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Short Note
Representation of seismic sources in the one-way wave equations

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INTRODUCTION

One-way extrapolation of downgoing and upgoing acoustic waves plays an essential role in the current practice of seismic migration (Berkhout, 1985; Stolt and Benson, 1986; Claerbout, 1985; Gardner, 1985). Generally, one-way wave equations are derived for the source-free situation. Sources are then represented as boundary conditions for the one-way extrapolation problem. This approach is valid provided the source representation is done with utmost care. For instance, it is not correct to represent a monopole source by a spatial delta function and to use this as input data for a standard one-way extrapolation scheme. This yields an erroneous directivity pattern as illustrated below.

One-way wave equations are derived below that include a source term. From these equations it becomes clear how seismic sources can be properly represented in one-way extrapolation schemes. An important application is pre-stack migration. Particularly when one tries to image the *angle-dependent* properties of subsurface reflectors, a proper treatment of source directivity is essential. This is also illustrated.

ACOUSTIC TWO-WAY WAVE EQUATION

For an inhomogeneous fluid, the linearized equations of continuity and motion in the frequency domain read

$$\frac{i\omega}{K} P + \nabla \cdot \mathbf{V} = i\omega I \tag{1a}$$

and

$$i\omega\rho \, \mathbf{V} + \nabla P = \mathbf{F}, \tag{1b}$$

where $P(\mathbf{r},\omega)$ is acoustic pressure, $\mathbf{V}(\mathbf{r},\omega)$ is particle velocity, $K(\mathbf{r})$ is adiabatic compression modulus, $\rho(\mathbf{r})$ is volume density of mass, $I(\mathbf{r},\omega)$ is volume density of volume injection, $\mathbf{F}(\mathbf{r},\omega)$ is volume density of external force, $\mathbf{r}$ is Cartesian coordinate vector (x,y,z), and ω is circular frequency.

Eliminating $\mathbf{V}$ from equations (1a) and (1b) and separating the vertical derivatives from the horizontal derivatives yields the two-way wave equation

$$\rho \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial P}{\partial z}\right) = -\mathcal{H}_2 P - S, \tag{2a}$$

where the second derivative operator $\mathcal{H}_2(\mathbf{r},\omega)$ is defined as

$$\mathcal{H}_2 = \frac{\omega^2\rho}{K} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial \ln\rho}{\partial x}\frac{\partial}{\partial x} - \frac{\partial \ln\rho}{\partial y}\frac{\partial}{\partial y}, \tag{2b}$$

and where the source term $S(\mathbf{r},\omega)$ is defined as

$$S = -\omega^2\rho \, I - \rho\nabla \cdot \left(\frac{1}{\rho}\mathbf{F}\right). \tag{2c}$$

ACOUSTIC ONE-WAY WAVE EQUATIONS

Define the first derivative operator $\mathcal{H}_1(\mathbf{r},\omega)$ and its inverse $\mathcal{H}_{-1}(\mathbf{r},\omega)$, such that

$$\mathcal{H}_2 P = \mathcal{H}_1\mathcal{H}_1 P \tag{3a}$$

and

$$P = \mathcal{H}_1\mathcal{H}_{-1} P. \tag{3b}$$

This formal operator notation is explained in the Appendix with the aid of generalized spatial convolution integrals. Operator $\mathcal{H}_1(\mathbf{r},\omega)$ represents the well-known square root operator in its general form. Define downgoing waves $P^+(\mathbf{r},\omega)$ and upgoing waves $P^-(\mathbf{r},\omega)$ that satisfy

$$P = P^+ + P^-. \tag{3c}$$

For a homogeneous source-free medium the following one-way wave equations apply:

$$\frac{\partial P^+}{\partial z} = -i\mathcal{H}_1 P^+ \tag{4a}$$

and

$$\frac{\partial P^-}{\partial z} = +i\mathcal{H}_1 P^-. \tag{4b}$$

This is easily verified by adding both equations and differentiating both sides of the resulting equation with respect to z, yielding

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$$\frac{\partial^2(P^+ + P^-)}{\partial z^2} = -i\mathcal{H}_1\left(\frac{\partial P^+}{\partial z} - \frac{\partial P^-}{\partial z}\right), \tag{5a}$$

or, upon substitution of one-way wave equations (4a) and (4b),

$$\frac{\partial^2(P^+ + P^-)}{\partial z^2} = -\mathcal{H}_1\mathcal{H}_1(P^+ + P^-), \tag{5b}$$

or, with definitions (3a) and (3c),

$$\frac{\partial^2 P}{\partial z^2} = -\mathcal{H}_2 P, \tag{5c}$$

which represents two-way wave equation (2a) for a homogeneous source-free medium. For an inhomogeneous medium with sources, the following coupled system of one-way wave equations applies:

$$\frac{\partial P^+}{\partial z} = -i\mathcal{H}_1 P^+ - \frac{1}{2}\mathcal{H}_{-1}\left[\rho\frac{\partial}{\partial z}\left(\frac{1}{\rho}\mathcal{H}_1\right)(P^+ - P^-)\right] + S^+ \tag{6a}$$

and

$$\frac{\partial P^-}{\partial z} = +i\mathcal{H}_1 P^- + \frac{1}{2}\mathcal{H}_{-1}\left[\rho\frac{\partial}{\partial z}\left(\frac{1}{\rho}\mathcal{H}_1\right)(P^+ - P^-)\right] - S^-; \tag{6b}$$

where

$$S^+ = \frac{1}{2}i\mathcal{H}_{-1}\left[\omega^2\rho I + \rho\frac{\partial}{\partial x}\left(\frac{F_x}{\rho}\right) + \rho\frac{\partial}{\partial y}\left(\frac{F_y}{\rho}\right)\right] + \frac{1}{2}F_z \tag{7a}$$

and

$$S^- = \frac{1}{2}i\mathcal{H}_{-1}\left[\omega^2\rho I + \rho\frac{\partial}{\partial x}\left(\frac{F_x}{\rho}\right) + \rho\frac{\partial}{\partial y}\left(\frac{F_y}{\rho}\right)\right] - \frac{1}{2}F_z. \tag{7b}$$

These results can be verified as above. $S^+(\mathbf{r},\omega)$ and $S^-(\mathbf{r},\omega)$ are the one-way representations of the source functions $I(\mathbf{r},\omega)$ and $\mathbf{F}(\mathbf{r},\omega)$. For a source function in terms of a volume density of vertical force $F_z(\mathbf{r},\omega)$ a very simple expression for $S^+(\mathbf{r},\omega)$ and $S^-(\mathbf{r},\omega)$ is obtained:

$$S^{\pm}(\mathbf{r},\omega) = \pm\frac{1}{2}F_z(\mathbf{r},\omega). \tag{8a}$$

On the other hand, for a source function in terms of a volume density of volume injection, $I(\mathbf{r},\omega)$ (e.g., an air gun) the expressions for $S^+(\mathbf{r},\omega)$ and $S^-(\mathbf{r},\omega)$ are not trivial. In the integral notation of the Appendix, expressions (7a) and (7b) read for this situation

$$S^{\pm}(\mathbf{r},\omega) = \frac{1}{2}i\omega^2\iint_{-\infty}^{\infty}\left[H_{-1}(\mathbf{r},\mathbf{r}',\omega)\rho(\mathbf{r}')\,I(\mathbf{r}',\omega)\right]_{z'=z}\mathrm{d}x'\,\mathrm{d}y'. \tag{8b}$$

This is the main result of this paper. This equation states that the one-way representation of a volume injection source is obtained by spatially convolving the source function with the inverse square root operator. This is generally ignored in prestack migration.

SOLUTION OF ACOUSTIC ONE-WAY WAVE EQUATIONS

Equations (6a) and (6b) decouple by neglecting $P^-(\mathbf{r},\omega)$ with respect to $P^+(\mathbf{r},\omega)$ in equation (6a) for downward propagation and by neglecting $P^+(\mathbf{r},\omega)$ with respect to $P^-(\mathbf{r},\omega)$ in equation (6b) for upward propagation. This means that in both equations multiple reflections are reflected. Hence, primary waves fulfill the decoupled one-way wave equations

$$\frac{\partial P^{\pm}}{\partial z} \approx \mp i\mathcal{H}_1^{\pm} P^{\pm} \pm S^{\pm} \tag{9a}$$

where

$$i\mathcal{H}_1^{\pm} = i\mathcal{H}_1 \pm \frac{1}{2}\mathcal{H}_{-1}\left[\rho\frac{\partial}{\partial z}\left(\frac{1}{\rho}\mathcal{H}_1\right)\right]. \tag{9b}$$

For the source-free situation the formal solution reads

$$P^{\pm}(x,y,z,\omega) \approx \mathcal{W}^{\pm}(x,y,z,z_0,\omega)\,P^{\pm}(x,y,z_0,\omega), \tag{10a}$$

or, in the integral notation of the Appendix,

$$P^{\pm}(\mathbf{r},\omega) \approx \iint_{-\infty}^{+\infty}\left[W^{\pm}(\mathbf{r},\mathbf{r}',\omega)\,P^{\pm}(\mathbf{r}',\omega)\right]_{z'=z_0}\mathrm{d}x'\,\mathrm{d}y' \tag{10b}$$

The one-way wave-field extrapolation operator $W^{\pm}$ is discussed by numerous authors and is therefore not derived in this paper. The formal solution of one-way wave equation (9), including the source term, reads

$$P^{\pm}(x,y,z,\omega) \approx \mathcal{W}^{\pm}(x,y,z,z_0,\omega)\,P^{\pm}(x,y,z_0,\omega) \pm \int_{z_0}^{z}\mathcal{W}^{\pm}(x,y,z,z',\omega)\,S^{\pm}(x,y,z',\omega)\,\mathrm{d}z', \tag{11a}$$

or, in the integral notation of the Appendix,

$$P^{\pm}(\mathbf{r},\omega) \approx \iint_{-\infty}^{+\infty}\left[W^{\pm}(\mathbf{r},\mathbf{r}',\omega)\,P^{\pm}(\mathbf{r}',\omega)\right]_{z'=z_0}\mathrm{d}x'\,\mathrm{d}y' \pm \int_{z_0}^{z}\left[\iint_{-\infty}^{+\infty}\left[W^{\pm}(\mathbf{r},\mathbf{r}',\omega)\,S^{\pm}(\mathbf{r}',\omega)\right]\mathrm{d}x'\,\mathrm{d}y'\right]\mathrm{d}z'. \tag{11b}$$

This equation describes one-way wave-field extrapolation to a depth level z in terms of a surface integral over the one-way wave field at z_0 and a volume integral over the one-way sources between z_0 and z. We illustrate these results with some simple 2-D examples. $\mathbf{r}$ denotes the 2-D Cartesian coordinate vector (x,z).

EXAMPLES

Example 1: Dipole source wave-field extrapolation

Consider a line source of vertical force at the origin ($\mathbf{r} = \mathbf{0}$) of an unbounded homogeneous fluid (propagation velocity: $c = 2000$ m/s), according to

$$F_z(\mathbf{r},\omega) = \delta(\mathbf{r})\,S_0(\omega), \tag{12a}$$

or, according to equation (8a),

$$S^{\pm}(\mathbf{r},\omega) = \pm\frac{1}{2}\delta(\mathbf{r})\,S_0(\omega). \tag{12b}$$

This one-way source representation is shown for $z = 0$ in the space-time domain (x,t) in Figure 1a. Choose depth level z_0 in the upper half-space ($z_0 < 0$) and assume that there is no downgoing wave field $P^+(\mathbf{r},\omega)$ at z_0. Then, in analogy with equation (11b), the expression for 2-D downward extrapolation of the source wave reads

$$P^+(\mathbf{r},\omega) = \frac{1}{2}\int_{z_0}^{z}\left[\int_{-\infty}^{+\infty}\left[W^+(\mathbf{r},\mathbf{r}',\omega)\,\delta(\mathbf{r}')\,S_0(\omega)\right]\mathrm{d}x'\right]\mathrm{d}z'. \tag{13a}$$

Here W^+ represents a 2-D operator for one-way wave-field extrapolation. Therefore we omitted the integral along the y'-axis. Equation (13a) yields

$$P^+(\mathbf{r},\omega) = 0 \qquad \text{for } z_0 \le z < 0 \tag{13b}$$

and

$$P^+(\mathbf{r},\omega) = \frac{1}{2}W^+(\mathbf{r},\mathbf{0},\omega)\,S_0(\omega) \qquad \text{for } z > 0. \tag{13c}$$

This one-way wave-field extrapolation result is shown for $z = 400$ m in the space-time domain (x,t) in Figure 1b. The maximum of each trace is shown as a function of x in Figure 1c. The amplitude shows the behavior of a *dipole* source function.

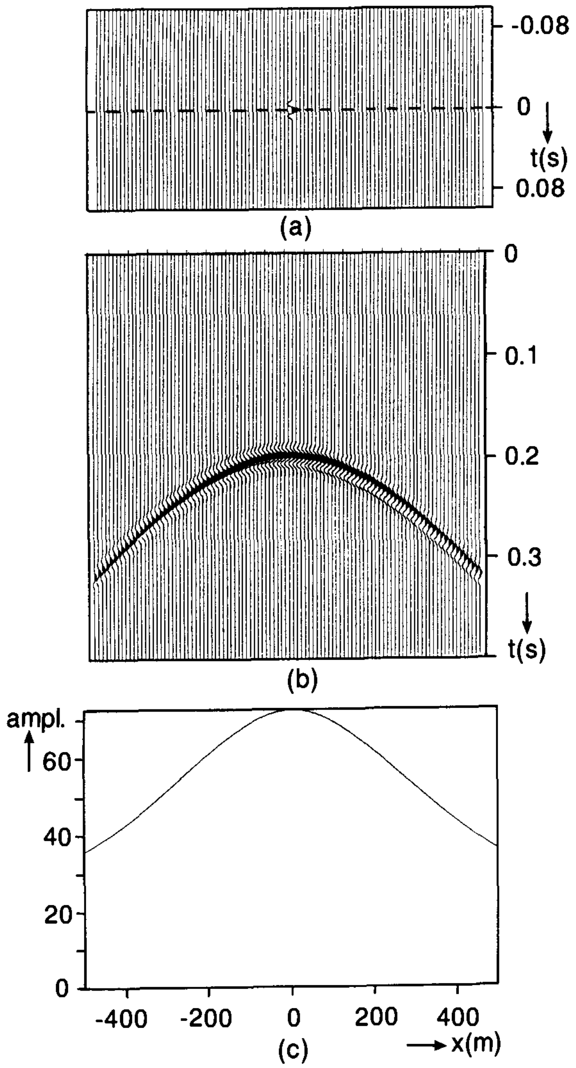

FIG. 1. Dipole source wave-field extrapolation. (a) One-way representation of a band-limited source of vertical force at $\mathbf{r} = \mathbf{0}$. (b) Downgoing wave field at $z = 400$ m obtained by applying one-way wave-field extrapolation to the data in (a). (c) Amplitude cross-section of (b).

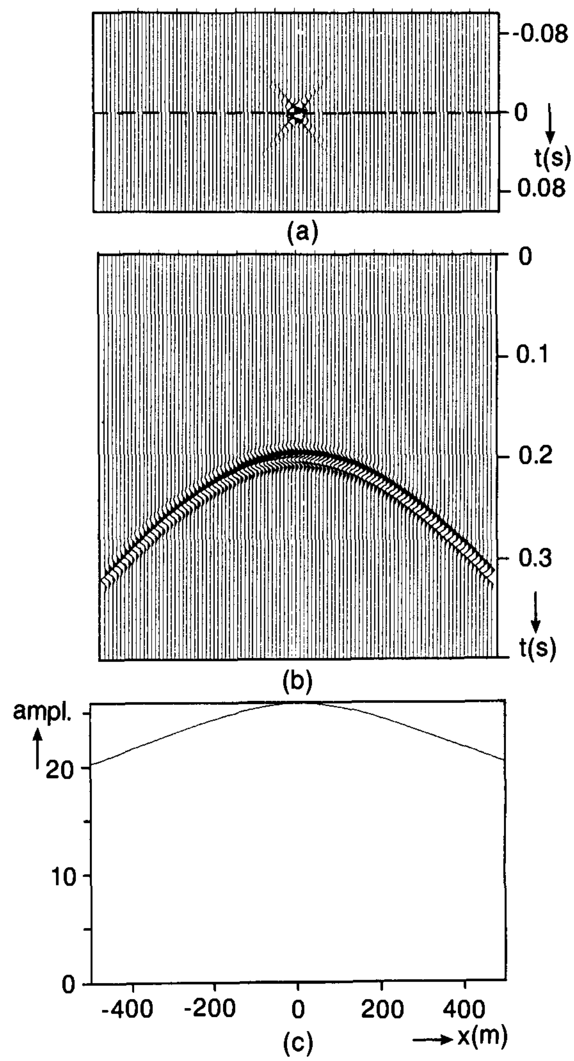

FIG. 2. Monopole source wave-field extrapolation. (a) One-way representation of a band-limited source of volume injection at $\mathbf{r} = \mathbf{0}$. (b) Downgoing wave field at $z = 400$ m obtained by applying one-way wave-field extrapolation to the data in (a). (c) Amplitude cross-section of (b).

Example 2: Monopole source wave-field extrapolation

Consider a line source of volume injection (e.g., an air gun) at the origin ($\mathbf{r} = \mathbf{0}$) of an unbounded homogeneous fluid, according to

$$I(\mathbf{r},\omega) = \delta(\mathbf{r})\, S_0(\omega) \ . \tag{14a}$$

Suppose we apply one-way wave-field extrapolation directly to this spatial delta function. Then the results would be very similar to Figure 1, which is obviously wrong for the monopole source considered here. According to the 2-D version of equation (8b), the correct source representation is given by

$$S^{\pm}(\mathbf{r},\omega) = \frac{1}{2}\, i\omega^2\rho\, [H_{-1}(\mathbf{r},\mathbf{0},\omega)]_{z=0}\delta(z)\, S_0(\omega) \ . \tag{14b}$$

This one-way source representation is shown for $z = 0$ in the space-time domain (x,t) in Figure 2a. Note that the spatial delta function is smeared due to convolution with the inverse square root operator. Choose depth level z_0 in the upper half-space ($z_0 < 0$) and assume that there is no downgoing wave field $P^{+}(\mathbf{r},\omega)$ at z_0. Then, in analogy with equation (11b), the expression for 2-D downward extrapolation of the source wave reads

$$P^{+}(\mathbf{r},\omega) = 0 \qquad \text{for } z_0 \le z < 0 \tag{15a}$$

and

$$P^{+}(\mathbf{r},\omega) = \frac{1}{2}\, i\omega^2\rho S_0(\omega) \int_{-\infty}^{\infty} [W^{+}(\mathbf{r},\mathbf{r}',\omega) H_{-1}(\mathbf{r}',\mathbf{0},\omega)]_{z'=0}\, dx' . \tag{15b}$$

for $z > 0$. This one-way wave field extrapolation result is shown for $z = 400$ m in the space-time domain (x,t) in Figure 2b. The maximum of each trace is shown as a function of x in Figure 2c. The amplitude shows the behavior of a *monopole* source function.

Example 3: Angle-dependent reflectivity imaging

Consider the simple 2-D subsurface configuration shown in Figure 3a. The model contains one reflector between two homogeneous half-spaces. Since there is density contrast only, the reflectivity function $R(\alpha)$ is independent of the angle α:

$$R(\alpha) = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} = \frac{3000 - 1000}{3000 + 1000} = 0.5 \,. \tag{16}$$

The reflection response related to a monopole source at ($x = 0$, $z = 0$) is shown in the space-time domain (x,t) in Figure 3b. Two prestack migration experiments were carried out, aimed at imaging the reflectivity as a function of angle (De Bruin et al., 1990). In the first migration experiment the monopole source was erroneously represented by the spatial delta function of Figure 1a. The migration result is shown in Figure 4a. It represents the reflectivity as a function of depth z for different values of the incidence angle α. Figure 4b shows the imaged reflectivity at the reflector depth as a function of α. Note that it deviates significantly from $R(\alpha) = 0.5$. This could be expected since the monopole source representation was not correct. In the second migration experiment, the monopole source was correctly represented by the smeared delta function of Figure 2a. The results are shown in Figure 5. Note that the imaged reflectivity is constant (as it should be) up to high incidence angles. The deviations for $\alpha \rightarrow 90$ degrees are due to the limited aperture.

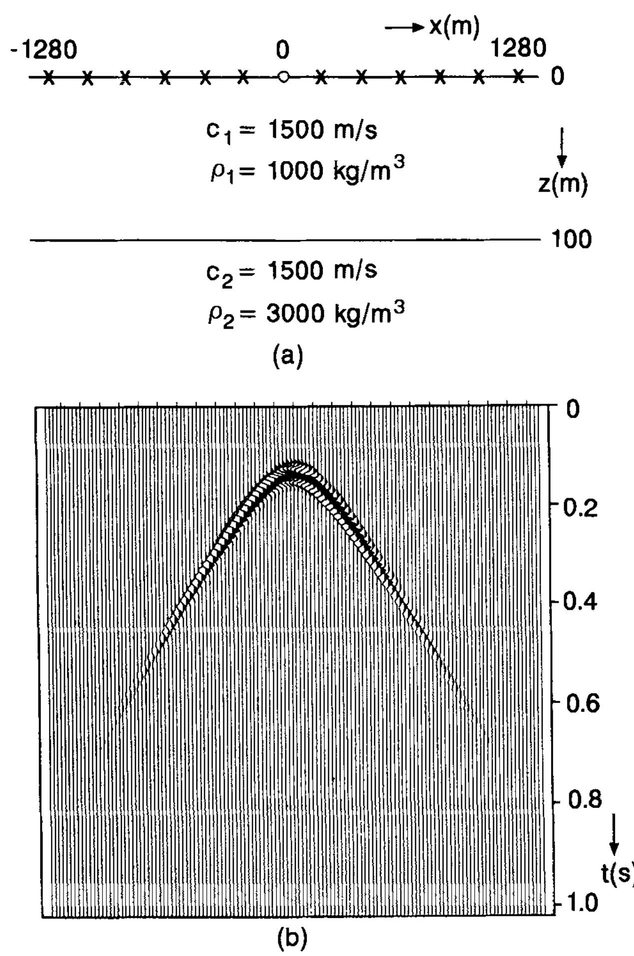

FIG. 3. Model for evaluating angle-dependent reflectivity imaging. (a) Subsurface configuration with a density contrast at z = 100 m, $R(\alpha) = 0.5$. (b) One-shot record for a monopole source at ($x = 0$, $z = 0$).

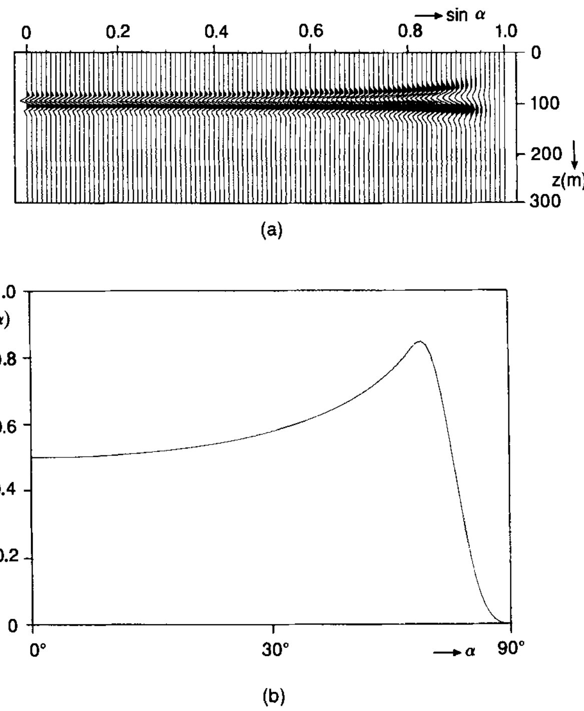

FIG. 4. Prestack migration with an erroneous source representation (Figure 1a). (a) Angle-dependent reflectivity image. Each trace shows the reflectivity as a function of depth for a specific incidence angle. (b) Angle-dependent reflectivity at z = 100 m. This result was retrieved from (a) after envelope detection. Note that this result deviates significantly from $R(\alpha) = 0.5$.

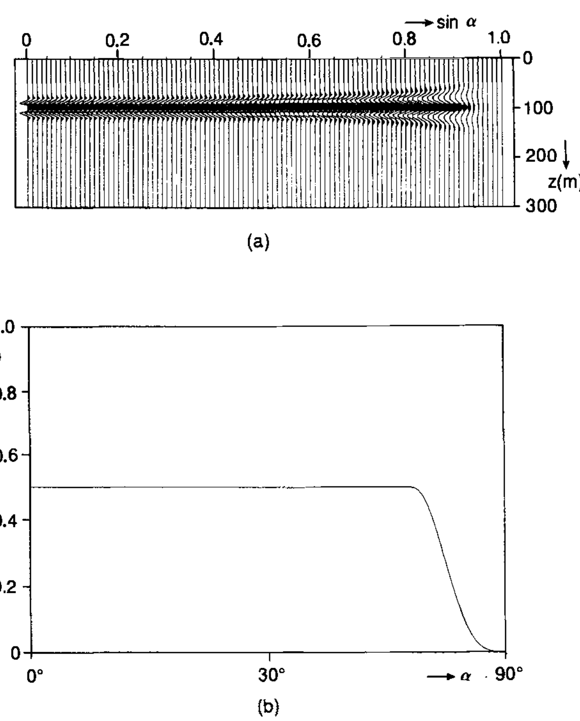

FIG. 5. Prestack migration with the correct source representation (Figure 2a). (a) Angle-dependent reflectivity image. (b) Angle-dependent reflectivity at z = 100 m. Note that $R(\alpha)$ = 0.5 up to high incidence angles.

CONCLUSIONS

In seismic literature one-way wave equations for downgoing and upgoing waves are generally derived for the source-free situation. In this paper, one-way wave equations are derived, including one-way representations of the sources. For sources of the volume injection type (like air guns), one-way representations are obtained by spatially convolving the source function with the inverse square root operator [equation (8b)]. The effect of this spatial convolution applied to a monopole source of volume injection is a lateral smearing (Figure 2a). This smearing appears to be necessary, so that after standard one-way wave-field extrapolation the true monopole response is obtained (Figures 2b and 2c). Ignoring the smearing effect of the inverse square root operator would result in a dipole-like response (as in Figures 1b and 1c), which is obviously wrong for a monopole source of the volume injection type. Figures 3, 4, and 5 show that a correct source representation is essential for angle-dependent relectivity imaging by prestack migration. In this paper I only considered the acoustic situation. A derivation of the seismic source representation in *elastic* one-way wave equations can be found in Wapenaar and Berkhout (1989).

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APPENDIX

GENERALIZED SPATIAL CONVOLUTION INTEGRALS

Differentiations with respect to x and y can be written as conventional spatial convolution integrals, according to

$$\frac{\partial^m P(x,y,z,\omega)}{\partial x^m} = \int_{-\infty}^{\infty} d_m(x-x')\, P(x',y,z,\omega)\, \mathrm{d}x' \quad \text{(A-1a)}$$

and

$$\frac{\partial^m P(x,y,z,\omega)}{\partial y^m} = \int_{-\infty}^{\infty} d_m(y-y')\, P(x,y',z,\omega)\, \mathrm{d}y' , \quad \text{(A-1b)}$$

with

$$d_m(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{d}_m(k_x)\, e^{-ik_x x}\, \mathrm{d}k_x \quad \text{(A-1c)}$$

and

$$d_m(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{d}_m(k_y)\, e^{-ik_y y}\, \mathrm{d}k_y , \quad \text{(A-1d)}$$

where $\tilde{d}_m(k_x)$ and $\tilde{d}_m(k_y)$ represent properly chosen band-limited versions of $(-ik_x)^m$ and $(-ik_y)^m$, respectively (Berkhout, 1985). Equations (A-1a) and (A-1b) are exact when $P(x,y,z,\omega)$ is a spatially band-limited function. With these definitions, the operation $\mathcal{H}_2 P$ in wave equation (2) can be written as a generalized spatial convolution integral, according to

$$\mathcal{H}_2(\mathbf{r},\omega)\, P(\mathbf{r},\omega) = \iint_{-\infty}^{\infty} [H_2(\mathbf{r},\mathbf{r}',\omega)\, P(\mathbf{r}',\omega)]_{z'=z}\, \mathrm{d}x'\mathrm{d}y' , \quad \text{(A-2a)}$$

where

$$H_2(\mathbf{r},\mathbf{r}',\omega) = \frac{\omega^2\rho(\mathbf{r})}{K(\mathbf{r})}\,\delta(x-x')\,\delta(y-y') + d_2(x-x')\,\delta(y-y') + \delta(x-x')\,d_2(y-y') - \frac{\partial \ln\rho(\mathbf{r})}{\partial x}\, d_1(x-x')\,\delta(y-y') - \frac{\partial \ln\rho(\mathbf{r})}{\partial y}\,\delta(x-x')\,d_1(y-y') . \quad \text{(A-2b)}$$

Similarly, the operation $\mathcal{H}_1 P$ in equation (3a) can be written as

$$\mathcal{H}_1(\mathbf{r},\omega)\, P(\mathbf{r},\omega) = \iint_{-\infty}^{\infty} [H_1(\mathbf{r},\mathbf{r}',\omega)\, P(\mathbf{r}',\omega)]_{z'=z}\mathrm{d}x'\mathrm{d}y' , \quad \text{(A-3a)}$$

with H_1 defined implicitly by

$$H_2(\mathbf{r},\mathbf{r}'',\omega) = \iint_{-\infty}^{\infty} H_1(\mathbf{r},\mathbf{r}',\omega)\, H_1(\mathbf{r}',\mathbf{r}'',\omega)\, \mathrm{d}x'\mathrm{d}y' . \quad \text{(A-3b)}$$

Finally, the operation $\mathcal{H}_{-1}P$ in equation (3b) can be written as

$$\mathcal{H}_{-1}(\mathbf{r},\omega)\, P(\mathbf{r},\omega) = \iint_{-\infty}^{\infty} [H_{-1}(\mathbf{r},\mathbf{r}',\omega)\, P(\mathbf{r}',\omega)]_{z'=z}\mathrm{d}x'\mathrm{d}y' , \quad \text{(A-4a)}$$

with H_{-1} defined implicitly by

$$\delta(x-x'')\delta(y-y'') = \iint_{-\infty}^{\infty} H_1(\mathbf{r},\mathbf{r}',\omega)\, H_{-1}(\mathbf{r}',\mathbf{r}'',\omega)\, \mathrm{d}x'\mathrm{d}y' . \quad \text{(A-4b)}$$