows that suboperators $\tilde{W}_I$ and $\tilde{W}_{II}$, given by equations (16b) and (16c), can be expanded as

$$\tilde{W}_I(z, z_0) = \sum_{m=0}^{\infty} \alpha_m \tilde{D}_2^m, \tag{18a}$$

and

$$\tilde{W}_{II}(z, z_0) = \sum_{m=0}^{\infty} \beta_m \tilde{D}_2^m, \tag{18b}$$

with

$$\tilde{D}_2 = (-k_x^2 - k_y^2) \tag{18c}$$

and where α_m and β_m are given by equations (A-5f) and (A-5g) for all m. For $m = 0, 1$, etc.,

$$\alpha_0 = \cos k\Delta z, \tag{18d}$$

$$\alpha_1 = -\frac{\Delta z}{2k}\sin k\Delta z, \tag{18e}$$

etc., and

$$\beta_0 = \frac{i\omega\rho}{k}\sin k\Delta z, \tag{18f}$$

$$\beta_1 = -\frac{i\omega\rho}{2k^3}(\sin k\Delta z - k\Delta z\cos k\Delta z), \tag{18g}$$

etc.

The band-limited inverse spatial Fourier transform of operator $\tilde{D}_2 = (-k_x^2 - k_y^2)$ is given by the finite-difference spatial differentiation operator $D_2 = d_2(x) + d_2(y)$ (see also Figure 4; the band limits correspond to the spatial Nyquist frequencies). Hence, the band-limited inverse spatial Fourier transform of equation (17a) reads

$$\mathbf{Q}_\ell(z) = \mathbf{W}_\ell(z, z_0)\mathbf{Q}_\ell(z_0), \tag{19a}$$

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where

$$\mathbf{W}_\ell(z, z_0) = \begin{bmatrix} W_{\mathrm{I}}(z, z_0)* & W_{\mathrm{II}}(z, z_0)* \\ W_{\mathrm{III}}(z, z_0)* & W_{\mathrm{IV}}(z, z_0)* \end{bmatrix}, \tag{19b}$$

$$W_{\mathrm{I}}(z, z_0) = \sum_{m=0}^{\infty} \alpha_m D_{2m}, \tag{19c}$$

$$W_{\mathrm{II}}(z, z_0) = \sum_{m=0}^{\infty} \beta_m D_{2m}, \tag{19d}$$

$$W_{\mathrm{III}}(z, z_0) = \frac{1}{(\omega\rho)^2} H_2 * W_{\mathrm{II}}(z, z_0), \tag{19e}$$

$$W_{\mathrm{IV}}(z, z_0) = W_{\mathrm{I}}(z, z_0), \tag{19f}$$

and where D_{2m} is defined recursively according to $D_{2m} = D_2 * D_{2m-2}$.

In this section we derived an explicit finite-difference two-way wave-field extrapolation operator in the space-frequency domain. Notice that for 2-D as well as 3-D applications, the operator is based on 1-D convolutions only. The only approximation we have made concerns the band limitation of the differentiation operators; hence, for band-limited seismic data the operator is exact. For practical implementation the infinite series expansions must be truncated. Wapenaar and Berkhout (1986a) discuss a recursive finite-difference scheme based on operator (19b) and show that the *first-order* approximation is accurate and stable for all propagation angles up to 90 degrees (for $k\Delta z \leq \pi/2$). The excellent convergence properties are due to the formulation of two-way wave equation (7), which is based on the explicit operator H_2. In contrast, conventional finite-difference schemes, based on one-way wave equations (11), suffer from the implicit character of the square-root operator H_1. Because we derived algorithm (19) in the wavenumber-frequency domain, it is valid for homogeneous media. However, it can be shown (Wapenaar and Berkhout, 1986a) that this algorithm may also be applied to inhomogeneous media, using space-dependent coefficients α_m and β_m based on $K(x, y, z)$ and $\rho(x, y, z)$. In this case it must be assumed that the derivatives of the medium parameters may be neglected.

For recursive two-way wave-field extrapolation in inhomogeneous media, based on (19), we propose to use the computationally convenient subsurface model shown in Figure 2. Notice that at the layer boundaries the boundary conditions are automatically fulfilled, because the total wave field $\mathbf{Q}_\ell$ is continuous for all depths. This means that transmission effects and multiple reflections are properly incorporated.

FULL ELASTIC TWO-WAY WAVE EQUATION

For full elastic two-way wave-field extrapolation along the depth coordinate, we consider a horizontally layered computational model, as shown in Figure 5. The medium parameters λ, μ, and ρ between two depth levels may be arbitrary functions of the space coordinates (x, y, z). Our aim is to derive operators that extrapolate the *total* full elastic wave field from one depth level to another. In the frequency domain, we define the total wave field in terms of the (monochromatic) traction $\mathbf{T}_z = [T_{zx}, T_{zy}, T_{zz}]^T$ and the (monochromatic) particle velocity $\mathbf{V} = [V_x, V_y, V_z]^T$, because both quantities are continuous across the computational layer interfaces. Hence, in order to derive the desired two-way wave-field extrapolation operator,

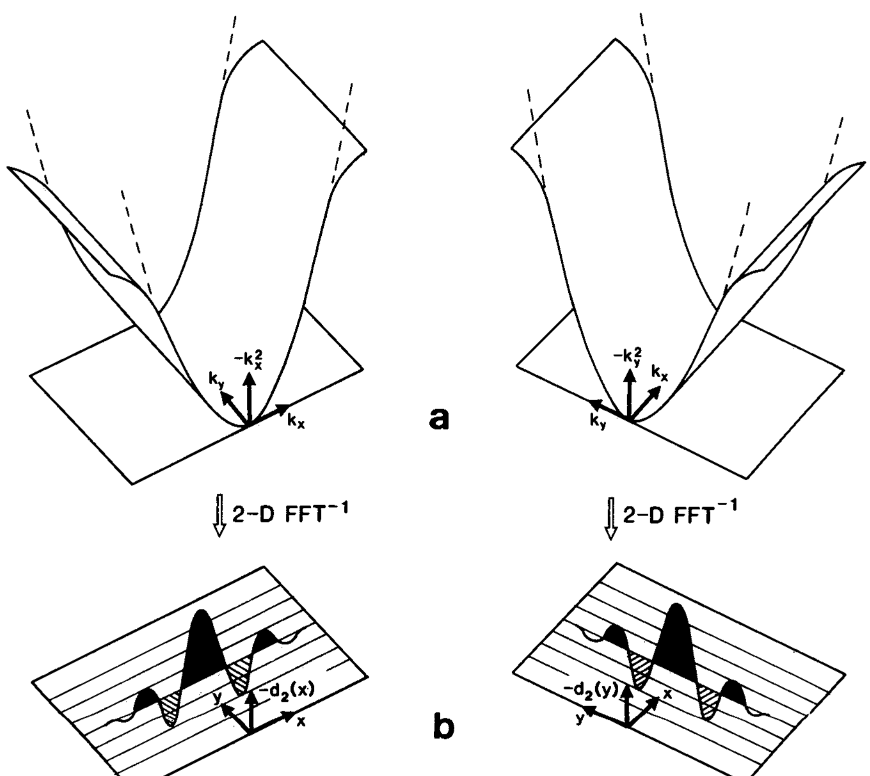

FIG. 4. Spatially band-limited differentiation operators. (a) Representation in the wavenumber domain. (b) Representation in the space domain.

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we first need to formulate the full elastic two-way wave equation for the wave vector $\mathbf{Q}_s$, defined as

$$\mathbf{Q}_s(x, y, z_i, \omega) = \begin{bmatrix} V_z(x, y, z_i, \omega) \\ T_{zx}(x, y, z_i, \omega) \\ T_{zy}(x, y, z_i, \omega) \\ T_{zz}(x, y, z_i, \omega) \\ V_x(x, y, z_i, \omega) \\ V_y(x, y, z_i, \omega) \end{bmatrix},$$

where subscript s refers to solids.

In the frequency domain the equation of motion (1) transforms to

$$i\omega\rho V_q = \sum_{r=1}^{3} \frac{\partial T_{qr}}{\partial r} \tag{20}$$

and the stress-displacement equation (2) transforms to

$$i\omega T_{qr} = i\omega T_{rq} = \lambda\delta_{qr}\nabla \cdot \mathbf{V} + \mu\left(\frac{\partial V_q}{\partial r} + \frac{\partial V_r}{\partial q}\right), \tag{21}$$

where q (or r) = 1, 2, 3 stands for x, y, z, respectively. T_{xx}, T_{xy}, T_{xz}, T_{yx}, T_{yy}, and T_{yz} must be eliminated because they are not present in the wave vector $\mathbf{Q}_s$. For the general 3-D case the result is given in Appendix B. In the following, we consider only 2-D wave propagation in 2-D inhomogeneous media, that is, we assume that $\lambda = \lambda(x, z)$, $\mu = \mu(x, z)$, and $\rho = \rho(x, z)$.

We further assume that the normals to the wavefronts lie in the vertical x, z plane. Under these assumptions, S-waves with horizontal polarization (SH-waves, with $V_x = V_z = 0$ and $T_{zx} = T_{zz} = 0$) propagate independently from P-waves and S-waves with polarization in the vertical x, z plane (P- and SV-waves, with $V_y = 0$ and $T_{zy} = 0$). We only consider waves with polarization in the vertical plane (the solution for waves with horizontal polarization is very similar to the solution for acoustic waves). Hence, the wave vector $\mathbf{Q}_s$ simplifies to

$$\mathbf{Q}_s(x, z_i, \omega) = \begin{bmatrix} V_z(x, z_i, \omega) \\ T_{zx}(x, z_i, \omega) \\ T_{zz}(x, z_i, \omega) \\ V_x(x, z_i, \omega) \end{bmatrix}.$$

From the set of coupled equations (B-1) we can derive the following two-way wave equation:

$$\frac{\partial \mathbf{Q}_s}{\partial z} = \underset{\sim}{\mathbf{A}}_s \mathbf{Q}_s, \tag{22}$$

where 4×4 matrix operator $\underset{\sim}{\mathbf{A}}_s$ is given in equation (B-2). (Note that this equation cannot be applied in liquids because operator $\underset{\sim}{\mathbf{A}}_s$ is singular for $\mu = 0$. Furthermore, in the derivation it was assumed for simplicity that the lateral derivatives of the medium parameters may be neglected.) Similar to the acoustic case, the operator $\underset{\sim}{\mathbf{A}}_s$ can be decomposed according to

$$\underset{\sim}{\mathbf{A}}_s = \underset{\sim}{\mathbf{L}}_s \underset{\sim}{\mathbf{\Lambda}}_s \underset{\sim}{\mathbf{L}}_s^{-1}, \tag{23}$$

with operators $\underset{\sim}{\mathbf{L}}_s$, $\underset{\sim}{\mathbf{\Lambda}}_s$, and $\underset{\sim}{\mathbf{L}}_s^{-1}$ given by equations (B-3b), (B-3c), and (B-3d). To discuss the physical meaning of these operators, we now define the Lamé potentials Φ and $\mathbf{\Psi}$ for the particle velocity:

$$\mathbf{V} = \nabla\Phi + \nabla \times \mathbf{\Psi}, \tag{24}$$

where Φ represents the scalar potential for dilatational or compressional waves and where $\mathbf{\Psi}$ represents the vector potential for distortional or shear waves. In the general inhomogeneous case, the potentials Φ and $\mathbf{\Psi}$ are coupled.

With the 2-D assumptions made before, the vector potential $\mathbf{\Psi} = [\Psi_x, \Psi_y, \Psi_z]^T$ can be chosen such that $\Psi_x = \Psi_z = 0$. We define $\Psi = \Psi_y$. Furthermore we define a choice of potentials for downgoing and upgoing P-waves, Φ^+ and Φ^-, as well as for downgoing and upgoing SV-waves, Ψ^+ and Ψ^-, such that [in accordance with the acoustic situation, see also equations (10a) and (10b))]

$$\Phi = \Phi^+ + \Phi^-, \tag{25a}$$

$$\frac{\partial \Phi}{\partial z} = -iH_1^{(P)} * [\Phi^+ - \Phi^-], \tag{25b}$$

$$\Psi = \Psi^+ + \Psi^-, \tag{25c}$$

$$\frac{\partial \Psi}{\partial z} = -iH_1^{(SV)} * [\Psi^+ - \Psi^-], \tag{25d}$$

with $H_1^{(P)}$ and $H_1^{(SV)}$ defined by equations (B-3l)–(B-3p). With the above definitions and equation (21), the following equations can be found:

$$\mathbf{Q}_s = \underset{\sim}{\mathbf{L}}_s \mathbf{P}_s, \tag{26a}$$

or

$$\mathbf{P}_s = \underset{\sim}{\mathbf{L}}_s^{-1} \mathbf{Q}_s, \tag{26b}$$

with

$$\mathbf{P}_s = \begin{bmatrix} \Phi^+ \\ \Psi^+ \\ -\Phi^- \\ \Psi^- \end{bmatrix}. \tag{26c}$$

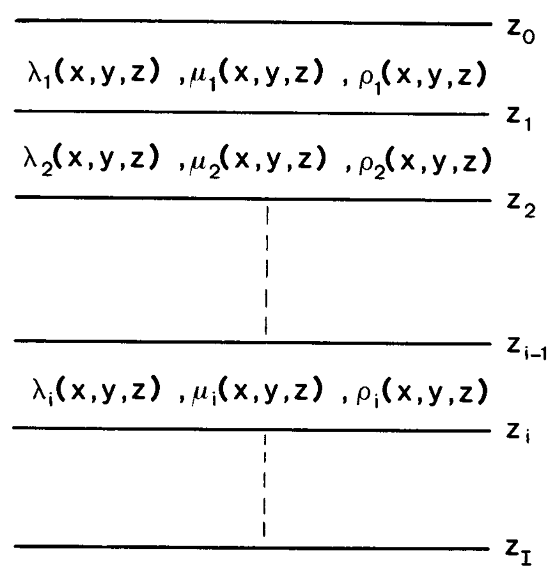

FIG. 5. A computationally convenient subsurface model for recursive full elastic two-way wave-field extrapolation in solids. The prestack migration scheme, described in a following section, is based on this model.

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Hence, operator $\mathbf{L}_s$ is a *composition* operator and $\mathbf{L}_s^{-1}$ is a *decomposition* operator. Substitution of equation (26a) into two-way wave equation (22), using property (23), yields a set of coupled one-way wave equations, according to

$$\frac{\partial \mathbf{P}_s}{\partial z} = \mathbf{B}_s \mathbf{P}_s, \tag{27a}$$

with

$$\mathbf{B}_s = \mathbf{\Lambda}_s - \mathbf{L}_s^{-1} \frac{\partial \mathbf{L}_s}{\partial z}. \tag{27b}$$

Notice that these equations fully decouple for homogeneous media.

The full elastic two-way wave equation for inhomogeneous solids can be described by the first-order matrix differential equation (22). By decomposing operator $\mathbf{A}_s$, given in equation (B-2), we showed the close relationship between the full elastic two-way equation (22) and the coupled full elastic one-way wave equations (27). According to equation (26b), the total wave field $\mathbf{Q}_s$ can be decomposed into coupled downgoing and upgoing P- and SV-waves by means of decomposition operator $\mathbf{L}_s^{-1}$, given by equation (B-3d). This operator plays an important role in full elastic prestack modeling and migration schemes.

FULL ELASTIC TWO-WAY WAVE-FIELD EXTRAPOLATION

Here we discuss the solution of the full elastic two-way wave equation. For convenience we consider 2-D waves in a homogeneous medium. Also we discuss the extension of the solution for inhomogeneous media.

For homogeneous media we may consider the full elastic two-way wave equation in the wavenumber-frequency domain

$$\frac{\partial \tilde{\mathbf{Q}}_s}{\partial z} = \tilde{\mathbf{A}}_s \tilde{\mathbf{Q}}_s, \tag{28a}$$

where

$$\tilde{\mathbf{Q}}_s = \begin{bmatrix} \tilde{V}_z \\ \tilde{T}_{zx} \\ \tilde{T}_{zz} \\ \tilde{V}_x \end{bmatrix}, \tag{28b}$$

and where operator $\tilde{\mathbf{A}}_s$ represents the spatial Fourier transform of operator $\mathbf{A}_s$, given in equation (B-2). Eigenvalue decomposition applied to operator $\tilde{\mathbf{A}}_s$ yields

$$\tilde{\mathbf{A}}_s = \tilde{\mathbf{L}}_s \tilde{\mathbf{\Lambda}}_s \tilde{\mathbf{L}}_s^{-1}, \tag{29}$$

where operators $\tilde{\mathbf{L}}_s$, $\tilde{\mathbf{\Lambda}}_s$, and $\tilde{\mathbf{L}}_s^{-1}$ represent the spatial Fourier transforms of operators $\mathbf{L}_s$, $\mathbf{\Lambda}_s$, and $\mathbf{L}_s^{-1}$, given by equations (B-3b), (B-3c), and (B-3d). Notice that this decomposition breaks down for $\tilde{H}_1^{(P)} \to 0$ and for $\tilde{H}_1^{(SV)} \to 0$, that is, for P- and SV-waves which propagate in the horizontal direction. Similar to the acoustic case, a solution of equation (28) can be given by

$$\tilde{\mathbf{Q}}_s(z) = \tilde{\mathbf{W}}_s(z, z_0)\tilde{\mathbf{Q}}_s(z_0), \tag{30a}$$

where (symbolically)

$$\tilde{\mathbf{W}}_s(z, z_0) = \exp(\tilde{\mathbf{A}}_s \Delta z),\ \Delta z = z - z_0, \tag{30b}$$

or, using series expansion,

$$\tilde{\mathbf{W}}_s(z, z_0) = \tilde{\mathbf{L}}_s(z)\tilde{\mathbf{V}}_s(z, z_0)\tilde{\mathbf{L}}_s^{-1}(z_0), \tag{30c}$$

with

$$\tilde{\mathbf{V}}_s(z, z_0) = \exp(\tilde{\mathbf{\Lambda}}_s \Delta z), \tag{30d}$$

or

$$\tilde{\mathbf{V}}_s(z, z_0) = \begin{bmatrix} \exp(-i\tilde{H}_1^{(P)}\Delta z) & 0 & 0 & 0 \\ 0 & \exp(-i\tilde{H}_1^{(SV)}\Delta z) & 0 & 0 \\ 0 & 0 & \exp(i\tilde{H}_1^{(P)}\Delta z) & 0 \\ 0 & 0 & 0 & \exp(i\tilde{H}_1^{(SV)}\Delta z) \end{bmatrix}. \tag{30e}$$

If we define full-elastic two-way operator $\tilde{\mathbf{W}}_s(z, z_0)$ as

$$\tilde{\mathbf{W}}_s(z, z_0) = \begin{bmatrix} \tilde{\mathbf{W}}_{\mathrm{I}}(z, z_0) & \tilde{\mathbf{W}}_{\mathrm{II}}(z, z_0) \\ \tilde{\mathbf{W}}_{\mathrm{III}}(z, z_0) & \tilde{\mathbf{W}}_{\mathrm{IV}}(z, z_0) \end{bmatrix}, \tag{31a}$$

then expressions for the submatrices $\tilde{\mathbf{W}}_{\mathrm{I}} \cdots \tilde{\mathbf{W}}_{\mathrm{IV}}$ follow directly from equation (30) and Appendix B:

$$\tilde{\mathbf{W}}_{\mathrm{I}}(z, z_0) = \tilde{\mathbf{L}}_1(\cosh \tilde{\mathbf{\Lambda}}\Delta z)\tilde{\mathbf{L}}_1^{-1}, \tag{31b}$$

$$\tilde{\mathbf{W}}_{\mathrm{II}}(z, z_0) = \tilde{\mathbf{L}}_1(\sinh \tilde{\mathbf{\Lambda}}\Delta z)\tilde{\mathbf{L}}_2^{-1}, \tag{31c}$$

$$\tilde{\mathbf{W}}_{\mathrm{III}}(z, z_0) = \tilde{\mathbf{L}}_2(\sinh \tilde{\mathbf{\Lambda}}\Delta z)\tilde{\mathbf{L}}_1^{-1}, \tag{31d}$$

$$\tilde{\mathbf{W}}_{\mathrm{IV}}(z, z_0) = \tilde{\mathbf{L}}_2(\cosh \tilde{\mathbf{\Lambda}}\Delta z)\tilde{\mathbf{L}}_2^{-1}, \tag{31e}$$

where operators $\tilde{\mathbf{L}}_1$, $\tilde{\mathbf{L}}_1^{-1}$, $\tilde{\mathbf{L}}_2$, $\tilde{\mathbf{L}}_2^{-1}$, and $\tilde{\mathbf{\Lambda}}$ are given by the spatial Fourier transforms of operators $\mathbf{L}_1$, $\mathbf{L}_1^{-1}$, $\mathbf{L}_2$, $\mathbf{L}_2^{-1}$, and $\mathbf{\Lambda}$, given by equations (B-3e)–(B-3i). Finally, if we define operator $\tilde{\mathbf{W}}_s(z, z_0)$ as

$$\tilde{\mathbf{W}}_s(z, z_0) = \begin{bmatrix} \tilde{W}_{11}(z, z_0) & \tilde{W}_{12}(z, z_0) & \tilde{W}_{13}(z, z_0) & \tilde{W}_{14}(z, z_0) \\ \tilde{W}_{21}(z, z_0) & \tilde{W}_{22}(z, z_0) & \tilde{W}_{23}(z, z_0) & \tilde{W}_{24}(z, z_0) \\ \tilde{W}_{31}(z, z_0) & \tilde{W}_{32}(z, z_0) & \tilde{W}_{33}(z, z_0) & \tilde{W}_{34}(z, z_0) \\ \tilde{W}_{41}(z, z_0) & \tilde{W}_{42}(z, z_0) & \tilde{W}_{43}(z, z_0) & \tilde{W}_{44}(z, z_0) \end{bmatrix}, \tag{32a}$$

then expressions for the suboperators $W_{11} \cdots W_{44}$ follow directly from equation (31) and Appendix B:

$$\tilde{W}_{11}(z, z_0) = (1 - 2k_x^2/k_{SV}^2)\tilde{W}_1^{(P)}(z, z_0) + (2k_x^2/k_{SV}^2)\tilde{W}_1^{(SV)}(z, z_0), \tag{32b}$$

$$\tilde{W}_{12}(z, z_0) = \Big[-k_x/(\omega\rho)\Big]\tilde{W}_1^{(P)}(z, z_0) + \Big[k_x/(\omega\rho)\Big]\tilde{W}_1^{(SV)}(z, z_0), \tag{32c}$$

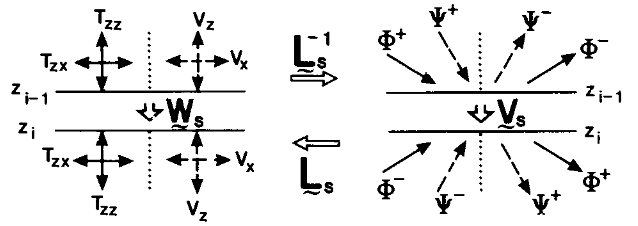

FIG. 6. Diagram showing the relationship between full elastic two-way and full elastic one-way wave-field extrapolation.

etc., where

$$\tilde{W}_{\mathrm{I}}^{(P)}(z, z_0) = \cos(\tilde{H}_{\mathrm{I}}^{(P)}\Delta z), \tag{32d}$$

$$\tilde{W}_{\mathrm{I}}^{(SV)}(z, z_0) = \cos(\tilde{H}_{\mathrm{I}}^{(SV)}\Delta z), \tag{32e}$$

etc., and

$$\tilde{H}_{\mathrm{I}}^{(P)} = \sqrt{\tilde{H}_2^{(P)}} = \sqrt{k_P^2 - k_x^2}, \tag{32f}$$

$$k_P^2 = \omega^2/c_P^2 = \omega^2\rho/(\lambda + 2\mu) \tag{32g}$$

$$\tilde{H}_{\mathrm{I}}^{(SV)} = \sqrt{\tilde{H}_2^{(SV)}} = \sqrt{k_{SV}^2 - k_x^2}, \tag{32h}$$

$$k_{SV}^2 = \omega^2/c_{SV}^2 = \omega^2\rho/\mu. \tag{32i}$$

Operator $\tilde{\mathbf{W}}_s$, given by equation (32), describes full elastic two-way wave-field extrapolation in the wavenumber-frequency domain. Notice that these operators do *not* break down for $\tilde{H}_{\mathrm{I}}^{(P)} \to 0$ and $\tilde{H}_{\mathrm{I}}^{(SV)} \to 0$.

We may conclude that, for the special case of a homogeneous layer, full elastic wave-field extrapolation can be carried out in two ways. Direct two-way wave-field extrapolation is described according to equation (30a) by

$$\tilde{\mathbf{Q}}_s(z_i) = \tilde{\mathbf{W}}_s(z_i, z_{i-1})\tilde{\mathbf{Q}}_s(z_{i-1}), \tag{33a}$$

with operator $\tilde{\mathbf{W}}_s$ defined by equation (32). Alternatively, according to equation (30c), this algorithm can be replaced by three subprocesses:

$$\tilde{\mathbf{Q}}_s(z_i) = \tilde{\mathbf{L}}_s(z_i)\tilde{\mathbf{V}}_s(z_i, z_{i-1})\tilde{\mathbf{L}}_s^{-1}(z_{i-1})\tilde{\mathbf{Q}}_s(z_{i-1}). \tag{33b}$$

These algorithms were already discussed for the acoustic case. For the full elastic case, algorithms (33a) and (33b) are shown in Figure 6.

As for the acoustic case, we derive a finite-difference formulation for full elastic two-way wave-field extrapolation in the space-frequency domain, based on the direct two-way formulation (33a). Therefore all suboperators $\tilde{W}_{11} \cdots \tilde{W}_{44}$ given in equation (32) need to be expanded as Taylor series and their inverse spatial Fourier transforms need to be computed. Fortunately, all suboperators $\tilde{W}_{11} \cdots \tilde{W}_{44}$ can be written as linear combinations of operators $\tilde{W}_{\mathrm{I}}^{(P)} \cdots \tilde{W}_{\mathrm{III}}^{(P)}$ and $\tilde{W}_{\mathrm{I}}^{(SV)} \cdots \tilde{W}_{\mathrm{III}}^{(SV)}$. These operators represent the 2-D versions ($k_y^2 = 0$) of the acoustic suboperators $\tilde{W}_{\mathrm{I}} \cdots \tilde{W}_{\mathrm{III}}$, given by equations (16b)–(16d), with k replaced by k_P or k_{SV}. Hence, if we define the band-limited inverse spatial Fourier transform of equation (33a) as

$$\mathbf{Q}_s(z) = \mathbf{W}_s(z, z_0)\mathbf{Q}_s(z_0), \tag{34a}$$

where

$$\mathbf{W}_s(z, z_0) = \begin{bmatrix} W_{11}(z, z_0)* & W_{12}(z, z_0)* & W_{13}(z, z_0)* & W_{14}(z, z_0)* \\ W_{21}(z, z_0)* & W_{22}(z, z_0)* & W_{23}(z, z_0)* & W_{24}(z, z_0)* \\ W_{31}(z, z_0)* & W_{32}(z, z_0)* & W_{33}(z, z_0)* & W_{34}(z, z_0)* \\ W_{41}(z, z_0)* & W_{42}(z, z_0)* & W_{43}(z, z_0)* & W_{44}(z, z_0)* \end{bmatrix}, \tag{34b}$$

then all suboperators $W_{11} \cdots W_{44}$ can be written as linear combinations of the band-limited inverse spatial Fourier transforms of operators $\tilde{W}_{\mathrm{I}}^{(P)} \cdots \tilde{W}_{\mathrm{III}}^{(P)}$ and $\tilde{W}_{\mathrm{I}}^{(SV)} \cdots \tilde{W}_{\mathrm{III}}^{(SV)}$, according to

$$W_{11}(z, z_0) = \left[d_0(x) + 2k_{SV}^{-2}d_2(x)\right] * W_{\mathrm{I}}^{(P)}(z, z_0) - \left[2k_{SV}^{-2}d_2(x)\right] * W_{\mathrm{I}}^{(SV)}(z, z_0), \tag{34c}$$

$$W_{12}(z, z_0) = \left[(i\omega\rho)^{-1}d_1(x)\right] * W_{\mathrm{I}}^{(P)}(z, z_0) - \left[(i\omega\rho)^{-1}d_1(x)\right] * W_{\mathrm{I}}^{(SV)}(z, z_0), \tag{34d}$$

etc. Here the suboperators $W_{\mathrm{I}}^{(P)} \cdots W_{\mathrm{III}}^{(P)}$ and $W_{\mathrm{I}}^{(SV)} \cdots W_{\mathrm{III}}^{(SV)}$ represent the 2-D versions $[d_2(y) = 0]$ of the acoustic suboperators $W_{\mathrm{I}} \cdots W_{\mathrm{III}}$, given by equations (19c)–(19e) with k replaced by k_P or k_{SV}.

Consequently, a recursive explicit finite-difference extrapolation scheme can be designed, based on full elastic two-way operator (34b), which has the same convergence properties as the previous finite-difference scheme, based on acoustic two-way operator (19b). Hence, the first-order approximation is accurate and stable for all propagation angles up to 90 degrees (for $k^{(SV)}\Delta z \le \pi/2$). Because we derived algorithm (34) in the wavenumber-frequency domain, it is valid for homogeneous media. However, for the acoustic case operators $W_{\mathrm{I}} \cdots W_{\mathrm{III}}$, given by equations (19c)–(19e), may be applied to inhomogeneous media as well if the derivatives of the medium parameters may be neglected. Because full elastic algorithm (34) is based on the same operators, it may also be applied to inhomogeneous media, using spatially dependent coefficients based on the medium parameters $\lambda(x, z)$, $\mu(x, z)$, and $\rho(x, z)$. Of course, it must also be assumed in this case that the derivatives of the medium parameters may be neglected. In conclusion, for recursive full elastic two-way wave-field extrapolation in inhomogeneous media, based on equation (34), we propose the (2-D version of) the computationally convenient subsurface model, shown in Figure 5. Notice that at the layer boundaries the boundary conditions are automatically fulfilled, because the total wave field $\mathbf{Q}_s$ is continuous for all depths. This means that transmission effects, multiple reflections, and wave conversions are properly incorporated.

PRESTACK MODELING SCHEME BASED ON THE FULL ELASTIC TWO-WAY WAVE EQUATION

We have derived a relatively simple, manageable matrix formalism which describes the full elastic wave propagation effects between two depth levels. This is schematically repre-

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sented by

$$\begin{bmatrix} V_z \\ T_{zx} \\ T_{zz} \\ V_x \end{bmatrix}_{z_{i-1}} \rightarrow \mathbf{W}_s \rightarrow \begin{bmatrix} V_z \\ T_{zx} \\ T_{zz} \\ V_x \end{bmatrix}_{z_i}.$$

Because migration is the main subject of this paper, we discuss the theory of two-way modeling only for simple subsurface geometries. It is obvious that the above relation is not particularly suited for modeling purposes because it requires knowledge of the total wave field at a specific depth *before* modeling. Here we show how this paradox can be solved for 1-D inhomogeneous media. Therefore we consider a horizontally layered medium consisting of I homogeneous layers, as shown in Figure 7. We assume homogeneous half-spaces for $z < z_0$ and $z \geq z_I$. In layer i, with $z_{i-1} \leq z < z_i$, the medium parameters are given by the Lamé constants λ_i, μ_i and the mass density ρ_i. As long as no boundary conditions are specified, then within each solid layer the total wave field $\mathbf{Q}_s = [V_z, T_{zx}, T_{zz}, V_x]^T$ can be expressed as a linear combination of four arbitrary, linearly independent basic wave fields $\mathbf{Q}_s^{(1)}$, $\mathbf{Q}_s^{(2)}$, $\mathbf{Q}_s^{(3)}$, and $\mathbf{Q}_s^{(4)}$. Due to the radiation condition at $z = z_I$ (downgoing waves only), the number of linearly independent basic wave fields reduces to two ($\mathbf{Q}_s^{(1)}$ and $\mathbf{Q}_s^{(2)}$).

In the wavenumber-frequency domain (k_x, ω), modeling of P- and SV-waves consists of the following steps.

Wave-field extrapolation

Given the basic wave fields at z_i, the basic wave fields at z_{i-1} can be found by applying the full elastic two-way wave-field extrapolation operator for layer i, according to

$$\tilde{\mathbf{Q}}_s^{(1)}(z_{i-1}) = \tilde{\mathbf{W}}_s(z_{i-1}, z_i)\tilde{\mathbf{Q}}_s^{(1)}(z_i) \tag{35a}$$

and

$$\tilde{\mathbf{Q}}_s^{(2)}(z_{i-1}) = \tilde{\mathbf{W}}_s(z_{i-1}, z_i)\tilde{\mathbf{Q}}_s^{(2)}(z_i), \tag{35b}$$

with extrapolation operator $\tilde{\mathbf{W}}_s$ given by equation (32).

This step, which describes the total modeling procedure for layer i, can be applied recursively to model a sequence of I layers. Note that the boundary conditions are automatically fullfilled because $\tilde{\mathbf{Q}}_s^{(1)}$ and $\tilde{\mathbf{Q}}_s^{(2)}$ are continuous across the layer interfaces.

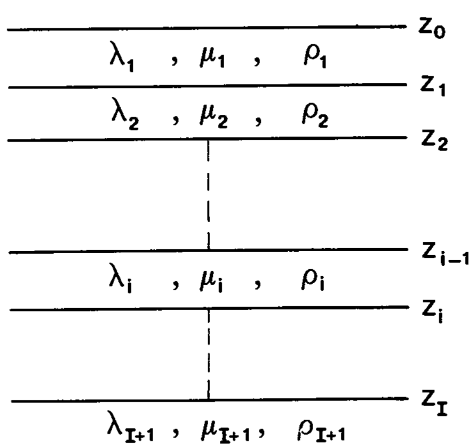

FIG. 7. Subsurface model for the modeling scheme based on the full elastic two-way wave equation.

Specification of the radiation condition

The procedure (for each k_x and ω value) starts at $z = z_I$ by specifying the basic wave fields, assuming only downgoing waves in the homogeneous lower half-space $z \geq z_I$. We define

$$\tilde{\mathbf{Q}}_s^{(1)}(z_I) = \tilde{\mathbf{L}}_s(z_I)\tilde{\mathbf{P}}_s^{(1)}(z_I), \tag{36a}$$

$$\tilde{\mathbf{Q}}_s^{(2)}(z_I) = \tilde{\mathbf{L}}_s(z_I)\tilde{\mathbf{P}}_s^{(2)}(z_I), \tag{36b}$$

with

$$\tilde{\mathbf{P}}_s^{(1)}(z_I) = \left[\tilde{\Phi}^+ = 1, \tilde{\Psi}^+ = 0, -\tilde{\Phi}^- = 0, \tilde{\Psi}^- = 0\right]^T, \tag{36c}$$

$$\tilde{\mathbf{P}}_s^{(2)}(z_I) = \left[\tilde{\Phi}^+ = 0, \tilde{\Psi}^+ = 1, -\tilde{\Phi}^- = 0, \tilde{\Psi}^- = 0\right]^T, \tag{36d}$$

$\tilde{\mathbf{L}}_s$ being the spatial Fourier transform of $\mathbf{L}_s$, given by equation (B-3b).

Computation of the impulse response

When the surface $z = z_0$ has been reached, then the plane-wave impulse response of the subsurface can be calculated. We consider three cases:

(1) Surface z_0 is reflection-free (the homogeneous upper half-space is a solid with the same properties as layer 1). Because at a reflection-free surface the total downgoing waves are given by the source waves only, an impulse response $\tilde{\mathbf{X}}^{(0)}(z_0)$ may be defined according to

$$\tilde{\mathbf{X}}^{(0)}(z_0) = \begin{bmatrix} \tilde{X}^{(0)}_{P,P} & \tilde{X}^{(0)}_{P,SV} \\ \tilde{X}^{(0)}_{SV,P} & \tilde{X}^{(0)}_{SV,SV} \end{bmatrix}_{z_0} = \begin{bmatrix} \tilde{\Phi}_1^- & \tilde{\Phi}_2^- \\ \tilde{\Psi}_1^- & \tilde{\Psi}_2^- \end{bmatrix}_{z_0} \begin{bmatrix} \tilde{\Phi}_1^+ & \tilde{\Phi}_2^+ \\ \tilde{\Psi}_1^+ & \tilde{\Psi}_2^+ \end{bmatrix}_{z_0}^{-1}, \tag{37a}$$

where $\tilde{\Phi}^{\pm}_{1,2}(z_0)$ and $\tilde{\Psi}^{\pm}_{1,2}(z_0)$ follow from

$$\tilde{\mathbf{P}}_s^{(1)}(z_0) = \left[\tilde{\Phi}_1^+, \tilde{\Psi}_1^+, -\tilde{\Phi}_1^-, \tilde{\Psi}_1^-\right]_{z_0}^T = \tilde{\mathbf{L}}_s^{-1}(z_0)\tilde{\mathbf{Q}}_s^{(1)}(z_0), \tag{37b}$$

and

$$\tilde{\mathbf{P}}_s^{(2)}(z_0) = \left[\tilde{\Phi}_2^+, \tilde{\Psi}_2^+, -\tilde{\Phi}_2^-, \tilde{\Psi}_2^-\right]_{z_0}^T = \tilde{\mathbf{L}}_s^{-1}(z_0)\tilde{\mathbf{Q}}_s^{(2)}(z_0), \tag{37c}$$

$\tilde{\mathbf{L}}_s^{-1}$ being the spatial Fourier transform of $\mathbf{L}_s^{-1}$, given by equation (B-3d). The subscripts in matrix $\tilde{\mathbf{X}}^{(0)}$ should be read from right to left; for instance, $\tilde{X}^{(0)}_{SV,P}$ denotes the P-SV impulse response.

(2) Surface z_0 is characterized by $\tilde{\mathbf{R}}^-(z_0)$, which represents the reflectivity matrix (Aki and Richards, 1980) for the lower side of surface z_0. An impulse response $\tilde{\mathbf{X}}(z_0)$ may be defined as

$$\tilde{\mathbf{X}}(z_0) = \left[\mathbf{I} - \tilde{\mathbf{X}}^{(0)}(z_0)\tilde{\mathbf{R}}^-(z_0)\right]^{-1}\tilde{\mathbf{X}}^{(0)}(z_0), \tag{38}$$

with $\tilde{\mathbf{X}}^{(0)}(z_0)$ given by equation (37) and where $\mathbf{I}$ represents a 2×2 identity matrix. Surface-related multiple generation, as described by equation (38), is schematically represented by the feedback system in Figure 8.

(3) Surface z_0 is traction-free (the upper half-space is vacuum). This situation can be handled as in (2). Note that in this case det $[\tilde{\mathbf{R}}^-(z_0)] = 1$. As an alternative we may define an admittance impulse response $\tilde{\mathbf{Y}}(z_0)$ which describes the detected total particle velocity due to an impulsive traction source. Because at a traction-free surface the total traction is given by the source traction only, we may write

$$\tilde{\mathbf{Y}}(z_0) = \begin{bmatrix} \tilde{Y}_{x,x} & \tilde{Y}_{x,z} \\ \tilde{Y}_{z,x} & \tilde{Y}_{z,z} \end{bmatrix}_{z0}$$

$$= -\begin{bmatrix} \tilde{V}_x^{(1)} & \tilde{V}_x^{(2)} \\ \tilde{V}_z^{(1)} & \tilde{V}_z^{(2)} \end{bmatrix}_{z0} \begin{bmatrix} \tilde{T}_{zx}^{(1)} & \tilde{T}_{zx}^{(2)} \\ \tilde{T}_{zz}^{(1)} & \tilde{T}_{zz}^{(2)} \end{bmatrix}_{z0}^{-1}, \quad (39a)$$

where $\tilde{V}_{x,z}^{(1,2)}(z_0)$ and $\tilde{T}_{zx,zz}^{(1,2)}(z_0)$ follow from

$$\tilde{\mathbf{Q}}_s^{(1)}(z_0) = \left[\tilde{V}_z^{(1)}, \tilde{T}_{zx}^{(1)}, \tilde{T}_{zz}^{(1)}, \tilde{V}_x^{(1)} \right]_{z0}^T, \quad (39b)$$

and

$$\tilde{\mathbf{Q}}_s^{(2)}(z_0) = \left[\tilde{V}_z^{(2)}, \tilde{T}_{zx}^{(2)}, \tilde{T}_{zz}^{(2)}, \tilde{V}_x^{(2)} \right]_{z0}^T. \quad (39c)$$

Specification of the acquisition configuration

Again we consider three cases.

(1) Surface z_0 is reflection-free. For the upgoing reflected P- and SV-waves $\tilde{\mathbf{P}}_{\text{refl}}^- = [\tilde{\Phi}_{\text{refl}}^-, \tilde{\Psi}_{\text{refl}}^-]^T$, due to downgoing incident P- and SV-waves $\tilde{\mathbf{P}}_{\text{inc}}^+ = [\tilde{\Phi}_{\text{inc}}^+, \tilde{\Psi}_{\text{inc}}^+]^T$, we may write

$$\tilde{\mathbf{P}}_{\text{refl}}^-(z_0) = \tilde{\mathbf{X}}^{(0)}(z_0)\tilde{\mathbf{P}}_{\text{inc}}^+(z_0), \quad (40)$$

where $\tilde{\mathbf{X}}^{(0)}(z_0)$ represents the impulse response given by equation (37).

(2) Surface z_0 is characterized by $\tilde{\mathbf{R}}^-(z_0)$. Again we may apply equation (40), with $\tilde{\mathbf{X}}^{(0)}(z_0)$ replaced by $\tilde{\mathbf{X}}(z_0)$, given by equation (38).

(3) Surface z_0 is traction-free. For the detected particle velocity $\tilde{\mathbf{V}}_{\text{CSP}}(z_0) = [\tilde{V}_{x,\text{CSP}}, \tilde{V}_{z,\text{CSP}}]^T$ in a common shotpoint (CSP) gather we may write

$$\tilde{\mathbf{V}}_{\text{CSP}}(z_0) = -\tilde{\mathbf{D}}_V(z_0)\tilde{\mathbf{Y}}(z_0)\tilde{\mathbf{T}}_{z,\text{SRC}}(z_0), \quad (41)$$

where $\tilde{\mathbf{T}}_{z,\text{SRC}}(z_0) = [\tilde{T}_{zx,\text{SRC}}, \tilde{T}_{zz,\text{SRC}}]^T$ represents the traction source, $\tilde{\mathbf{Y}}(z_0)$ represents the impulse response given by equation (39), and $\tilde{\mathbf{D}}_V(z_0)$ represents the particle velocity detector transfer function. [Ideally, $\tilde{\mathbf{D}}_V(z_0)$ is a function of ω only: $\tilde{\mathbf{D}}_V(z_0) = D_V(\omega)\mathbf{I}$.]

When this modeling procedure has been applied for all wavenumbers and frequencies, then the space-time data (one shot record) are obtained after inverse temporal and spatial Fourier transforms.

Note that the scheme is valid for modeling of land data only, because operator $\tilde{\mathbf{W}}_s$ may not be applied in liquids. For modeling of marine data, a small modification must be made. At the sea bottom a linear combination of $\tilde{\mathbf{Q}}_s^{(1)}$ and $\tilde{\mathbf{Q}}_s^{(2)}$ should be used so that the shear stress $\tilde{T}_{zx}$ vanishes. Subsequently, the propagation effects in the water layer may be modeled by extrapolation operator $\tilde{\mathbf{W}}_\ell$, given by equation (16). A modified scheme which handles arbitrary sequences of liquid layers and solid layers is discussed by Wapenaar (1986).

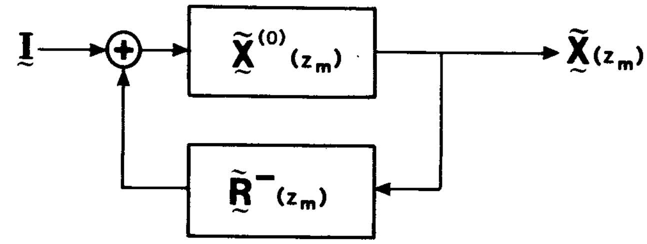

FIG. 8. Feedback system for surface-related multiple generation. $m = 0$.

Examples

A CSP gather was modeled for the marine subsurface configuration shown in Figure 9a. The first layer is a liquid (sea water), and the lower half-space is a solid. Figure 9b shows the CSP gather in the space-time domain (x, t). Notice that besides the ordinary reflected P-wave, there are P- and SV-head waves in these data, because evanescent waves in the solid were included in the computations (see also Figure 9c). (The pseudo-Rayleigh wave is not visible due to its very low amplitude.) Multiple reflections are absent because the surface was assumed reflection-free.

A CSP gather was also modeled for the land subsurface configuration shown in Figure 10a. A vertical traction is imposed at $x = 0$ at the free surface. The resulting particle velocity at the surface is shown in the (x, t) domain in Figures 10b (vertical component) and 10c (horizontal component; note the opposite polarities at both sides of the vertical traction source). Notice that primary P-waves [P-P(1) and P-P(2)] and primary SV-waves [e.g., SV-SV(1)], as well as many converted [e.g., P-SV(1)] and multiply reflected waves are visible in these figures. Head waves and Rayleigh waves are absent, because evanescent waves were excluded from the computations (see also Figure 10d).

The advantage of the modeling scheme discussed here is that all primary and multiple P-waves as well as SV-waves are included in the simple recursive algorithms [equations (35a) and (35b)] and that the boundary conditions at the layer interfaces are solved automatically. For 2-D and 3-D inhomogeneous media, modeling techniques based on full elastic two-way depth extrapolation become very complicated and two-way time extrapolation techniques are preferred. A 2-D application is presented by Kosloff et al. (1984). An extensive discussion of modeling algorithms, however, is beyond the scope of this paper.

PRESTACK MIGRATION SCHEME BASED ON THE FULL ELASTIC TWO-WAY WAVE EQUATION

In this section we introduce a recursive prestack migration scheme, based on the full elastic two-way wave equation, which properly handles primary and multiple energy as well as wave conversion. First we briefly review the principle of prestack migration by single-shot record inversion (SSRI), as proposed by Berkhout (1984). This scheme basically consists of the following three steps:

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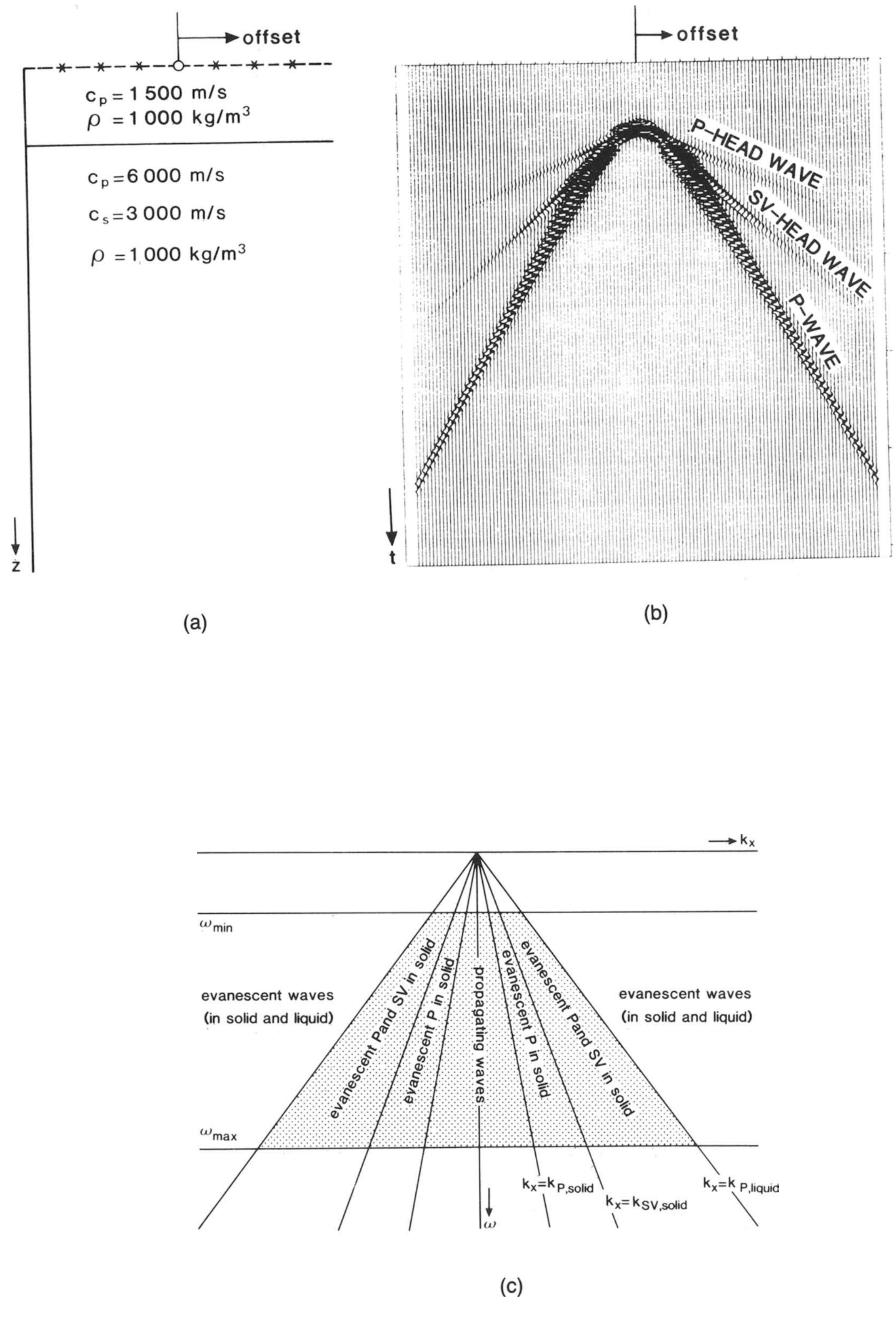

FIG. 9. Marine data modeled with the full elastic two-way wave equation. (a) Subsurface model. (b) CSP gather in (x, t) domain. (c) The shaded area shows the location of computations in the wavenumber-frequency domain.

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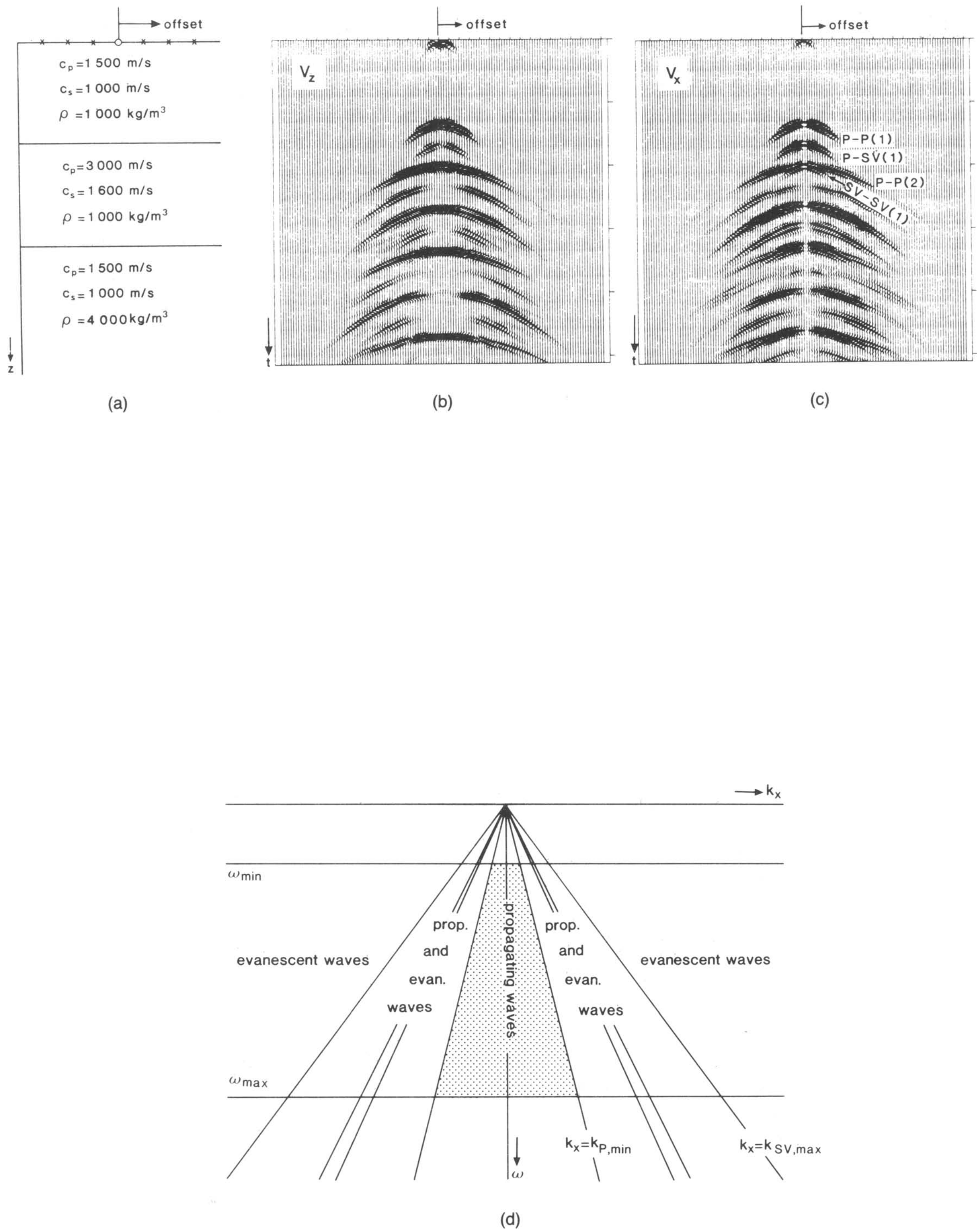

FIG. 10. Land data modeled with the full elastic two-way wave equation. (a) Subsurface model. (b) Vertical component of the particle velocity in the (x, t) domain. (c) Horizontal component of the particle velocity in the (x, t) domain. (d) The shaded area shows the location of computations in the wavenumber-frequency domain.

(1) Downward extrapolation of a single CSP gather to depth level z_i. Forward extrapolation of the downgoing source wave, based on the acoustic one-way wave equation for downgoing waves, yields $P^+_{SRC}(x, z_i, \omega)$. Inverse extrapolation of the detected upgoing wave, based on the acoustic one-way wave equation for upgoing waves, yields $P^-(x, z_i, \omega)$.

(2) "Correlation" at depth level z_i. Assume that the downgoing and upgoing waves at z_i are related to each other via a spatial convolution, according to

$$P^-(x, z_i, \omega) = X(x, z_i, \omega) * P^+_{SRC}(x, z_i, \omega), \quad (42)$$

where $X(x, z_i, \omega)$ describes the spatial impulse response of the subsurface. This relation cannot be inverted simply. It can be shown (Berkhout, 1984; Wapenaar, 1986) that a true amplitude estimate of the (single-fold) zero-offset (ZO) impulse response is given by

$$\langle X_{ZO}(x, z_i, \omega)\rangle = \frac{1}{s^2} P^-(x, z_i, \omega)\left[P^+_{SRC}(x, z_i, \omega)\right]^*, \quad (43a)$$

where

$$s^2 = \int P^+_{SRC}(x, z_i, \omega)\left[P^+_{SRC}(x, z_i, \omega)\right]^* dx. \quad (43b)$$

In one-way wave-equation migration the energy scaling factor s^{-2} is generally omitted because transmission effects are neglected in one-way wave-field extrapolation.

(3) Imaging at depth level z_i. An estimate of the (single-fold) zero-offset reflectivity at the current level (downgoing and upgoing waves are time-coincident) is obtained by integrating the zero-offset impulse response over all frequencies, according to

$$\langle R_{ZO}(x, z_i)\rangle = \frac{1}{2\pi}\int_\omega \langle X_{ZO}(x, z_i, \omega)\rangle\, d\omega. \quad (44)$$

The procedure is repreated (recursively or nonrecursively) for all depths z_i and for all CSP gathers. The individual migration results can be summed afterward over all CSP gathers (true common-depth-point stacking), optionally after a residual NMO correction if the input velocity model is slightly in error. The computational diagram is shown in Figure 11a. Results obtained by De Graaff (1984) demonstrate the high lateral resolution properties of this scheme. For the acoustic case, the scheme is further developed by Wapenaar and Berkhout (1986b) to incorporate critical-angle events and multiple reflections. Here we discuss the extension of the scheme for the incorporation of multiple reflections and wave conversion. In the downward extrapolation step we replace the acoustic one-

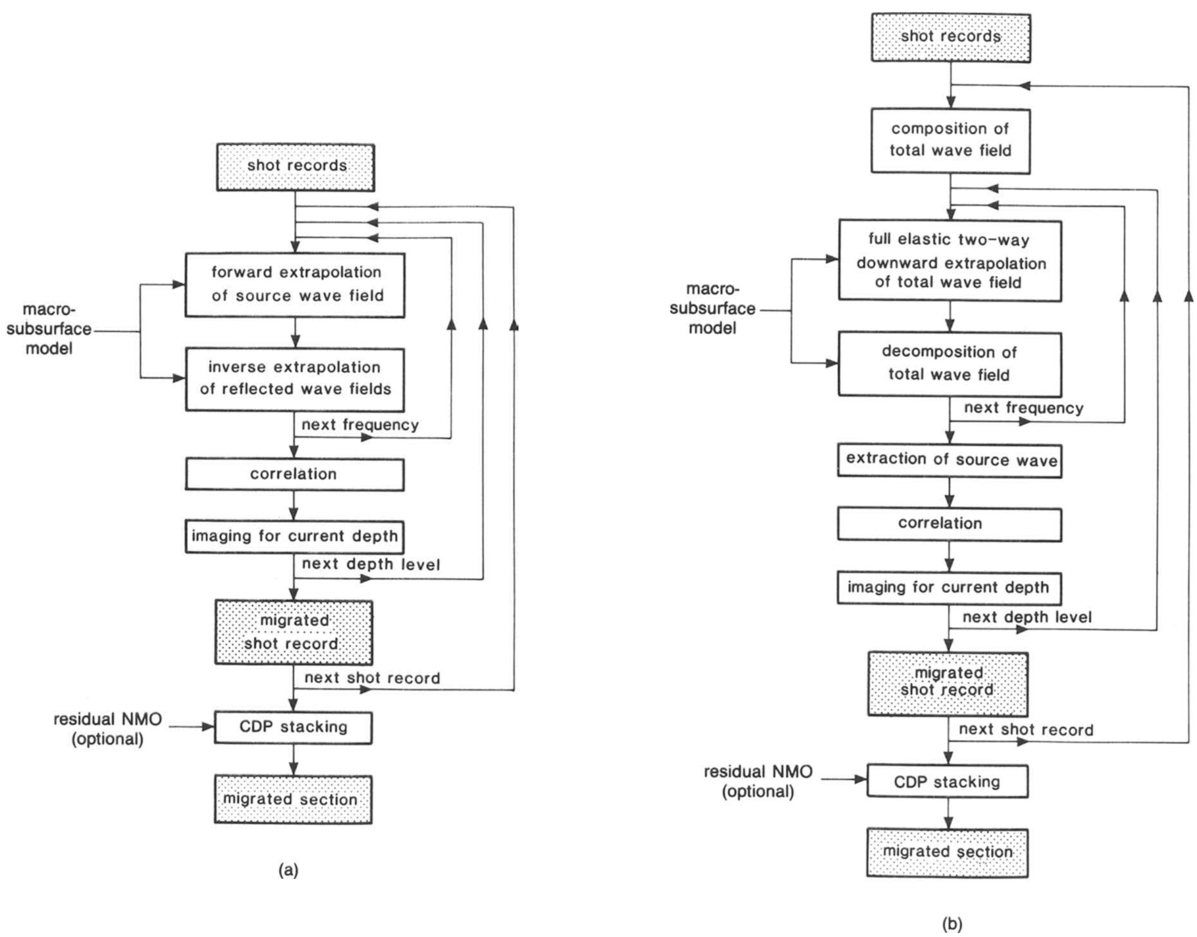

FIG. 11. Computational diagrams for shot record-oriented, prestack migration. (a) Scheme based on the acoustic one-way wave equations (Berkhout, 1984). (b) Scheme based on the full elastic two-way wave equation.

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way wave equations by the full elastic two-way wave equation, so downgoing and upgoing P- and SV-waves are downward extrapolated simultaneously. The total scheme that we propose consists of the following steps.

Composition of the total wave field at the surface

We consider a traction-free surface and assume that [analogous to relation (41)] the detected particle velocity vector $\mathbf{V}_{\mathrm{CSP},m}$ in the mth CSP gather is related to the mth source traction vector $\mathbf{T}_{z,\mathrm{SRC},m}$ according to

$$\mathbf{V}_{\mathrm{CSP},m}(z_0) = -\underset{\sim}{\mathbf{D}}_v(z_0)\underset{\sim}{\mathbf{Y}}(z_0)\mathbf{T}_{z,\mathrm{SRC},m}(z_0), \tag{45a}$$

or

$$\begin{bmatrix} V_{x,\mathrm{CSP},m}(x, z_0, \omega) \\ V_{z,\mathrm{CSP},m}(x, z_0, \omega) \end{bmatrix} = -\begin{bmatrix} D_v(\omega)\delta(x)* & 0 \\ 0 & D_v(\omega)\delta(x)* \end{bmatrix} \times \begin{bmatrix} Y_{x,x}(x, z_0, \omega)* & Y_{x,z}(x, z_0, \omega)* \\ Y_{z,x}(x, z_0, \omega)* & Y_{z,z}(x, z_0, \omega)* \end{bmatrix} \times \begin{bmatrix} \delta(x - x_m)S(\omega)T_{zx}(z_0) \\ \delta(x - x_m)S(\omega)T_{zz}(z_0) \end{bmatrix}. \tag{45b}$$

Acquisition parameters and medium parameters can be described separately as follows:

$$\begin{bmatrix} V_{x,\mathrm{CSP},m}(x, z_0, \omega) \\ V_{z,\mathrm{CSP},m}(x, z_0, \omega) \end{bmatrix} = \begin{bmatrix} D_v(\omega)S(\omega) \end{bmatrix} \begin{bmatrix} V_{x,m}(x, z_0, \omega) \\ V_{z,m}(x, z_0, \omega) \end{bmatrix}, \tag{46a}$$

where

$$\begin{bmatrix} V_{x,m}(x, z_0, \omega) \\ V_{z,m}(x, z_0, \omega) \end{bmatrix} = -\underset{\sim}{\mathbf{Y}}(z_0)\begin{bmatrix} \delta(x - x_m)T_{zx}(z_0) \\ \delta(x - x_m)T_{zz}(z_0) \end{bmatrix}. \tag{46b}$$

Notice that $V_{x,m}(x, z_0, \omega)$ and $V_{z,m}(x, z_0, \omega)$ can be resolved from the measured data by inverting equation (46a) in a band-limited way (deconvolution). Equation (46b) describes the mth seismic experiment, corrected for the source and detector spectra $S(\omega)$ and $D_v(\omega)$, respectively. Accordingly, the total wave field at the surface reads

$$\mathbf{Q}_{s,m}(z_0) = \begin{bmatrix} V_{z,m}(x, z_0, \omega) \\ \delta(x - x_m)T_{zx}(z_0) \\ \delta(x - x_m)T_{zz}(z_0) \\ V_{x,m}(x, z_0, \omega) \end{bmatrix}. \tag{47}$$

Full elastic wave-field extrapolation

Recursive, full elastic, downward extrapolation of the total wave field is formally described by

$$\mathbf{Q}_{s,m}(z_i) = \underset{\sim}{\mathbf{W}}_s(z_i, z_{i-1})\mathbf{Q}_{s,m}(z_{i-1}). \tag{48}$$

For the 2-D inhomogeneous version of the computationally convenient macrosubsurface model of Figure 5, operator $\underset{\sim}{\mathbf{W}}_s$ represents a truncated version of the fast converging finite-difference operator (34b) in the space-frequency domain. For the special situation in which lateral variations of the medium parameters may be neglected, this step should preferably be applied in the wavenumber-frequency domain, with operator $\tilde{\underset{\sim}{\mathbf{W}}}_s$ given by equation (32).

Decomposition

Decomposition of the total wave field into downgoing and upgoing P- and SV-waves is formally described by

$$\mathbf{P}_{s,m}(z_i) = \underset{\sim}{\mathbf{L}}_s^{-1}(z_i)\mathbf{Q}_{s,m}(z_i), \tag{49a}$$

with

$$\mathbf{P}_{s,m}(z_i) = \begin{bmatrix} \Phi_m^+(x, z_i, \omega) \\ \Psi_m^+(x, z_i, \omega) \\ -\Phi_m^-(x, z_i, \omega) \\ \Psi_m^-(x, z_i, \omega) \end{bmatrix}, \tag{49b}$$

and where decomposition operator $\underset{\sim}{\mathbf{L}}_s^{-1}$ is given by equation (B-3d). Unfortunately the finite-difference expansion of this operator converges slowly. However, in practice it will often be sufficient to apply the decomposition in the wavenumber-frequency domain for a reference medium only, by means of operator $\tilde{\underset{\sim}{\mathbf{L}}}_s^{-1}$. Note that errors in the decomposed wave field $\mathbf{P}_{s,m}$ do not contribute to deeper depth levels, because the total wave field $\mathbf{Q}_{s,m}$ is downward extrapolated independently in equation (48).

The downgoing and upgoing waves at the current depth level are related to each other through a spatial convolution, according to

$$\mathbf{P}_{s,m}^-(z_i) = \underset{\sim}{\mathbf{X}}(z_i)\mathbf{P}_{s,m}^+(z_i), \tag{50a}$$

or

$$\begin{bmatrix} \Phi_m^-(x, z_i, \omega) \\ \Psi_m^-(x, z_i, \omega) \end{bmatrix} = \begin{bmatrix} X_{P,P}(x, z_i, \omega)* & X_{P,SV}(x, z_i, \omega)* \\ X_{SV,P}(x, z_i, \omega)* & X_{SV,SV}(x, z_i, \omega)* \end{bmatrix} \begin{bmatrix} \Phi_m^+(x, z_i, \omega) \\ \Psi_m^+(x, z_i, \omega) \end{bmatrix}, \tag{50b}$$

where $\underset{\sim}{\mathbf{X}}(z_i)$ describes the spatial impulse response of the subsurface. Inversion of this relation requires at least one more independent seismic experiment being carried out at the same source location x_m, such that the vectors $\mathbf{P}_{s,m}^+$ and $\mathbf{P}_{s,m}^-$ can be extended to square matrices. This is not very attractive from a practical point of view; therefore we propose an inversion procedure analogous to the acoustic one-way procedure described above.

We first need to extract the downgoing source wave from the total downgoing wave field. Because $c_P > c_{SV}$ in all layers, we may extract the dilatational part (P-wave) of the downgoing source wave from the downgoing wave $\Phi_m^+(x, z_i, \omega)$ by means of a "first arrival time window," yielding $\Phi_{\mathrm{SRC},m}^+(x, z_i, \omega)$. This actually means that multiply reflected waves are excluded from the imaging procedure, as demonstrated below.

Correlation

Correlation of the downgoing source P-wave and the upgoing reflected P- and SV-waves yields the single-fold zero-offset P-P and P-SV impulse responses, according to

$$\langle X_{\mathrm{ZO};P,P}(x, z_i, \omega)\rangle_m = \frac{1}{s_m^2}\,\Phi_m^-(x, z_i, \omega)\left[\Phi_{\mathrm{SRC},m}^+(x, z_i, \omega)\right]^*, \tag{51a}$$

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and

$$\langle X_{\text{ZO};SV,P}(x, z_i, \omega)\rangle_m = \frac{1}{s_m^2}\,\Psi_m^-(x, z_i, \omega)\left[\Phi_{\text{SRC},m}^+(x, z_i, \omega)\right]^*, \tag{51b}$$

where

$$s_m^2 = \int \Phi_{\text{SRC},m}^+(x, z_i, \omega)\left[\Phi_{\text{SRC},m}^+(x, z_i, \omega)\right]^* dx. \tag{51c}$$

In two-way wave-equation migration (acoustic or full elastic) the energy scaling factor s_m^{-2} may not be omitted because transmission effects are automatically incorporated in the downward extrapolation step. Notice that s_m^{-2} is stable because the deconvolution for the source wavelet was already carried out in the first step. Also notice that we cannot define the SV-P and the SV-SV impulse responses because we do not have an expression for the downgoing SV source wave. The fact that only two out of four impulse responses can be resolved is in agreement with an observation we made before: full inversion requires additional information obtained from an extra, independent seismic experiment, carried out at the same source location.

Imaging

Imaging is carried out by summing over all frequencies, yielding the single-fold zero-offset P-P and P-SV reflectivities:

$$\langle R_{\text{ZO};P,P}(x, z_i)\rangle_m = \frac{\Delta\omega}{2\pi}\sum_{\omega}\langle X_{\text{ZO};P,P}(x, z_i, \omega)\rangle_m, \tag{52a}$$

$$\langle R_{\text{ZO};SV,P}(x, z_i)\rangle_m = \frac{\Delta\omega}{2\pi}\sum_{\omega}\langle X_{\text{ZO};SV,P}(x, z_i, \omega)\rangle_m. \tag{52b}$$

CDP stacking

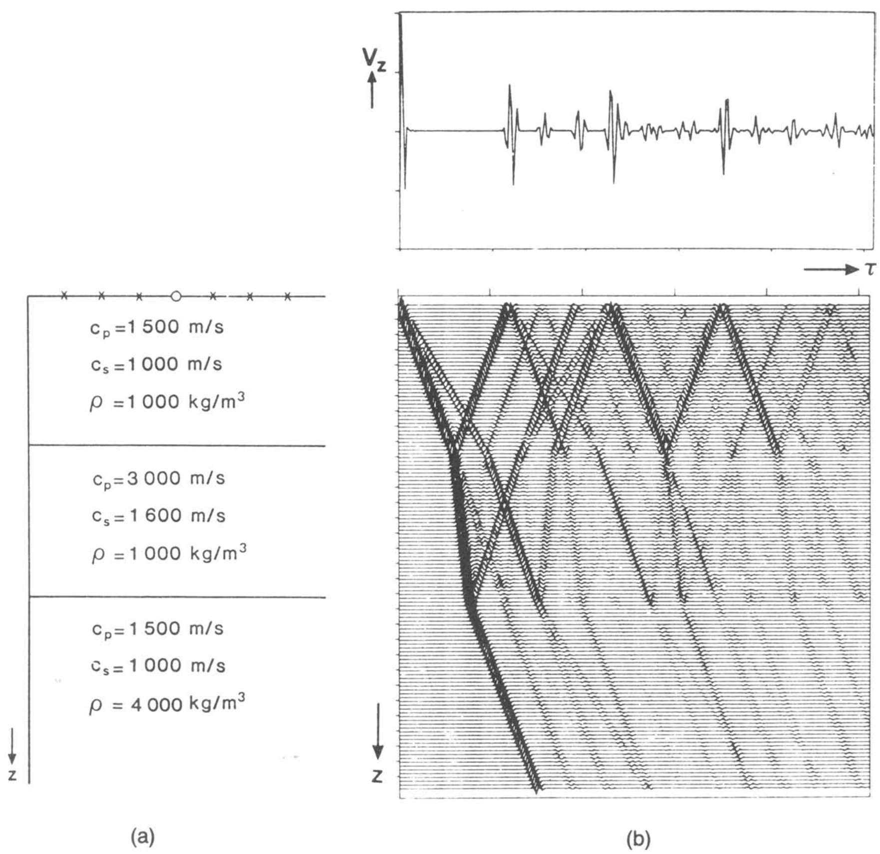
CDP stacking is carried out by summing all single-fold reflectivity functions, yielding the multifold wide-angle zero-offset P-P and P-SV reflectivities

$$\langle R_{\text{ZO};P,P}^{\text{CDP}}(x, z_i)\rangle = \sum_m \langle R_{\text{ZO};P,P}(x, z_i)\rangle_m, \tag{53a}$$

and

$$\langle R_{\text{ZO};SV,P}^{\text{CDP}}(x, z_i)\rangle = \sum_m \langle R_{\text{ZO};SV,P}(x, z_i)\rangle_m. \tag{53b}$$

The computational diagram is shown in Figure 11b.



FIGS. 12a, 12b. Results of full elastic two-way wave equation migration for a horizontally layered medium. (a) Subsurface model. (b) Vertical component of the particle velocity (incidence angle 25 degrees) as a function of intercept time registered at the surface (top frame) and downward extrapolated (bottom frame).

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Example

The performance of the algorithm is demonstrated with a simple numerical example. We consider a horizontally layered 1-D inhomogeneous medium. In this case the whole procedure can be applied in the wavenumber-frequency domain. As an alternative, we replace the zero-offset imaging step by a plane-wave imaging step, according to

$$\langle \tilde{R}_{P,P}(p, z_i)\rangle = \frac{\Delta\omega}{2\pi} \sum_{\omega}{}' \tilde{X}_{P,P}(k_x, z_i, \omega), \tag{54a}$$

$$\langle \tilde{R}_{SV,P}(p, z_i)\rangle = \frac{\Delta\omega}{2\pi} \sum_{\omega}{}' \tilde{X}_{SV,P}(k_x, z_i, \omega), \tag{54b}$$

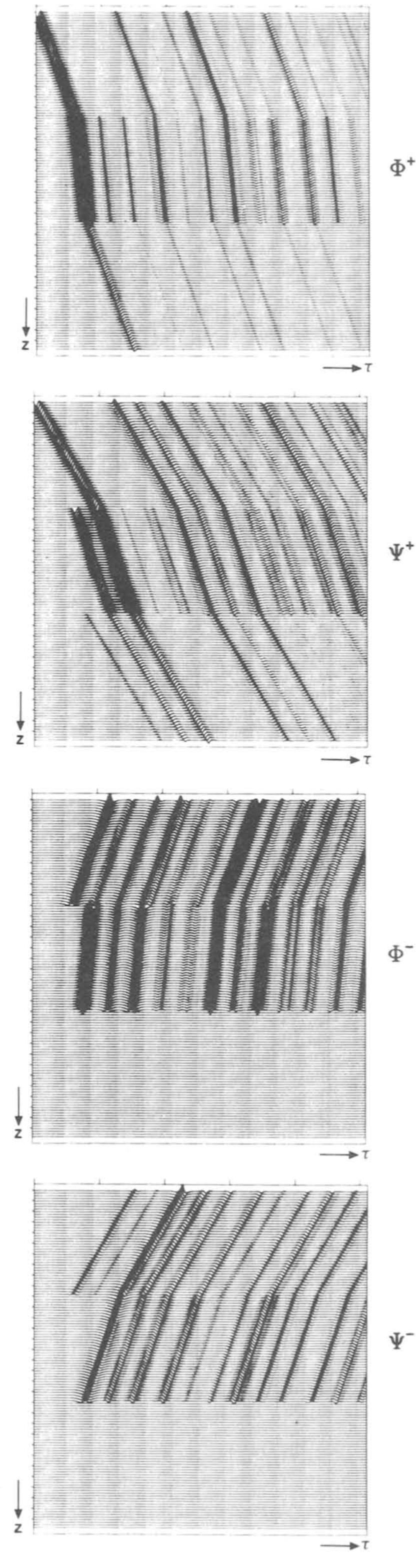
where

$$\tilde{X}_{P,P}(k_x, z_i, \omega) \triangleq \tilde{\phi}^-(k_x, z_i, \omega)/\tilde{\phi}^+_{SRC}(k_x, z_i, \omega), \tag{54c}$$

$$\tilde{X}_{SV,P}(k_x, z_i, \omega) \triangleq \tilde{\psi}^-(k_x, z_i, \omega)/\tilde{\phi}^+_{SRC}(k_x, z_i, \omega) \tag{54d}$$

in some stable sense. The symbol $\sum'$ denotes that the summation is carried out for constant ray parameter $p = k_x/\omega$ (constant propagation direction).

Figure 12a shows again the subsurface configuration of the second modeling example (Figure 10a). The top frame of Figure 12b shows one trace of a τ-p map of the vertical component of the particle velocity data (Figure 10b) for one constant p value (constant k_x/ω), such that $\sin\theta(z_0) = pc_P(z_0) = .43$, that is, for one oblique plane wave with incidence angle $\theta(z_0) = 25$ degrees. Because the data are presented as a function of the intercept time τ, all reflection events related to this single plane wave are visible. Notice that besides primary P-waves, many multiple reflections and converted waves are present, which makes the trace difficult to interpret even for this simple two-layer model. However, the interpretability improves significantly when the data are downward extrapolated, using the full elastic two-way wave-field extrapolation operator. The downward extrapolated vertical component of the particle velocity is shown in the bottom frame of Figure 12b as a function of depth and intercept time, again for one p value. Similar pictures could be shown for the horizontal component of the particle velocity and for the two components of the traction. Notice that, similarly to a vertical seismic profile (VSP) recorded in a vertical borehole, all wave types can be clearly recognized. The decomposed data are shown in Figure 12c, again as a function of depth and intercept time for one p value. The left frame of Figure 12d shows the compressional downgoing source wave which has been retrieved from the upper frame of Figure 12c by means of a first-arrival time window. Also in Figure 12d the plane-wave imaged P-P and P-SV reflectivities are presented as a function of depth. Notice that this plane-wave image shows the two reflectors only, in spite of the complex nature of the input trace (top frame of Figure 12b), because the downward extrapolated upgoing waves (Figure 12c) are properly terminated at the reflectors (which also explains the half-wavelets of the images). We have shown the migration procedure for one plane wave only. The procedure can be repeated for all angles of incidence. In Figure 12e the plane-wave imaged P-P and P-SV reflectivities are shown as a function of depth for various ray parameters (the computations were restricted to the shaded area in Figure



(c)

FIG. 12c. Decomposed data; from top to bottom: downgoing P-waves, downgoing S-waves, upgoing P-waves and upgoing S-waves.

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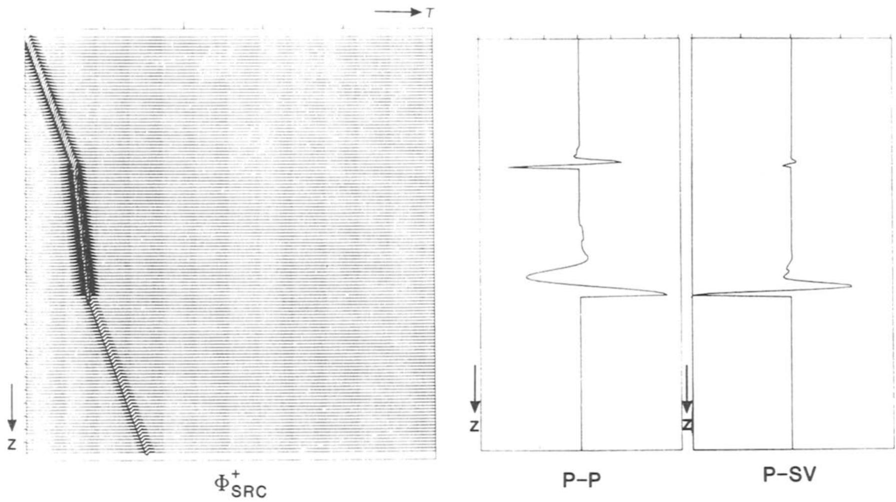
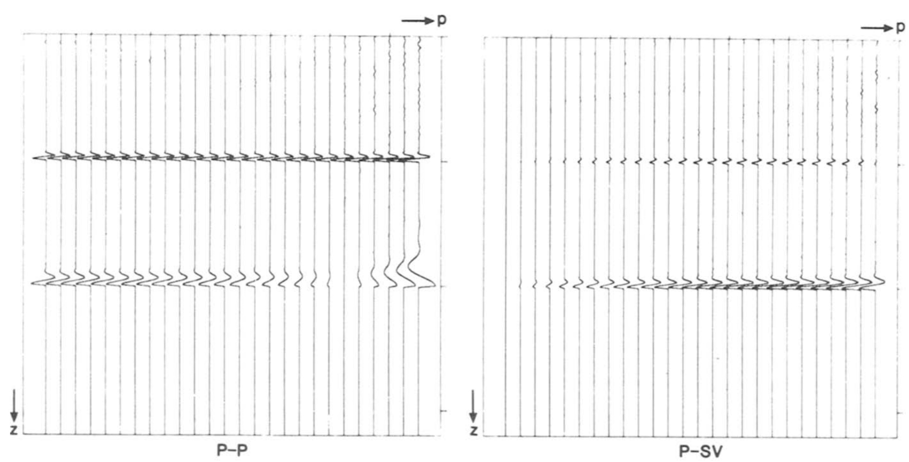


(d)



(e)

Figs. 12d, 12e. (d) Downgoing *P* source wave (left frame), plane wave imaged *P-P* reflectivity (middle frame) and plane wave imaged *P-SV* reflectivity (right frame). (e) Plane wave imaged results for all incidence angles; *P-P* reflectivity (left frame) and *P-SV* reflectivity (right frame).

10d). The angle-dependent reflection functions can be accurately retrieved from these data, as shown in Figure 12f.

For this 1-D inhomogeneous medium, plane-wave imaging was carried out outside the depth extrapolation loop, and only one shot record was considered. For 2-D inhomogeneous media, zero-offset imaging should be carried out inside the depth extrapolation loop and the procedure should be repeated for all shot records, conforming to the computational diagram shown in Figure 11b.

We have introduced a prestack migration scheme based on the full elastic two-way wave equation. Notice the following advantages of this scheme in comparison with conventional acoustic one-way schemes: use of the square-root operator is avoided; true CDP stacking is achieved; transmission effects are automatically included; multiply reflected waves may be properly handled; and converted waves may be properly handled.

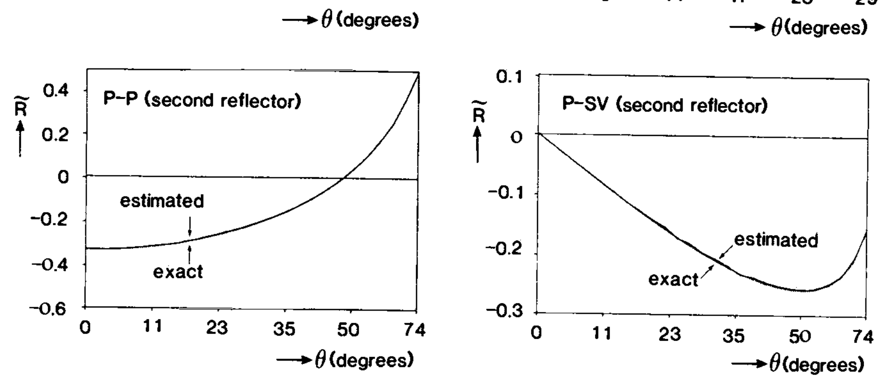
The transmission effects, multiply reflected waves, and converted waves are properly handled because the extrapolated total wave field $\mathbf{Q}_s$ is continuous for all depths as long as only solid layers are considered. Problems occur if the macrosubsurface model consists of liquid as well as solid layers because the horizontal component V_x of the particle velocity may be discontinuous at liquid-solid interfaces. Of course, given the source configuration as well as a complete description of the subsurface, V_x just below a liquid-solid interface is uniquely defined. However, it cannot be obtained from V_x just above the liquid-solid interface. For this reason full elastic two-way migration of marine data, for instance, is impractical. [A full elastic *one-way* extrapolation operator for marine data is discussed by Wapenaar and Peels (1987).]

For proper handling of multiply reflected and converted waves (in the case of land data), accurate knowledge of the macrosubsurface model is required. Of course, not every detail of the subsurface need be known. However, the macrosubsurface model must be specified such that the traveltimes related to the major reflecting boundaries match the traveltimes in the data as accurately as possible (errors should be significantly smaller than a cycle time related to the central frequency). As in conventional multiple elimination schemes, a small mispositioning of the major reflecting boundaries may result in an increase of undesired reflection events, so the migration scheme should be applied iteratively. On the other hand, generation of undesired reflection events may be avoided by spatially filtering (smoothing) the abrupt changes in the macrosubsurface model before migration. Of course, in this case multiply reflected and converted waves will no longer be handled properly.

DISCUSSION

The seismic literature includes only a small number of full elastic migration approaches. We briefly review these methods and discuss how they compare with our approach.

Kuo and Dai (1984) discuss a migration approach based on Kirchhoff-Helmholtz type integrals for elastic waves. Their *one-way* scheme properly images reflectors in an otherwise homogeneous solid. However, multiple reflections and wave conversion in layered media are neglected for the most part, and waves in arbitrarily inhomogeneous media cannot be handled. For comparison, our examples show that the full elastic *two-way* prestack migration scheme properly handles multiple reflections and wave conversion in layered media. Furthermore, we showed theoretically that the operator for downward extrapolation may be applied in arbitrarily inho-



(f)

FIG. 12f. Angle-dependent reflection functions, retrieved from the migrated data (solid lines) and computed analytically (dotted lines).

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mogeneous media, provided the derivatives of the medium parameters may be neglected.

Kennett (1984) discusses an operator approach to full elastic migration founded on straightforward inversion of the forward model of a complete seismic data set. Therefore, in his approach *all* CSP gathers must be handled simultaneously. In addition, two independent seismic experiments are required for each source location. This approach is unattractive from a practical point of view, particularly for the 3-D case. In contrast, our full elastic two-way prestack migration scheme (including multiple elimination) is carried out on *single shot* records, similar to the acoustic one-way prestack (primary) migration scheme as proposed by Berkhout (1984).

Tarantola (1986) solves the nonlinear inverse problem for full elastic seismic data, using a generalized least-squares criterion. He derives an iterative algorithm and shows elegantly that each iteration step essentially consists of a forward modeling of the sources in the current model and a reverse time migration of the data residuals. Multiply reflected and converted waves are taken into account. Practical implementation of this reverse time migration scheme causes serious problems: for each time step the total data volume must be kept in memory, which is particularly difficult in the 3-D case. In comparison, in our scheme only one frequency component of the data need be kept in memory for each depth step. The latter aspect is critical in the design of (acoustic or full elastic) 3-D prestack migration algorithms.

CONCLUSIONS

We have discussed the following matrix representations of the acoustic wave equation:

$$\frac{\partial \mathbf{Q}_\ell}{\partial z} = \mathbf{A}_\ell \mathbf{Q}_\ell \tag{55a}$$

and

$$\frac{\partial \mathbf{P}_\ell}{\partial z} = \mathbf{B}_\ell \mathbf{P}_\ell, \tag{55b}$$

where subscript ℓ refers to liquids. Expression (55a) represents the *two-way* wave equation for the total wave field $\mathbf{Q}_\ell = [-P,\ V_z]^T$. Expression (55b) represents the *one-way* wave equation for the decomposed wave field $\mathbf{P}_\ell = [-P^+,\ -P^-]^T$. The decomposition is described by

$$\mathbf{P}_\ell = \mathbf{L}_\ell^{-1}\mathbf{Q}_\ell. \tag{55c}$$

In addition, we discussed the following matrix representations of the full elastic wave equation:

$$\frac{\partial \mathbf{Q}_s}{\partial z} = \mathbf{A}_s \mathbf{Q}_s \tag{56a}$$

and

$$\frac{\partial \mathbf{P}_s}{\partial z} = \mathbf{B}_s \mathbf{P}_s, \tag{56b}$$

where subscript s refers to solids. Expression (56a) represents the *two-way* wave equation for the total wave field $\mathbf{Q}_s = [V_z,\ T_{zx},\ T_{zz},\ V_x]^T$. Expression (56b) represents the *one-way* wave equations for the decomposed wave field $\mathbf{P}_s = [\Phi^+, \Psi^+, -\Phi^-, \Psi^-]^T$. The decomposition is described by

$$\mathbf{P}_s = \mathbf{L}_s^{-1}\mathbf{Q}_s. \tag{56c}$$

Because the square-root operator is avoided in the two-way wave equations (55a) and (56a), we derived explicit finite-difference two-way wave field extrapolation operators $\mathbf{W}_\ell$ and $\mathbf{W}_s$ for $\mathbf{Q}_\ell$ and $\mathbf{Q}_s$, respectively, which converge rapidly. These operators can be used recursively in inhomogeneous media, assuming that the derivatives of the medium parameters can be neglected in each layer of the background medium (the "macrosubsurface model").

Prestack migration based on the full elastic two-way wave equation is, in principle, founded on two-way downward extrapolation, which is schematically represented by

$$\begin{bmatrix} V_z \\ T_{zx} \\ T_{zz} \\ V_x \end{bmatrix}_{z_{i-1}} \rightarrow \mathbf{W}_s \rightarrow \begin{bmatrix} V_z \\ T_{zx} \\ T_{zz} \\ V_x \end{bmatrix}_{z_i}, \tag{57a}$$

followed by decomposition

$$\begin{bmatrix} V_z \\ T_{zx} \\ T_{zz} \\ V_x \end{bmatrix}_{z_i} \rightarrow \mathbf{L}_s^{-1} \rightarrow \begin{bmatrix} \Phi^+ \\ \Psi^+ \\ -\Phi^- \\ \Psi^- \end{bmatrix}_{z_i} \xrightarrow{\text{window}} \begin{bmatrix} \Phi^+_{\mathrm{SRC}} \\ \Psi^+ \\ -\Phi^- \\ \Psi^- \end{bmatrix}_{z_i}, \tag{57b}$$

and correlation and imaging

$$\Phi^-(z_i) \rightarrow \langle R_{P,P}(z_i)\rangle \leftarrow \Phi^+_{\mathrm{SRC}}(z_i), \tag{57c}$$

$$\Psi^-(z_i) \rightarrow \langle R_{SV,P}(z_i)\rangle \leftarrow \Phi^+_{\mathrm{SRC}}(z_i). \tag{57d}$$

This scheme (which should be applied for all depths z_i and for all shot records) assumes 2-D wave propagation in 1-D or 2-D inhomogeneous macrosubsurface models. It properly handles multiple reflections and wave conversion if the macrosubsurface model is accurately known. For smoothed macrosubsurface models, large dip angles for primary waves may be handled. For 3-D wave propagation in 1-D, 2-D, and 3-D inhomogeneous macrosubsurface models, the scheme is not valid because the conversion between P- and SV-waves on one hand and SH-waves on the other cannot be neglected. Hence, for the 3-D case the scheme should be based on a modified full elastic, two-way wave equation which describes all wave types (P, SV, and SH) simultaneously. The matrix formulation of this 3-D wave equation follows directly from equation (B-1):

$$\frac{\partial \mathbf{Q}_s}{\partial z} = \mathbf{A}_s \mathbf{Q}_s, \tag{58a}$$

with

$$\mathbf{Q}_s = \begin{bmatrix} V_z \\ T_{zx} \\ T_{zy} \\ T_{zz} \\ V_x \\ V_y \end{bmatrix}, \tag{58b}$$

and where $\mathbf{A}_s$ represents a 6×6 matrix operator. The solution for 1-D inhomogeneous media is given by Ursin (1983), assuming constant medium parameters in a certain depth interval. The solution for 2-D and 3-D inhomogeneous media can

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be found in a similar way as described earlier. Based on this solution, a full elastic prestack migration scheme for 3-D inhomogeneous macrosubsurface models can be designed in a similar way as described here.

The practical realization of full elastic migration in complicated subsurface situations will be quite difficult. The demand for accurate macrosubsurface information is particularly severe. It is our opinion, however, that with our method the feasibility is improved significantly (particularly for the 3-D case) compared with other fully elastic migration or inversion schemes (Kennett, 1984; Tarantola, 1986), because the method we propose is carried out per shot record and per frequency component.

ACKNOWLEDGMENTS

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REFERENCES

Achenbach, J. D., 1973, Wave propagation in elastic solids: North Holland Publ. Co.

Aki, K., and Richards, P. G., 1980, Quantitative seismology: W. H. Freeman and Co.

Berkhout, A. J., 1982, Seismic migration: Imaging of acoustic energy by wave field extrapolation. A. Theoretical aspects: Elsevier Science Publ. Co., Inc.

——— 1984, Seismic migration: Imaging of acoustic energy by wave field extrapolation. B. Practical aspects: Elsevier Science Publ. Co., Inc.

De Graaff, M. P., 1984, Pre-stack migration by single shot record inversion: Doctoral thesis, Delft Univ. of Tech.

Kennett, B. L. N., 1984, An operator approach to forward modeling, data processing and migration: Geophys. Prosp., **32**, 1074–1090.

Kosloff, D. D., Reshef, M., and Loewenthal, D., 1984, Elastic wave calculation by the Fourier method: Bull., Seis. Soc. Am., **74**, 875–891.

Kuo, J. T., and Dai, T. F., 1984, Kirchhoff elastic wave migration for the case of noncoincident source and receiver: Geophysics, **49**, 1223–1238.

Pilant, W. L., 1979, Elastic waves in the earth: Elsevier Science Publ. Co., Inc.

Tarantola, A., 1986, A strategy for nonlinear elastic inversion of seismic reflection data: Geophysics, 1893–1903.

Ursin, B., 1983, Review of elastic and electromagnetic wave propagation in horizontally layered media: Geophysics, **48**, 1063–1081.

Wapenaar, C. P. A., 1986, Pre-stack migration in two and three dimensions: Doctoral thesis, Delft Univ. of Tech.

Wapenaar, C. P. A., and Berkhout, A. J., 1986a, Wave field extrapolation techniques for inhomogeneous media which include critical angle events. Part II: Methods using the two-way wave equation: Geophys. Prosp., **34**, 147–179.

——— 1986b, Wave field extrapolation techniques for inhomogeneous media which include critical angle events. Part III: Applications in modeling, migration and inversion: Geophys. Prosp., **34**, 180–207.

Wapenaar, C. P. A., and Peels, G. L., 1987, True amplitude wave-field extrapolation of primary waves: Submitted to Geophysics.

APPENDIX A

SERIES EXPANSION OF THE ACOUSTIC TWO-WAY OPERATOR

With the aid of Taylor series expansion of equations (16b)–(16e), operator $\tilde{\mathbf{W}}_\ell$ can be written as

$$\tilde{\mathbf{W}}_\ell(z, z_0) = \sum_{n=0}^{\infty} \left[(a_n \mathbf{I} + b_n \tilde{\mathbf{A}}_\ell) \tilde{\mathbf{H}}_2^n \right], \quad \text{(A-1a)}$$

where

$$\tilde{\mathbf{H}}_2^n = \tilde{H}_2^n \mathbf{I}, \quad \text{(A-1b)}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{(A-1c)}$$

$$a_n = \frac{\Delta z^{2n}}{(2n)!}(-1)^n, \quad \text{(A-1d)}$$

$$b_n = \frac{\Delta z^{2n+1}}{(2n+1)!}(-1)^n, \quad \text{(A-1e)}$$

$$\tilde{H}_2 = k^2 - k_x^2 - k_y^2, \quad \text{(A-1f)}$$

where $\tilde{\mathbf{A}}_\ell$ is given by equation (12b).

Applying a binomial expansion for $\tilde{\mathbf{H}}_2^n$,

$$\tilde{\mathbf{H}}_2^n = (\mathbf{K} + \tilde{\mathbf{D}}_2)^n = \sum_{m=0}^{n} \left[\frac{1}{m!} \left(\frac{\partial^m}{\partial \kappa^m} \mathbf{K}^n \right) \tilde{\mathbf{D}}_2^m \right], \quad \text{(A-2a)}$$

where

$$\mathbf{K} = \kappa \mathbf{I}, \quad \text{(A-2b)}$$

$$\tilde{\mathbf{D}}_2 = \tilde{D}_2 \mathbf{I}, \quad \text{(A-2c)}$$

$$\kappa = k^2, \quad \text{(A-2d)}$$

and

$$\tilde{D}_2 = (-k_x^2 - k_y^2). \quad \text{(A-2e)}$$

Substituting equation (A-2a) in equation (A-1a), changing the order of summations, and using the property $\partial_\kappa^m \kappa^n = 0$ for $m > n$ yield

$$\tilde{\mathbf{W}}_\ell(z, z_0) = \sum_{m=0}^{\infty} \left[\frac{1}{m!} \left(\frac{\partial^m}{\partial \kappa^m} \mathbf{M} + \tilde{\mathbf{A}}_\ell \frac{\partial^m}{\partial \kappa^m} \mathbf{N} \right) \tilde{\mathbf{D}}_2^m \right], \quad \text{(A-3a)}$$

where

$$\mathbf{M} = \sum_{n=0}^{\infty} \left[a_n \mathbf{K}^n \right] \quad \text{(A-3b)}$$

and

$$\mathbf{N} = \sum_{n=0}^{\infty} \left[b_n \mathbf{K}^n \right]. \quad \text{(A-3c)}$$

Notice that the infinite series (A-3b)–(A-3c) can be replaced by closed expressions, according to

$$\mathbf{M} = W_{\mathrm{I},0} \mathbf{I}, \quad \text{(A-4a)}$$

and

$$\mathbf{N} = (i\omega\rho)^{-1} W_{\mathrm{II},0} \mathbf{I}, \quad \text{(A-4b)}$$

where

$$W_{\mathrm{I},0} = \cos(k\Delta z) \quad \text{(A-4c)}$$

and

$$W_{\text{II},0} = \frac{i\omega\rho}{k} \sin(k\Delta z). \quad \text{(A-4d)}$$

Notice that $W_{\text{I},0}$ and $W_{\text{II},0}$ equal the operators W_{I} and W_{II}, respectively, given by equations (16b) and (16c) for a horizontal plane wave, that is, for $k_x^2 = k_y^2 = 0$.

By substituting equation (A-4) in equation (A-3), it follows that operator $\tilde{\mathbf{W}}_\ell(z, z_0)$ is given by

$$\tilde{\mathbf{W}}_\ell(z, z_0) = \begin{bmatrix} \tilde{W}_{\text{I}}(z, z_0) & \tilde{W}_{\text{II}}(z, z_0) \\ \tilde{W}_{\text{III}}(z, z_0) & \tilde{W}_{\text{IV}}(z, z_0) \end{bmatrix}, \quad \text{(A-5a)}$$

where

$$\tilde{W}_{\text{I}}(z, z_0) = \sum_{m=0}^{\infty} \alpha_m \tilde{D}_2^m, \quad \text{(A-5b)}$$

$$\tilde{W}_{\text{II}}(z, z_0) = \sum_{m=0}^{\infty} \beta_m \tilde{D}_2^m, \quad \text{(A-5c)}$$

$$\tilde{W}_{\text{III}}(z, z_0) = \frac{1}{(\omega\rho)^2} \tilde{H}_2 \tilde{W}_{\text{II}}(z, z_0), \quad \text{(A-5d)}$$

and

$$\tilde{W}_{\text{IV}}(z, z_0) = \tilde{W}_{\text{I}}(z, z_0) \quad \text{(A-5e)}$$

with

$$\alpha_m = \frac{1}{m!}\left(\frac{\partial^m}{\partial \kappa^m} W_{\text{I},0}\right), \quad \text{(A-5f)}$$

and

$$\beta_m = \frac{1}{m!}\left(\frac{\partial^m}{\partial \kappa^m} W_{\text{II},0}\right). \quad \text{(A-5g)}$$

APPENDIX B

MATHEMATICS OF THE FULL ELASTIC TWO-WAY WAVE EQUATION

By eliminating $\mathbf{T}_x$ and $\mathbf{T}_y$ from equations (20) and (21), we obtain the following set of coupled equations:

$$\frac{\partial V_z}{\partial z} = \frac{1}{\lambda + 2\mu}\left(i\omega T_{zz} - \lambda \frac{\partial V_x}{\partial x} - \lambda \frac{\partial V_y}{\partial y}\right), \quad \text{(B-1a)}$$

$$\frac{\partial T_{zx}}{\partial z} = -\frac{1}{i\omega}\left[\frac{\partial}{\partial x}\left(i\omega\chi_1 T_{zz} + 4\mu\chi_2 \frac{\partial V_x}{\partial x} + 2\mu\chi_1 \frac{\partial V_y}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial V_x}{\partial y} + \mu \frac{\partial V_y}{\partial x}\right) + \omega^2 \rho V_x\right], \quad \text{(B-1b)}$$

$$\frac{\partial T_{zy}}{\partial z} = -\frac{1}{i\omega}\left[\frac{\partial}{\partial y}\left(i\omega\chi_1 T_{zz} + 4\mu\chi_2 \frac{\partial V_y}{\partial y} + 2\mu\chi_1 \frac{\partial V_x}{\partial x}\right) + \frac{\partial}{\partial x}\left(\mu \frac{\partial V_y}{\partial x} + \mu \frac{\partial V_x}{\partial y}\right) + \omega^2 \rho V_y\right], \quad \text{(B-1c)}$$

$$\frac{\partial T_{zz}}{\partial z} = i\omega\rho V_z - \frac{\partial T_{zx}}{\partial x} - \frac{\partial T_{zy}}{\partial y}, \quad \text{(B-1d)}$$

$$\frac{\partial V_x}{\partial z} = -\frac{\partial V_z}{\partial x} + \frac{i\omega}{\mu} T_{zx}, \quad \text{(B-1e)}$$

and

$$\frac{\partial V_y}{\partial z} = -\frac{\partial V_z}{\partial y} + \frac{i\omega}{\mu} T_{zy}, \quad \text{(B-1f)}$$

with

$$\chi_1 = \frac{\lambda}{\lambda + 2\mu} \quad \text{(B-1g)}$$

and

$$\chi_2 = \frac{\lambda + \mu}{\lambda + 2\mu}. \quad \text{(B-1h)}$$

So far no approximations have been made; hence the medium parameters (λ, μ, ρ) may be arbitrary functions of the space coordinates (x, y, z).

Next we combine the above set of coupled equations in one matrix equation, and we replace the lateral derivatives by convolution operators. For simplicity we consider waves with polarization in the vertical plane ($V_y = 0$ and $T_{zy} = 0$) in 2-D inhomogeneous media. We further assume that the lateral derivatives of the medium parameters λ, μ, and ρ may be neglected. We thus obtain

$$\frac{\partial \mathbf{Q}_s}{\partial z} = \mathbf{A}_s \mathbf{Q}_s, \quad \text{(B-2a)}$$

where

$$\mathbf{A}_s = \begin{bmatrix} \mathbf{O} & \mathbf{A}_1 \\ \mathbf{A}_2 & \mathbf{O} \end{bmatrix}, \quad \text{(B-2b)}$$

$$\mathbf{Q}_s = \begin{bmatrix} V_z \\ T_{zx} \\ T_{zz} \\ V_x \end{bmatrix}, \quad \text{(B-2c)}$$

with

$$\mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{(B-2d)}$$

$$\mathbf{A}_1 = \begin{bmatrix} \left(\frac{i\omega}{\lambda + 2\mu}\right) d_0(x) * & -\left(\frac{\lambda}{\lambda + 2\mu}\right) d_1(x) * \\ -\left(\frac{\lambda}{\lambda + 2\mu}\right) d_1(x) * & \left\{ i\omega\rho d_0(x) - \frac{4\mu}{i\omega}\left(\frac{\lambda + \mu}{\lambda + 2\mu}\right) d_2(x) \right\} * \end{bmatrix}, \quad \text{(B-2e)}$$

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$$\underset{\sim}{\mathbf{A}}_2 = \begin{bmatrix} i\omega\rho d_0(x)* & -d_1(x)* \\ -d_1(x)* & \frac{i\omega}{\mu} d_0(x)* \end{bmatrix}. \tag{B-2f}$$

Operator $\underset{\sim}{\mathbf{A}}_s$ can be decomposed as follows

$$\underset{\sim}{\mathbf{A}}_s = \underset{\sim}{\mathbf{L}}_s \underset{\sim}{\mathbf{\Lambda}}_s \underset{\sim}{\mathbf{L}}_s^{-1}, \tag{B-3a}$$

where

$$\underset{\sim}{\mathbf{L}}_s = \begin{bmatrix} \underset{\sim}{\mathbf{L}}_1 & \underset{\sim}{\mathbf{L}}_1 \\ \underset{\sim}{\mathbf{L}}_2 & -\underset{\sim}{\mathbf{L}}_2 \end{bmatrix}, \tag{B-3b}$$

$$\underset{\sim}{\mathbf{\Lambda}}_s = \begin{bmatrix} \underset{\sim}{\mathbf{\Lambda}} & \underset{\sim}{\mathbf{O}} \\ \underset{\sim}{\mathbf{O}} & -\underset{\sim}{\mathbf{\Lambda}} \end{bmatrix} \tag{B-3c}$$

and

$$\underset{\sim}{\mathbf{L}}_s^{-1} = \frac{1}{2}\begin{bmatrix} \underset{\sim}{\mathbf{L}}_1^{-1} & \underset{\sim}{\mathbf{L}}_2^{-1} \\ \underset{\sim}{\mathbf{L}}_1^{-1} & -\underset{\sim}{\mathbf{L}}_2^{-1} \end{bmatrix}, \tag{B-3d}$$

with

$$\underset{\sim}{\mathbf{L}}_1 = \begin{bmatrix} -iH_1^{(P)}* & d_1(x)* \\ -2\frac{\mu}{\omega} d_1(x)*H_1^{(P)}* & -\left[i\omega\rho d_0(x) + 2i\frac{\mu}{\omega} d_2(x)\right]* \end{bmatrix}, \tag{B-3e}$$

$$\underset{\sim}{\mathbf{L}}_2 = \begin{bmatrix} \left[i\omega\rho d_0(x) + 2i\frac{\mu}{\omega} d_2(x)\right]* & -2\frac{\mu}{\omega} d_1(x)*H_1^{(SV)}* \\ d_1(x)* & iH_1^{(SV)}* \end{bmatrix}, \tag{B-3f}$$

$$\underset{\sim}{\mathbf{\Lambda}} = \begin{bmatrix} -iH_1^{(P)}* & 0 \\ 0 & -iH_1^{(SV)}* \end{bmatrix}, \tag{B-3g}$$

$$\underset{\sim}{\mathbf{L}}_1^{-1} = \frac{1}{i\omega\rho}\underset{\sim}{\mathbf{\Lambda}}^{-1}\underset{\sim}{\mathbf{L}}_2^T, \tag{B-3h}$$

$$\underset{\sim}{\mathbf{L}}_2^{-1} = \frac{1}{i\omega\rho}\underset{\sim}{\mathbf{\Lambda}}^{-1}\underset{\sim}{\mathbf{L}}_1^T, \tag{B-3i}$$

$$\underset{\sim}{\mathbf{\Lambda}}^{-1} = \begin{bmatrix} iH_{-1}^{(P)}* & 0 \\ 0 & iH_{-1}^{(SV)}* \end{bmatrix}, \tag{B-3j}$$

$$H_{-1}^{(P)} * H_1^{(P)} = \delta(x), \tag{B-3k}$$

$$H_1^{(P)} * H_1^{(P)} = H_2^{(P)}, \tag{B-3l}$$

$$H_2^{(P)} = \left(\frac{\omega^2\rho}{\lambda + 2\mu}\right) d_0(x) + d_2(x), \tag{B-3m}$$

$$H_{-1}^{(SV)} * H_1^{(SV)} = \delta(x), \tag{B-3n}$$

$$H_1^{(SV)} * H_1^{(SV)} = H_2^{(SV)}, \tag{B-3p}$$

and

$$H_2^{(SV)} = \left(\frac{\omega^2\rho}{\mu}\right) d_0(x) + d_2(x). \tag{B-3q}$$

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