A unified optical theorem for scalar and vectorial wave fields

Kees Wapenaar[a)]
Department of Geoscience and Engineering, Delft University of Technology, P.O. Box 5048, 2600 GA Delft, The Netherlands

Huub Douma
ION Geophysical, GXT Imaging Solutions, 225 East 16th Avenue, Suite 1200, Denver Colorado 80203

(Received 29 November 2011; revised 19 March 2012; accepted 21 March 2012)

The generalized optical theorem is an integral relation for the angle-dependent scattering amplitude of an inhomogeneous scattering object embedded in a homogeneous background. It has been derived separately for several scalar and vectorial wave phenomena. Here a unified optical theorem is derived that encompasses the separate versions for scalar and vectorial waves. Moreover, this unified theorem also holds for scattering by anisotropic elastic and piezoelectric scatterers as well as bianisotropic (non-reciprocal) EM scatterers. © *2012 Acoustical Society of America*. [http://dx.doi.org/10.1121/1.3701880]

PACS number(s): 43.20.Fn, 43.20.Gp [KML] Pages: 3611–3626

I. INTRODUCTION

The optical theorem finds its origin in the late nineteenth century, when Rayleigh[1] and others formulated the relation between the optical refraction index of a scattering object in a homogeneous embedding and its forward scattering amplitude. Later Heisenberg,[2] Glauber and Schomaker,[3] and others derived a more general theorem for the scattering amplitude in quantum mechanics and other scalar wave phenomena. This theorem, which has become known as the generalized optical theorem, is an integral relation for the scattering amplitude for any angle of incidence and any scattering angle. Both the optical theorem and the generalized optical theorem are a consequence of conservation of energy (or conservation of probability in quantum mechanics). For more extensive reviews, see Newton,[4] Marston,[5] and Carney *et al.*[6]

The generalized optical theorem is most often applied to scalar wave phenomena, but extensions for vectorial wave phenomena have been formulated as well. Snieder[7] and Halliday and Curtis[8,9] derive an optical theorem for multi-mode elastic surface waves in a layered medium bounded by a free surface. Tan,[10] de Hoop,[11] and Lu *et al.*[12] discuss the optical theorem for scattering of elastic body waves, and Torrungrueng *et al.*[13] and Lytle *et al.*[14] derive a version for electromagnetic waves.

It has recently been recognized that there is a close connection between the generalized optical theorem and the Green's function representations[15–17] that underlie the methodology of Green's function retrieval from ambient noise in open systems.[18–22] It has been shown that the optical theorem for scalar waves can be derived from the scalar Green's function representation,[23–27] whereas the optical theorems for surface waves and elastic body waves have been derived from elastodynamic Green's function representations for surface waves[8,9] and body waves,[12,28] respectively. Halliday and Curtis[9] and Douma *et al.*[26] suggested that a unified optical theorem for scalar and vectorial wave fields could possibly be derived from a unified Green's function representation.[29] The aim of this paper is to show that this is indeed the case. Starting with a unified wave equation for scalar and vectorial fields, unified Green's function representations are derived. Next, following a similar procedure as Douma *et al.*[26] for scalar wave fields, a unified optical theorem for scalar and vectorial wave fields is derived. This unified theorem captures most of the situations discussed above and in addition covers scattering by non-reciprocal materials and piezoelectric materials.

II. RECIPROCITY THEOREMS

The starting point is the following unified wave equation:[30–32]

$$\mathbf{A}\partial_t\mathbf{u} + \mathbf{B}\mathbf{u} + \mathbf{D_x}\mathbf{u} = \mathbf{s}, \tag{1}$$

in which $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is a $L \times 1$-vector containing space ($\mathbf{x}$) and time (t) dependent wave field quantities, $\mathbf{A} = \mathbf{A}(\mathbf{x})$ and $\mathbf{B} = \mathbf{B}(\mathbf{x})$ are $L \times L$ matrices containing space-dependent medium parameters, ∂_t denotes differentation with respect to time, $\mathbf{D_x}$ is a $L \times L$ matrix containing spatial differential operators ∂_1, ∂_2, ∂_3, and $\mathbf{s} = \mathbf{s}(\mathbf{x}, t)$ is a $L \times 1$ source vector. In Appendix A, these vectors and matrices are specified for acoustic waves (for which $L = 4$), quantum-mechanical waves ($L = 4$), electromagnetic waves in reciprocal and non-reciprocal materials ($L = 6$), elastodynamic body waves ($L = 9$) and coupled electromagnetic and elastodynamic waves in piezoelectric materials ($L = 15$). For all situations, matrix $\mathbf{D_x}$ obeys the following symmetry relations:

$$\mathbf{D_x} = \mathbf{D}_{\mathbf{x}}^T, \tag{2}$$

$$\mathbf{D_x} = -\mathbf{K}\mathbf{D_x}\mathbf{K}, \tag{3}$$

where superscript T denotes transposition and where $\mathbf{K}$ is a $L \times L$ diagonal matrix containing a specific ordering of 1's and -1's along the diagonal. Note that $\mathbf{K}$ obeys the property $\mathbf{K} = \mathbf{K}^{-1} = \mathbf{K}^T$.

[a)]Author to whom correspondence should be addressed. Electronic mail: c.p.a.wapenaar@tudelft.nl

Equation (1) also holds for diffusion phenomena, linearized flow, as well as (coupled) electromagnetic and elastodynamic waves in poroelastic media.[31] These cases are not considered here because they do not obey energy conservation and hence there is no optical theorem for these situations.

The temporal Fourier transform of a time-dependent function $f(t)$ is defined as follows:

$$f(\omega) = \int_{-\infty}^{\infty} f(t)\exp(i\omega t)\mathrm{d}t, \tag{4}$$

where ω is the angular frequency and i the imaginary unit $\left(i = \sqrt{-1}\right)$. To keep the notation simple, the same symbol is used for time- and frequency-domain functions (here f). In the remainder of the main text all functions are in the frequency domain. In the appendixes it is always clear from the context which domain is considered.

In the frequency domain, Eq. (1) becomes

$$\mathbf{D_x u} = i\omega\boldsymbol{\mathcal{A}}\mathbf{u} + \mathbf{s}, \tag{5}$$

where

$$\boldsymbol{\mathcal{A}} = \mathbf{A} - \frac{1}{i\omega}\mathbf{B}, \tag{6}$$

with $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}(\mathbf{x},\omega)$, $\mathbf{u} = \mathbf{u}(\mathbf{x},\omega)$, and $\mathbf{s} = \mathbf{s}(\mathbf{x},\omega)$. The spatial differential operator $\mathbf{D_x}$ is the same as in Eq. (1).

What follows is a brief review of the derivation of two unified reciprocity theorems for wave fields obeying the unified wave equation.[31] Consider a domain $\mathbb{D}$ enclosed by boundary $\partial\mathbb{D}$ with outward pointing normal vector $\mathbf{n}$, see Fig. 1. In this domain there are two independent physical states $\{\boldsymbol{\mathcal{A}}_A, \mathbf{u}_A, \mathbf{s}_A\}$ and $\{\boldsymbol{\mathcal{A}}_B, \mathbf{u}_B, \mathbf{s}_B\}$, respectively, each state obeying wave equation (5). In Appendix B, the following matrix-vector form of Gauss's theorem is derived:

$$\int_{\mathbb{D}} \left\{\mathbf{a}^T\mathbf{D_x b} + (\mathbf{D_x a})^T\mathbf{b}\right\}\mathrm{d}^3\mathbf{x} = \oint_{\partial\mathbb{D}} \mathbf{a}^T\mathbf{N_x b}\mathrm{d}^2\mathbf{x}, \tag{7}$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary vector functions and $\mathbf{N_x}$ is a $L\times L$ matrix containing the components n_1, n_2, n_3 of the normal vector $\mathbf{n}$ on $\partial\mathbb{D}$, organized in the same way as ∂_1, ∂_2, ∂_3 in matrix $\mathbf{D_x}$. Substituting $\mathbf{a} = \mathbf{K}\mathbf{u}_A$, $\mathbf{b} = \mathbf{u}_B$, and using Eqs. (3) and (5), yields

$$\int_{\mathbb{D}} \{\mathbf{u}_A^T\mathbf{K}\mathbf{s}_B - \mathbf{s}_A^T\mathbf{K}\mathbf{u}_B\}\mathrm{d}^3\mathbf{x} = \oint_{\partial\mathbb{D}} \mathbf{u}_A^T\mathbf{K}\mathbf{N_x}\mathbf{u}_B\mathrm{d}^2\mathbf{x} - i\omega\int_{\mathbb{D}} \mathbf{u}_A^T\mathbf{K}(\boldsymbol{\mathcal{A}}_B - \boldsymbol{\mathcal{A}}_A^{(a)})\mathbf{u}_B\mathrm{d}^3\mathbf{x}, \tag{8}$$

where

$$\boldsymbol{\mathcal{A}}^{(a)} = \mathbf{K}\boldsymbol{\mathcal{A}}^T\mathbf{K}. \tag{9}$$

Equation (8) is the unified reciprocity theorem of the convolution type. $\boldsymbol{\mathcal{A}}^{(a)}$ is called the medium parameter matrix of the adjoint medium [which is to be distinguished from the adjoint matrix $\boldsymbol{\mathcal{A}}^\dagger$ appearing in Eq. (10)]. An adjoint medium is loosely defined as the medium in which, after interchanging a given source and receiver, the same response is obtained as in the original medium before the source and receiver were interchanged. For example, for acoustic waves in a flowing medium, the adjoint medium is the medium with reversed flow.[33,34] For all cases discussed in Appendix A, except for electromagnetic waves in bianisotropic materials,[35] it holds that $\boldsymbol{\mathcal{A}}^{(a)} = \boldsymbol{\mathcal{A}}$, which means that the medium parameters are self-adjoint for these cases. In Sec. III it is confirmed that self-adjointness of the medium parameters is equivalent to the medium being reciprocal. Self-adjointness of the medium parameters is not required for the derivation of the unified optical theorem, see Sec. V.

Next, substitute $\mathbf{a} = \mathbf{u}_A^*$ and $\mathbf{b} = \mathbf{u}_B$ into Gauss's theorem (7), where the asterisk (*) denotes complex conjugation. Using Eq. (5), this gives

$$\int_{\mathbb{D}} \left\{\mathbf{u}_A^\dagger\mathbf{s}_B + \mathbf{s}_A^\dagger\mathbf{u}_B\right\}\mathrm{d}^3\mathbf{x} = \oint_{\partial\mathbb{D}} \mathbf{u}_A^\dagger\mathbf{N_x}\mathbf{u}_B\mathrm{d}^2\mathbf{x} - i\omega\int_{\mathbb{D}} \mathbf{u}_A^\dagger\left(\boldsymbol{\mathcal{A}}_B - \boldsymbol{\mathcal{A}}_A^\dagger\right)\mathbf{u}_B\mathrm{d}^3\mathbf{x}, \tag{10}$$

where the dagger (†) denotes complex conjugation and transposition. Equation (10) is the unified reciprocity theorem of the correlation type. When state A is equal to state B, this equation simplifies to

$$2\Re\int_{\mathbb{D}} \mathbf{u}^\dagger\mathbf{s}\mathrm{d}^3\mathbf{x} = \oint_{\partial\mathbb{D}} \mathbf{u}^\dagger\mathbf{N_x}\mathbf{u}\mathrm{d}^2\mathbf{x} - i\omega\int_{\mathbb{D}} \mathbf{u}^\dagger\left(\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}^\dagger\right)\mathbf{u}\mathrm{d}^3\mathbf{x}, \tag{11}$$

where $\Re$ denotes the real part. The left-hand side represents the energy injected into the system by the sources in $\mathbb{D}$. The first integral on the right-hand side is the energy leaving the system through the boundary $\partial\mathbb{D}$ and the second integral on the right-hand side quantifies the energy loss in $\mathbb{D}$. Energy is conserved when $\boldsymbol{\mathcal{A}}^\dagger = \boldsymbol{\mathcal{A}}$, i.e., when matrix $\boldsymbol{\mathcal{A}}$ is self-adjoint (for quantum-mechanical waves, replace "energy" by

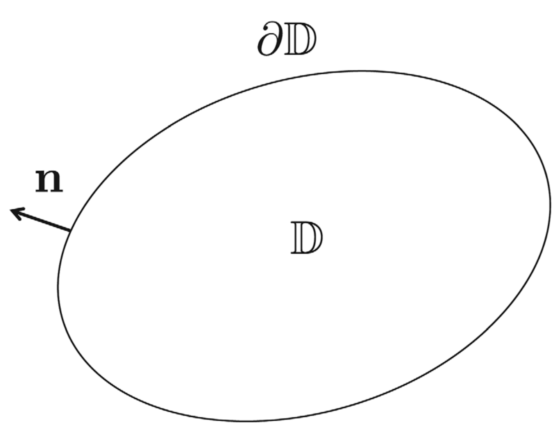

FIG. 1. Configuration for the reciprocity theorems.

"probability"). Hence, for the derivation of the unified optical theorem in section V it is required that $\boldsymbol{\mathcal{A}}$ is self-adjoint, since the optical theorem is related to the conservation of energy. However, for the moment (i.e., in Secs. III and IV), self-adjointness of $\boldsymbol{\mathcal{A}}$ is not assumed.

III. GREEN'S FUNCTION REPRESENTATIONS

A Green's function is defined as the wave field that would be obtained if the source were an impulsive point source $\delta(\mathbf{x}-\mathbf{x}')\delta(t)$, or, in the frequency domain, a point source $\delta(\mathbf{x}-\mathbf{x}')$ with unit spectrum. Because the source vector $\mathbf{s}$ in Eq. (5) contains L different source functions, there exist L different Green's wave field vectors. The lth Green's wave field vector (with $1 \le l \le L$) is defined as the causal solution of Eq. (5), with source vector $\mathbf{s}$ replaced by $\boldsymbol{i}_l\delta(\mathbf{x}-\mathbf{x}')$, where $\boldsymbol{i}_l$ is the $L \times 1$ unit vector $(0,\ldots,1,\ldots,0)^T$, with "1" on the lth position. Hence, the Green's wave vector obeys the following equation:

$$\mathbf{D}_{\mathbf{x}}\mathbf{g}_l = i\omega\boldsymbol{\mathcal{A}}\mathbf{g}_l + \boldsymbol{i}_l\delta(\mathbf{x}-\mathbf{x}'), \tag{12}$$

where $\mathbf{g}_l = \mathbf{g}_l(\mathbf{x}, \mathbf{x}', \omega)$ is the lth $L \times 1$ Green's wave vector observed at $\mathbf{x}$, due to a point source of the lth type at $\mathbf{x}'$. In the following, ω is suppressed in the argument list but the coordinate vectors $\mathbf{x}$ and $\mathbf{x}'$ are retained where appropriate. Equation (12) represents L matrix-vector equations for the L Green's wave vectors $\mathbf{g}_l$. The L Green's vectors are combined into a Green's matrix and the L source vectors into a source matrix, according to

$$(\mathbf{g}_1,\ldots,\mathbf{g}_l,\ldots,\mathbf{g}_L)(\mathbf{x},\mathbf{x}') = \mathbf{G}(\mathbf{x},\mathbf{x}'), \tag{13}$$

$$(\boldsymbol{i}_1,\ldots,\boldsymbol{i}_l,\ldots,\boldsymbol{i}_L)\delta(\mathbf{x}-\mathbf{x}') = \mathbf{I}\delta(\mathbf{x}-\mathbf{x}'), \tag{14}$$

where $\mathbf{G}(\mathbf{x}, \mathbf{x}')$ is the $L \times L$ Green's wave field matrix and $\mathbf{I}$ is the $L \times L$ identity matrix. With this notation, Eq. (12) for $l = 1,\ldots,L$ can be combined into

$$\mathbf{D}_{\mathbf{x}}\mathbf{G} = i\omega\boldsymbol{\mathcal{A}}\mathbf{G} + \mathbf{I}\delta(\mathbf{x}-\mathbf{x}'). \tag{15}$$

The convolution-type reciprocity theorem (8) is now used to derive the reciprocity properties of the Green's matrix. To this end, replace $\{\boldsymbol{\mathcal{A}}_A, \mathbf{u}_A, \mathbf{s}_A\}$ by $\{\boldsymbol{\mathcal{A}}(\mathbf{x}), \mathbf{G}(\mathbf{x},\mathbf{x}'), \mathbf{I}\delta(\mathbf{x}-\mathbf{x}')\}$ and $\{\boldsymbol{\mathcal{A}}_B, \mathbf{u}_B, \mathbf{s}_B\}$ by $\{\boldsymbol{\mathcal{A}}^{(a)}(\mathbf{x}), \mathbf{G}^{(a)}(\mathbf{x},\mathbf{x}''), \mathbf{I}\delta(\mathbf{x}-\mathbf{x}'')\}$. Because the medium in state B is chosen as the adjoint of the medium in state A, the second integral on the right-hand side of Eq. (8) vanishes. Replacing $\mathbb{D}$ by $\mathbb{R}^3$ and assuming that outside some sphere with finite radius the medium is homogeneous, isotropic and self-adjoint, the first integral on the right-hand side vanishes as well (Sommerfeld's radiation conditions[36–38]). This leaves

$$\int_{\mathbb{R}^3}\{\mathbf{G}^T(\mathbf{x},\mathbf{x}')\mathbf{K}\delta(\mathbf{x}-\mathbf{x}'') \\ -\delta(\mathbf{x}-\mathbf{x}')\mathbf{K}\mathbf{G}^{(a)}(\mathbf{x},\mathbf{x}'')\}\mathrm{d}^3\mathbf{x} = \mathbf{O} \tag{16}$$

or

$$\mathbf{G}^{(a)}(\mathbf{x}',\mathbf{x}'') = \mathbf{K}\mathbf{G}^T(\mathbf{x}'',\mathbf{x}')\mathbf{K}. \tag{17}$$

Note that $\mathbf{G}^{(a)}(\mathbf{x}',\mathbf{x}'')$ is defined in a medium which is the adjoint of the medium in which $\mathbf{G}(\mathbf{x}'',\mathbf{x}')$ is defined. For a self-adjoint medium equation (17) simplifies to

$$\mathbf{G}(\mathbf{x}',\mathbf{x}'') = \mathbf{K}\mathbf{G}^T(\mathbf{x}'',\mathbf{x}')\mathbf{K}. \tag{18}$$

This equation quantifies source-receiver reciprocity. Hence, self-adjointness of the medium is equivalent to the medium being reciprocal.

Next, two unified Green's function representations are derived. For state A, choose $\{\bar{\boldsymbol{\mathcal{A}}}(\mathbf{x}), \bar{\mathbf{G}}(\mathbf{x},\mathbf{x}'), \mathbf{I}\delta(\mathbf{x}-\mathbf{x}')\}$, where the bars denote a reference state, and take for state B the actual state, i.e., $\{\boldsymbol{\mathcal{A}}(\mathbf{x}), \mathbf{G}(\mathbf{x},\mathbf{x}''), \mathbf{I}\delta(\mathbf{x}-\mathbf{x}'')\}$. Substitution of these states in the convolution-type and correlation-type reciprocity theorems (8) and (10), respectively, yields [using Eq. (17) for the reference Green's function]

$$\chi_{\mathbb{D}}(\mathbf{x}')\mathbf{G}(\mathbf{x}',\mathbf{x}'') - \chi_{\mathbb{D}}(\mathbf{x}'')\bar{\mathbf{G}}^{(a)}(\mathbf{x}',\mathbf{x}'') \\ = -\oint_{\partial\mathbb{D}}\bar{\mathbf{G}}^{(a)}(\mathbf{x}',\mathbf{x})\mathbf{N}_{\mathbf{x}}\mathbf{G}(\mathbf{x},\mathbf{x}'')\mathrm{d}^2\mathbf{x} \\ + i\omega\int_{\mathbb{D}}\bar{\mathbf{G}}^{(a)}(\mathbf{x}',\mathbf{x})\{\boldsymbol{\mathcal{A}} - \bar{\boldsymbol{\mathcal{A}}}^{(a)}\}(\mathbf{x})\mathbf{G}(\mathbf{x},\mathbf{x}'')\mathrm{d}^3\mathbf{x} \tag{19}$$

and

$$\chi_{\mathbb{D}}(\mathbf{x}')\mathbf{G}(\mathbf{x}',\mathbf{x}'') + \chi_{\mathbb{D}}(\mathbf{x}'')\bar{\mathbf{G}}^{\dagger}(\mathbf{x}'',\mathbf{x}') \\ = \oint_{\partial\mathbb{D}}\bar{\mathbf{G}}^{\dagger}(\mathbf{x},\mathbf{x}')\mathbf{N}_{\mathbf{x}}\mathbf{G}(\mathbf{x},\mathbf{x}'')\mathrm{d}^2\mathbf{x} \\ - i\omega\int_{\mathbb{D}}\bar{\mathbf{G}}^{\dagger}(\mathbf{x},\mathbf{x}')\left\{\boldsymbol{\mathcal{A}} - \bar{\boldsymbol{\mathcal{A}}}^{\dagger}\right\}(\mathbf{x})\mathbf{G}(\mathbf{x},\mathbf{x}'')\mathrm{d}^3\mathbf{x}, \tag{20}$$

respectively, where $\chi_{\mathbb{D}}(\mathbf{x}')$ is the characteristic function for domain $\mathbb{D}$, defined as

$$\chi_{\mathbb{D}}(\mathbf{x}') = \begin{cases} 1 & \text{for } \mathbf{x}' \in \mathbb{D}, \\ \frac{1}{2} & \text{for } \mathbf{x}' \in \partial\mathbb{D}, \\ 0 & \text{for } \mathbf{x}' \in \mathbb{R}^3\backslash\{\mathbb{D}\cup\partial\mathbb{D}\}. \end{cases} \tag{21}$$

The convolution-type Green's function representation (19) is a basis, for example, for iterative forward modeling of scattered wave fields, using boundary and/or volume integral methods. The correlation-type representation (20) is a basis for the methodology of Green's function retrieval by cross-correlation of ambient noise in its most general form.[29] A further discussion of these applications is beyond the scope of this paper. Both representations are used in the following sections in the derivation of the unified optical theorem.

IV. INTEGRAL RELATION FOR THE GREEN'S FUNCTION OF THE SCATTERED WAVE FIELD

The generalized optical theorem is an integral relation for the angle-dependent scattering amplitude of a scattering object. Here an integral relation for the Green's function of the scattered wave field is derived, which will be used as the basis for the derivation of the generalized optical theorem in the next section.

The total Green's function $\mathbf{G}(\mathbf{x}, \mathbf{x}')$ in the actual medium $\boldsymbol{\mathcal{A}}(\mathbf{x})$ is the sum of the reference Green's function $\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}')$

in the reference medium $\bar{\mathcal{A}}(\mathbf{x})$ and the Green's function $\mathbf{G}^s(\mathbf{x},\mathbf{x}')$ of the scattered wave field, hence

$$\mathbf{G}(\mathbf{x},\mathbf{x}') = \bar{\mathbf{G}}(\mathbf{x},\mathbf{x}') + \mathbf{G}^s(\mathbf{x},\mathbf{x}'). \tag{22}$$

In Sec. V the reference medium will be taken homogeneous, isotropic, reciprocal, and lossless, but for the moment the choice of the reference medium is arbitrary. The correlation-type representation (20) will now be used to find an expression for the following integral:

$$\oint_{\partial\mathbb{D}} \{\mathbf{G}^s(\mathbf{x},\mathbf{x}')\}^\dagger \mathbf{N}_\mathbf{x}\mathbf{G}^s(\mathbf{x},\mathbf{x}'')\mathrm{d}^2\mathbf{x}, \tag{23}$$

with $\mathbf{x}'$ and $\mathbf{x}''$ both in $\mathbb{D}$. A compact notation to represent integrals of this form is

$$\mathcal{L}(\mathbf{G}_1,\mathbf{G}_2) = \oint_{\partial\mathbb{D}} \mathbf{G}^\dagger(\mathbf{x},\mathbf{x}')\mathbf{N}_\mathbf{x}\mathbf{G}(\mathbf{x},\mathbf{x}'')\mathrm{d}^2\mathbf{x}. \tag{24}$$

Here the subscripts 1 and 2 at the left-hand side correspond to the source positions $\mathbf{x}'$ and $\mathbf{x}''$, respectively, of the two Green's functions. Substitution of Eq. (22) into Eq. (24) yields

$$\mathcal{L}(\mathbf{G}_1,\mathbf{G}_2) = \mathcal{L}(\bar{\mathbf{G}}_1,\bar{\mathbf{G}}_2) + \mathcal{L}(\bar{\mathbf{G}}_1,\mathbf{G}_2^s) + \mathcal{L}(\mathbf{G}_1^s,\bar{\mathbf{G}}_2) + \mathcal{L}(\mathbf{G}_1^s,\mathbf{G}_2^s). \tag{25}$$

Using Eq. (22) again, the second and third term in the right-hand side of Eq. (25) can be expressed as

$$\mathcal{L}(\bar{\mathbf{G}}_1,\mathbf{G}_2^s) = \mathcal{L}(\bar{\mathbf{G}}_1,\mathbf{G}_2) - \mathcal{L}(\bar{\mathbf{G}}_1,\bar{\mathbf{G}}_2), \tag{26}$$

$$\mathcal{L}(\mathbf{G}_1^s,\bar{\mathbf{G}}_2) = \mathcal{L}(\mathbf{G}_1,\bar{\mathbf{G}}_2) - \mathcal{L}(\bar{\mathbf{G}}_1,\bar{\mathbf{G}}_2). \tag{27}$$

Substituting this into Eq. (25) and bringing the last term to the left-hand side gives

$$\mathcal{L}(\mathbf{G}_1^s,\mathbf{G}_2^s) = \mathcal{L}(\mathbf{G}_1,\mathbf{G}_2) + \mathcal{L}(\bar{\mathbf{G}}_1,\bar{\mathbf{G}}_2) - \mathcal{L}(\bar{\mathbf{G}}_1,\mathbf{G}_2) - \mathcal{L}(\mathbf{G}_1,\bar{\mathbf{G}}_2). \tag{28}$$

Note that the left-hand side is the sought integral of Eq. (23), which has now been expressed in terms of integrals containing the total and the reference Green's functions. The right-hand side is evaluated term by term. Taking the reference medium equal to the actual medium in Eq. (20), and using the fact that $\mathbf{x}'$ and $\mathbf{x}''$ are both situated in $\mathbb{D}$, yields for the first term on the right-hand side of Eq. (28)

$$\mathcal{L}(\mathbf{G}_1,\mathbf{G}_2) = \mathbf{G}(\mathbf{x}',\mathbf{x}'') + \mathbf{G}^\dagger(\mathbf{x}'',\mathbf{x}') + i\omega\int_{\mathbb{D}} \mathbf{G}^\dagger(\mathbf{x},\mathbf{x}')\{\mathcal{A}-\mathcal{A}^\dagger\}(\mathbf{x})\mathbf{G}(\mathbf{x},\mathbf{x}'')\mathrm{d}^3\mathbf{x}. \tag{29}$$

The same relation holds for the second term, with the total Green's functions in the actual medium replaced by the reference Green's functions in the reference medium, i.e.,

$$\mathcal{L}(\bar{\mathbf{G}}_1,\bar{\mathbf{G}}_2) = \bar{\mathbf{G}}(\mathbf{x}',\mathbf{x}'') + \bar{\mathbf{G}}^\dagger(\mathbf{x}'',\mathbf{x}') + i\omega\int_{\mathbb{D}} \bar{\mathbf{G}}^\dagger(\mathbf{x},\mathbf{x}')\left\{\bar{\mathcal{A}}-\bar{\mathcal{A}}^\dagger\right\}(\mathbf{x})\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}'')\mathrm{d}^3\mathbf{x}. \tag{30}$$

The third term on the right-hand side of Eq. (28) follows directly from Eq. (20), hence

$$\mathcal{L}(\bar{\mathbf{G}}_1,\mathbf{G}_2) = \mathbf{G}(\mathbf{x}',\mathbf{x}'') + \bar{\mathbf{G}}^\dagger(\mathbf{x}'',\mathbf{x}') + i\omega\int_{\mathbb{D}} \bar{\mathbf{G}}^\dagger(\mathbf{x},\mathbf{x}')\left\{\mathcal{A}-\bar{\mathcal{A}}^\dagger\right\}(\mathbf{x})\mathbf{G}(\mathbf{x},\mathbf{x}'')\mathrm{d}^3\mathbf{x}. \tag{31}$$

Finally, interchanging the roles of the total and reference Green's functions, yields for the fourth term

$$\mathcal{L}(\mathbf{G}_1,\bar{\mathbf{G}}_2) = \bar{\mathbf{G}}(\mathbf{x}',\mathbf{x}'') + \mathbf{G}^\dagger(\mathbf{x}'',\mathbf{x}') + i\omega\int_{\mathbb{D}} \mathbf{G}^\dagger(\mathbf{x},\mathbf{x}')\{\bar{\mathcal{A}}-\mathcal{A}^\dagger\}(\mathbf{x})\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}'')\mathrm{d}^3\mathbf{x}. \tag{32}$$

Substituting Eqs. (29)–(32) into the right-hand side of Eq. (28) and replacing the left-hand side by expression (23) gives

$$\begin{aligned}
&\oint_{\partial\mathbb{D}} \{\mathbf{G}^s(\mathbf{x},\mathbf{x}')\}^\dagger \mathbf{N}_\mathbf{x}\mathbf{G}^s(\mathbf{x},\mathbf{x}'')\mathrm{d}^2\mathbf{x}\\
&= +i\omega\int_{\mathbb{D}} \mathbf{G}^\dagger(\mathbf{x},\mathbf{x}')\{\mathcal{A}-\mathcal{A}^\dagger\}(\mathbf{x})\mathbf{G}(\mathbf{x},\mathbf{x}'')\mathrm{d}^3\mathbf{x}\\
&\quad + i\omega\int_{\mathbb{D}} \bar{\mathbf{G}}^\dagger(\mathbf{x},\mathbf{x}')\left\{\bar{\mathcal{A}}-\bar{\mathcal{A}}^\dagger\right\}(\mathbf{x})\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}'')\mathrm{d}^3\mathbf{x}\\
&\quad - i\omega\int_{\mathbb{D}} \bar{\mathbf{G}}^\dagger(\mathbf{x},\mathbf{x}')\left\{\mathcal{A}-\bar{\mathcal{A}}^\dagger\right\}(\mathbf{x})\mathbf{G}(\mathbf{x},\mathbf{x}'')\mathrm{d}^3\mathbf{x}\\
&\quad - i\omega\int_{\mathbb{D}} \mathbf{G}^\dagger(\mathbf{x},\mathbf{x}')\{\bar{\mathcal{A}}-\mathcal{A}^\dagger\}(\mathbf{x})\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}'')\mathrm{d}^3\mathbf{x}.
\end{aligned} \tag{33}$$

V. THE UNIFIED OPTICAL THEOREM

From here onward, consider a small scattering domain $\mathbb{D}_s$ around the origin, embedded in a reference domain $\mathbb{R}^3$, see Fig. 2. The scattering domain may be arbitrary inhomogeneous, anisotropic, and non-reciprocal, but it is assumed to be lossless, hence $\mathcal{A}(\mathbf{x}) = \mathcal{A}^\dagger(\mathbf{x})$. The reference state is taken homogeneous, isotropic, reciprocal, and lossless, hence $\bar{\mathcal{A}} = \bar{\mathcal{A}}^{(a)} = \bar{\mathcal{A}}^\dagger$. Outside the scattering domain $\mathbb{D}_s$, centered at the origin, it holds that $\mathcal{A}(\mathbf{x}) = \bar{\mathcal{A}}$. For $\partial\mathbb{D}$, choose a large spherical boundary, centered at the origin.[39] Define a unit vector $\hat{\mathbf{x}}$ in the direction of $\mathbf{x}$, according to $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$. Hence, the normal $\mathbf{n}$ on $\partial\mathbb{D}$ equals $\hat{\mathbf{x}}$, for $\mathbf{x}$ on $\partial\mathbb{D}$. Using all this in Eq. (33) yields

$$\begin{aligned}
&\oint_{\partial\mathbb{D}} \{\mathbf{G}^s(\mathbf{x},\mathbf{x}')\}^\dagger \mathbf{M}(\hat{\mathbf{x}})\mathbf{G}^s(\mathbf{x},\mathbf{x}'')\mathrm{d}^2\mathbf{x}\\
&= -i\omega\int_{\mathbb{D}_s} \bar{\mathbf{G}}^\dagger(\mathbf{x},\mathbf{x}')\{\mathcal{A}(\mathbf{x})-\bar{\mathcal{A}}\}\mathbf{G}(\mathbf{x},\mathbf{x}'')\mathrm{d}^3\mathbf{x}\\
&\quad + i\omega\int_{\mathbb{D}_s} \mathbf{G}^\dagger(\mathbf{x},\mathbf{x}')\{\mathcal{A}(\mathbf{x})-\bar{\mathcal{A}}\}\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}'')\mathrm{d}^3\mathbf{x},
\end{aligned} \tag{34}$$

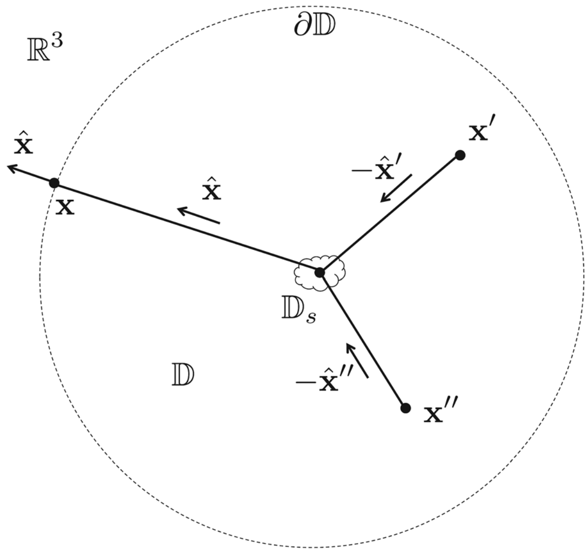

FIG. 2. Configuration for the optical theorem.

with $\mathbf{M}(\hat{\mathbf{x}})$ defined as $\mathbf{N}_\mathbf{x}$, but with all n_i replaced by $\hat{x}_i$. Express the far field of the Green's function of the scattered wave field as

$$\mathbf{G}^s(\mathbf{x},\mathbf{x}') = i\zeta\bar{\mathbf{G}}(\mathbf{x},\mathbf{0})\mathbf{F}(\hat{\mathbf{x}},-\hat{\mathbf{x}}')\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}') \tag{35}$$

and a similar expression for $\mathbf{G}^s(\mathbf{x},\mathbf{x}'')$, with $\mathbf{x}$ on $\partial\mathbb{D}$, and $\mathbf{x}'$, $\mathbf{x}''$ in $\mathbb{D}$, all far from the scattering domain $\mathbb{D}_s$, see Fig. 2. Here $\mathbf{F}(\hat{\mathbf{x}},-\hat{\mathbf{x}}')$ is a $L\times L$ matrix containing angle-dependent scattering amplitudes. Similar to $\hat{\mathbf{x}}$, vectors $\hat{\mathbf{x}}'$ and $\hat{\mathbf{x}}''$ are unit vectors in the direction of $\mathbf{x}'$ and $\mathbf{x}''$, respectively. Finally, $i\zeta$ is a conveniently chosen normalization factor that compensates for factors in the reference Green's function, see Appendix C for details. Next, the optical theorem for the scattering matrix $\mathbf{F}(\hat{\mathbf{x}},-\hat{\mathbf{x}}')$ is derived.

Step 1: substitution of Eq. (35) and a similar expression for $\mathbf{G}^s(\mathbf{x},\mathbf{x}'')$ into the left-hand side (LHS) of Eq. (34) gives

$$\text{LHS of Eq. (34)} = \zeta^2\bar{\mathbf{G}}^\dagger(\mathbf{0},\mathbf{x}')\oint_{\partial\mathbb{D}}\mathbf{F}^\dagger(\hat{\mathbf{x}},-\hat{\mathbf{x}}')\bar{\mathbf{G}}^\dagger(\mathbf{x},\mathbf{0}) \times\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x},\mathbf{0})\mathbf{F}(\hat{\mathbf{x}},-\hat{\mathbf{x}}'')\mathrm{d}^2\mathbf{x}\,\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}''). \tag{36}$$

In Appendix C it is shown that

$$\bar{\mathbf{G}}^\dagger(\mathbf{x},\mathbf{0})\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x},\mathbf{0}) = \frac{2}{\zeta}\frac{\Theta(\hat{\mathbf{x}})}{|\mathbf{x}|^2}, \tag{37}$$

where $\Theta(\hat{\mathbf{x}})$ is a function of the unit vector $\hat{\mathbf{x}}$ and the parameters of the embedding medium. Hence

$$\text{LHS of Eq. (34)} = 2\zeta\bar{\mathbf{G}}^\dagger(\mathbf{0},\mathbf{x}')\oint\mathbf{F}^\dagger(\hat{\mathbf{x}},-\hat{\mathbf{x}}')\Theta(\hat{\mathbf{x}}) \times\mathbf{F}(\hat{\mathbf{x}},-\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}}\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}''), \tag{38}$$

with $\mathrm{d}\Omega_{\hat{\mathbf{x}}} = \mathrm{d}^2\mathbf{x}/|\mathbf{x}|^2$.

Step 2: Eq. (19) is used to derive an explicit expression for the scattering matrix $\mathbf{F}$. Because $\bar{\boldsymbol{\mathcal{A}}}^{(a)} = \bar{\boldsymbol{\mathcal{A}}}$ in the reference state, it holds that $\bar{\mathbf{G}}^{(a)} = \bar{\mathbf{G}}$. Hence, taking into account that $\mathbf{x}'$ and $\mathbf{x}''$ are situated in $\mathbb{D}$, the left-hand side of Eq. (19) is equal to the Green's function $\mathbf{G}^s(\mathbf{x}',\mathbf{x}'')$ for the scattered wave field. Because outside $\partial\mathbb{D}$ the parameters of the reference state as well as of the actual state are homogeneous, isotropic, reciprocal and lossless, the boundary integral on the right-hand side of Eq. (19) vanishes on account of Sommerfeld's radiation conditions. This leaves

$$\mathbf{G}^s(\mathbf{x}',\mathbf{x}'') = i\omega\int_{\mathbb{D}_s}\bar{\mathbf{G}}(\mathbf{x}',\mathbf{x})\{\boldsymbol{\mathcal{A}}(\mathbf{x})-\bar{\boldsymbol{\mathcal{A}}}\}\mathbf{G}(\mathbf{x},\mathbf{x}'')\mathrm{d}^3\mathbf{x}. \tag{39}$$

For all $\mathbf{x}$ in the integration domain $\mathbb{D}_s$ it holds that $|\mathbf{x}| \ll |\mathbf{x}'|$ and $|\mathbf{x}| \ll |\mathbf{x}''|$, see Fig. 3. Approximate $\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}')$ by

$$\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}') = \bar{\mathbf{P}}(\mathbf{x},\hat{\mathbf{x}}')\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}'), \quad |\mathbf{x}| \ll |\mathbf{x}'|, \tag{40}$$

where $\bar{\mathbf{P}}(\mathbf{x},\hat{\mathbf{x}}')$ is a matrix containing plane-wave functions, see Appendix C for details. Similarly,

$$\bar{\mathbf{G}}(\mathbf{x}',\mathbf{x}) = \bar{\mathbf{G}}(\mathbf{x}',\mathbf{0})\bar{\mathbf{P}}(\hat{\mathbf{x}}',\mathbf{x}), \quad |\mathbf{x}| \ll |\mathbf{x}'|. \tag{41}$$

Applying symmetry relation (18) for the reference Green's function, yields

$$\bar{\mathbf{P}}(\hat{\mathbf{x}}',\mathbf{x}) = \mathbf{K}\bar{\mathbf{P}}^T(\mathbf{x},\hat{\mathbf{x}}')\mathbf{K}. \tag{42}$$

Approximate $\mathbf{G}(\mathbf{x},\mathbf{x}'')$ by

$$\mathbf{G}(\mathbf{x},\mathbf{x}'') = \mathbf{P}(\mathbf{x},\hat{\mathbf{x}}'')\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}''), \quad |\mathbf{x}| \ll |\mathbf{x}''|, \tag{43}$$

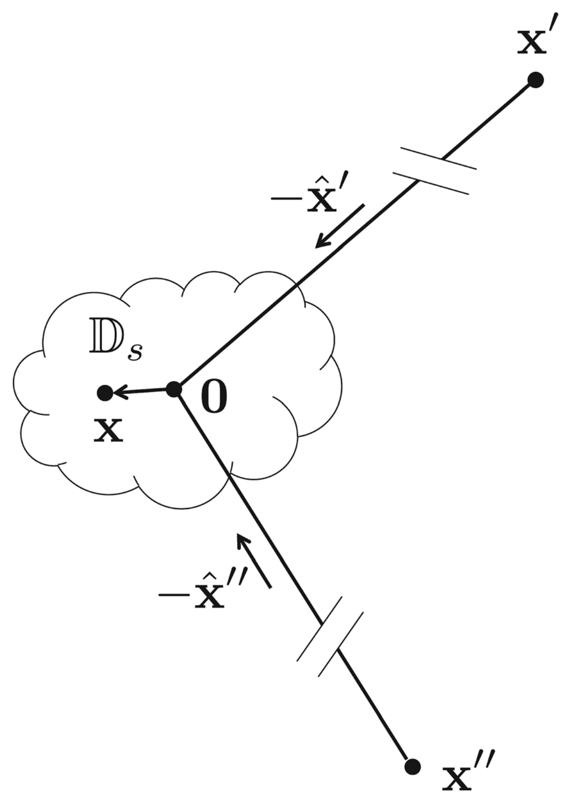

FIG. 3. As in Fig. 2, but zoomed-in on the scattering domain $\mathbb{D}_s$.

where $\mathbf{P}(\mathbf{x},\hat{\mathbf{x}}'')$ contains plane-wave functions $\bar{\mathbf{P}}(\mathbf{x},\hat{\mathbf{x}}'')$ for the direct-wave contribution plus non-linear scattering effects for the scattered-wave contribution. For the following analysis, $\mathbf{P}(\mathbf{x},\hat{\mathbf{x}}'')$ does not need to be specified further. Substituting Eqs. (41) and (43) into Eq. (39) and comparing the result with Eq. (35) gives

$$\mathbf{F}(\hat{\mathbf{x}}',-\hat{\mathbf{x}}'')=\frac{\omega}{\zeta}\int_{\mathbb{D}_s}\bar{\mathbf{P}}(\hat{\mathbf{x}}',\mathbf{x})\{\boldsymbol{\mathcal{A}}(\mathbf{x})-\bar{\boldsymbol{\mathcal{A}}}\}\mathbf{P}(\mathbf{x},\hat{\mathbf{x}}'')\mathrm{d}^3\mathbf{x}. \tag{44}$$

Step 3: substituting Eqs. (40) and (43) into the right-hand side (RHS) of Eq. (34) gives

$$\begin{aligned}&\text{RHS of (34)}\\&=\bar{\mathbf{G}}^{\dagger}(\mathbf{0},\mathbf{x}')\bigg[-i\omega\int_{\mathbb{D}_s}\bar{\mathbf{P}}^{\dagger}(\mathbf{x},\hat{\mathbf{x}}')\{\boldsymbol{\mathcal{A}}(\mathbf{x})-\bar{\boldsymbol{\mathcal{A}}}\}\mathbf{P}(\mathbf{x},\hat{\mathbf{x}}'')\mathrm{d}^3\mathbf{x}\\&\quad+i\omega\int_{\mathbb{D}_s}\mathbf{P}^{\dagger}(\mathbf{x},\hat{\mathbf{x}}')\{\boldsymbol{\mathcal{A}}(\mathbf{x})-\bar{\boldsymbol{\mathcal{A}}}\}\bar{\mathbf{P}}(\mathbf{x},\hat{\mathbf{x}}'')\mathrm{d}^3\mathbf{x}\bigg]\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}'').\end{aligned} \tag{45}$$

The integrals in this equation resemble that in Eq. (44), except for the daggers. In Appendix C it is shown that

$$\bar{\mathbf{P}}(\mathbf{x},\hat{\mathbf{x}}')=\mathbf{K}\bar{\mathbf{P}}^{*}(\mathbf{x},-\hat{\mathbf{x}}')\mathbf{K}. \tag{46}$$

Combining this with Eq. (42) gives

$$\bar{\mathbf{P}}^{\dagger}(\mathbf{x},\hat{\mathbf{x}}')=\bar{\mathbf{P}}(-\hat{\mathbf{x}}',\mathbf{x}). \tag{47}$$

Using symmetry relation (47) as well as $\boldsymbol{\mathcal{A}}(\mathbf{x})=\boldsymbol{\mathcal{A}}^{\dagger}(\mathbf{x})$ and $\bar{\boldsymbol{\mathcal{A}}}=\bar{\boldsymbol{\mathcal{A}}}^{\dagger}$ in Eq. (45) and comparing the result with Eq. (44) gives

$$\begin{aligned}\text{RHS of (34)}=-i\zeta\bar{\mathbf{G}}^{\dagger}(\mathbf{0},\mathbf{x}')\Big[&\mathbf{F}(-\hat{\mathbf{x}}',-\hat{\mathbf{x}}'')\\&-\mathbf{F}^{\dagger}(-\hat{\mathbf{x}}'',-\hat{\mathbf{x}}')\Big]\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}'').\end{aligned} \tag{48}$$

Combining this with Eq. (38) yields

$$\begin{aligned}&\oint\mathbf{F}^{\dagger}(\hat{\mathbf{x}},-\hat{\mathbf{x}}')\Theta(\hat{\mathbf{x}})\mathbf{F}(\hat{\mathbf{x}},-\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}}\\&\quad=\frac{1}{2i}\left(\mathbf{F}(-\hat{\mathbf{x}}',-\hat{\mathbf{x}}'')-\mathbf{F}^{\dagger}(-\hat{\mathbf{x}}'',-\hat{\mathbf{x}}')\right)\end{aligned} \tag{49}$$

or, renaming $-\hat{\mathbf{x}}'$ and $-\hat{\mathbf{x}}''$ as $\hat{\mathbf{x}}'$ and $\hat{\mathbf{x}}''$, respectively,

$$\oint\mathbf{F}^{\dagger}(\hat{\mathbf{x}},\hat{\mathbf{x}}')\Theta(\hat{\mathbf{x}})\mathbf{F}(\hat{\mathbf{x}},\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}}=\frac{1}{2i}\left(\mathbf{F}(\hat{\mathbf{x}}',\hat{\mathbf{x}}'')-\mathbf{F}^{\dagger}(\hat{\mathbf{x}}'',\hat{\mathbf{x}}')\right). \tag{50}$$

This is the unified optical theorem and the main result of this paper. In the next section this theorem is analyzed for the different types of wave fields discussed in the appendixes. On a case-by-case basis it is shown that the $L\times L$ matrix $\mathbf{F}(\hat{\mathbf{x}},\hat{\mathbf{x}}')$ can be replaced by a smaller sized scattering matrix $\mathbf{f}(\hat{\mathbf{x}},\hat{\mathbf{x}}')$. In particular, for scalar waves the 4×4 matrix $\mathbf{F}(\hat{\mathbf{x}},\hat{\mathbf{x}}')$ will be replaced by a scalar scattering function $f(\hat{\mathbf{x}},\hat{\mathbf{x}}')$, for which case Eq. (50) reduces to the well-known generalized optical theorem for scalar waves.

VI. OPTICAL THEOREMS FOR SCALAR AND VECTORIAL WAVE FIELDS

In the previous section the Green's function of the scattered wave field was defined as

$$\mathbf{G}^{s}(\mathbf{x},\mathbf{x}')=i\zeta\bar{\mathbf{G}}(\mathbf{x},\mathbf{0})\mathbf{F}(\hat{\mathbf{x}},-\hat{\mathbf{x}}')\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}'). \tag{51}$$

According to Appendix C, for acoustic, quantum-mechanical, and electromagnetic waves, the reference Green's functions in Eq. (51) can be written as

$$\bar{\mathbf{G}}(\mathbf{x},\mathbf{0})=\boldsymbol{\theta}(\hat{\mathbf{x}})\bar{\mathbf{G}}_0(\mathbf{x})\boldsymbol{\theta}^{T}(\hat{\mathbf{x}}), \tag{52}$$

$$\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}')=\boldsymbol{\theta}(-\hat{\mathbf{x}}')\bar{\mathbf{G}}_0(-\mathbf{x}')\boldsymbol{\theta}^{T}(-\hat{\mathbf{x}}'), \tag{53}$$

where $\boldsymbol{\theta}(\hat{\mathbf{x}})$ is a function of the unit vector $\hat{\mathbf{x}}$ and the parameters of the embedding medium. For acoustic and quantum-mechanical waves $\bar{\mathbf{G}}_0(\mathbf{x})$ is actually a scalar function, i.e., $\bar{G}_0(\mathbf{x})$, whereas for electromagnetic waves $\bar{\mathbf{G}}_0(\mathbf{x})$ is a 3×3 matrix. Substituting Eqs. (52) and (53) into Eq. (51) yields

$$\mathbf{G}^{s}(\mathbf{x},\mathbf{x}')=\boldsymbol{\theta}(\hat{\mathbf{x}})\mathbf{G}_0^{s}(\mathbf{x},\mathbf{x}')\boldsymbol{\theta}^{T}(-\hat{\mathbf{x}}'), \tag{54}$$

where

$$\mathbf{G}_0^{s}(\mathbf{x},\mathbf{x}')=i\zeta\bar{\mathbf{G}}_0(\mathbf{x})\mathbf{f}(\hat{\mathbf{x}},-\hat{\mathbf{x}}')\bar{\mathbf{G}}_0(-\mathbf{x}'), \tag{55}$$

with

$$\mathbf{f}(\hat{\mathbf{x}},-\hat{\mathbf{x}}')=\boldsymbol{\theta}^{T}(\hat{\mathbf{x}})\mathbf{F}(\hat{\mathbf{x}},-\hat{\mathbf{x}}')\boldsymbol{\theta}(-\hat{\mathbf{x}}'). \tag{56}$$

Note that Eq. (55) has the same form as Eq. (51), except that in Eq. (55) all functions are scalars (for acoustic and quantum-mechanical waves) or 3×3 matrices (for electromagnetic waves). Apply $\boldsymbol{\theta}^{T}(\hat{\mathbf{x}}')$ and $\boldsymbol{\theta}(\hat{\mathbf{x}}'')$ to both sides of the unified optical theorem [Eq. (50)], as follows:

$$\begin{aligned}&\oint\boldsymbol{\theta}^{T}(\hat{\mathbf{x}}')\mathbf{F}^{\dagger}(\hat{\mathbf{x}},\hat{\mathbf{x}}')\Theta(\hat{\mathbf{x}})\mathbf{F}(\hat{\mathbf{x}},\hat{\mathbf{x}}'')\boldsymbol{\theta}(\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}}\\&\quad=\frac{1}{2i}\boldsymbol{\theta}^{T}(\hat{\mathbf{x}}')\left(\mathbf{F}(\hat{\mathbf{x}}',\hat{\mathbf{x}}'')-\mathbf{F}^{\dagger}(\hat{\mathbf{x}}'',\hat{\mathbf{x}}')\right)\boldsymbol{\theta}(\hat{\mathbf{x}}'').\end{aligned} \tag{57}$$

According to Appendix C, for acoustic and quantum-mechanical waves, matrix $\Theta(\hat{\mathbf{x}})$ is given by

$$\Theta(\hat{\mathbf{x}})=\frac{k}{4\pi}\boldsymbol{\theta}(\hat{\mathbf{x}})\boldsymbol{\theta}^{T}(\hat{\mathbf{x}}). \tag{58}$$

Substituting this into Eq. (57), using Eq. (56), taking into account that $\boldsymbol{\theta}(\hat{\mathbf{x}})$ is real-valued and that $\mathbf{f}$ is a scalar function, yields

$$\frac{k}{4\pi}\oint f^{*}(\hat{\mathbf{x}},\hat{\mathbf{x}}')f(\hat{\mathbf{x}},\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}}=\frac{1}{2i}(f(\hat{\mathbf{x}}',\hat{\mathbf{x}}'')-f^{*}(\hat{\mathbf{x}}'',\hat{\mathbf{x}}')). \tag{59}$$

This is the well-known generalized optical theorem for scalar waves.[2–6] Usually it is assumed that the scattering domain $\mathbb{D}_s$ is characterized by a single parameter (e.g., a refraction-index contrast or a scattering potential). The derivation that led to Eq. (59) accounts for two contrast parameters. This can be seen as follows. The scattering function f is expressed in terms of the 4×4 matrix $\mathbf{F}$ via Eq. (56), which is related to the contrast matrix $\boldsymbol{\mathcal{A}}(\mathbf{x})-\bar{\boldsymbol{\mathcal{A}}}$ in domain $\mathbb{D}_s$ via Eq. (44). For the acoustic situation this contrast matrix contains, via Eq. (A7) (with $b^p = b^v = 0$), the compressibility and mass density contrasts. Douma *et al.*[26] also derived Eq. (59) for a scattering domain with two parameter contrasts, using the same method that is here extended using a unified notation.

For electromagnetic waves, matrix $\Theta(\hat{\mathbf{x}})$ is given by

$$\Theta(\hat{\mathbf{x}}) = \frac{\mu k}{4\pi}\boldsymbol{\theta}(\hat{\mathbf{x}})\{\mathbf{I}-\boldsymbol{\Gamma}(\hat{\mathbf{x}})\}\boldsymbol{\theta}^T(\hat{\mathbf{x}}). \tag{60}$$

Following the same procedure as above yields

$$\frac{\mu k}{4\pi}\oint \mathbf{f}^\dagger(\hat{\mathbf{x}},\hat{\mathbf{x}}')\{\mathbf{I}-\boldsymbol{\Gamma}(\hat{\mathbf{x}})\}\mathbf{f}(\hat{\mathbf{x}},\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}} = \frac{1}{2i}\left(\mathbf{f}(\hat{\mathbf{x}}',\hat{\mathbf{x}}'') - \mathbf{f}^\dagger(\hat{\mathbf{x}}'',\hat{\mathbf{x}}')\right), \tag{61}$$

with

$$\boldsymbol{\Gamma}(\hat{\mathbf{x}}) = \begin{pmatrix} \hat{x}_1^2 & \hat{x}_1\hat{x}_2 & \hat{x}_1\hat{x}_3 \\ \hat{x}_2\hat{x}_1 & \hat{x}_2^2 & \hat{x}_2\hat{x}_3 \\ \hat{x}_3\hat{x}_1 & \hat{x}_3\hat{x}_2 & \hat{x}_3^2 \end{pmatrix}. \tag{62}$$

Equation (61) is the generalized optical theorem for electromagnetic waves.[13,14] In its present compact form it holds for a scattering domain with arbitrary inhomogeneous, anisotropic, and possibly non-reciprocal parameters contained in matrix $\boldsymbol{\mathcal{A}}(\mathbf{x})-\bar{\boldsymbol{\mathcal{A}}}$, with $\boldsymbol{\mathcal{A}}$ defined in Eq. (A22).

For the elastodynamic situation, $\bar{\mathbf{G}}(\mathbf{x},\mathbf{0})$ and $\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}')$ in Eq. (51) are defined as

$$\bar{\mathbf{G}}(\mathbf{x},\mathbf{0}) = \boldsymbol{\theta}_P(\hat{\mathbf{x}})\bar{\mathbf{G}}_P(\mathbf{x})\boldsymbol{\theta}_P^T(\hat{\mathbf{x}}) + \boldsymbol{\theta}_S(\hat{\mathbf{x}})\bar{\mathbf{G}}_S(\mathbf{x})\boldsymbol{\theta}_S^T(\hat{\mathbf{x}}), \tag{63}$$

$$\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}') = \boldsymbol{\theta}_P(-\hat{\mathbf{x}}')\bar{\mathbf{G}}_P(-\mathbf{x}')\boldsymbol{\theta}_P^T(-\hat{\mathbf{x}}') + \boldsymbol{\theta}_S(-\hat{\mathbf{x}}')\bar{\mathbf{G}}_S(-\mathbf{x}')\boldsymbol{\theta}_S^T(-\hat{\mathbf{x}}'), \tag{64}$$

where $\bar{\mathbf{G}}_P(\mathbf{x})$ and $\bar{\mathbf{G}}_S(\mathbf{x})$ are 3×3 Green's matrices for P- and S-waves, respectively [Eqs. (C37) and (C38)]. Substituting Eqs. (63) and (64) into Eq. (51) gives

$$\begin{aligned}\mathbf{G}^s(\mathbf{x},\mathbf{x}') = {} & \boldsymbol{\theta}_P(\hat{\mathbf{x}})\mathbf{G}^s_{P,P}(\mathbf{x},\mathbf{x}')\boldsymbol{\theta}_P^T(-\hat{\mathbf{x}}') \\ & + \boldsymbol{\theta}_P(\hat{\mathbf{x}})\mathbf{G}^s_{P,S}(\mathbf{x},\mathbf{x}')\boldsymbol{\theta}_S^T(-\hat{\mathbf{x}}') \\ & + \boldsymbol{\theta}_S(\hat{\mathbf{x}})\mathbf{G}^s_{S,P}(\mathbf{x},\mathbf{x}')\boldsymbol{\theta}_P^T(-\hat{\mathbf{x}}') \\ & + \boldsymbol{\theta}_S(\hat{\mathbf{x}})\mathbf{G}^s_{S,S}(\mathbf{x},\mathbf{x}')\boldsymbol{\theta}_S^T(-\hat{\mathbf{x}}'),\end{aligned} \tag{65}$$

with

$$\mathbf{G}^s_{Q,R}(\mathbf{x},\mathbf{x}') = i\zeta\bar{\mathbf{G}}_Q(\mathbf{x})\mathbf{f}_{Q,R}(\hat{\mathbf{x}},-\hat{\mathbf{x}}')\bar{\mathbf{G}}_R(-\mathbf{x}') \tag{66}$$

and

$$\mathbf{f}_{Q,R}(\hat{\mathbf{x}},-\hat{\mathbf{x}}') = \boldsymbol{\theta}_Q^T(\hat{\mathbf{x}})\mathbf{F}(\hat{\mathbf{x}},-\hat{\mathbf{x}}')\boldsymbol{\theta}_R(-\hat{\mathbf{x}}'), \tag{67}$$

where each of the subscripts Q and R can stand for either P or S. Here $\mathbf{f}_{Q,R}(\hat{\mathbf{x}},-\hat{\mathbf{x}}')$ is a 3×3 scattering matrix for an incident R-type wave in the $-\hat{\mathbf{x}}'$ direction, scattered as a Q-type wave in the $\hat{\mathbf{x}}$ direction.

Apply $\boldsymbol{\theta}_Q^T(\hat{\mathbf{x}}')$ and $\boldsymbol{\theta}_R(\hat{\mathbf{x}}'')$ to both sides of the unified optical theorem [Eq. (50)], as follows:

$$\oint \boldsymbol{\theta}_Q^T(\hat{\mathbf{x}}')\mathbf{F}^\dagger(\hat{\mathbf{x}},\hat{\mathbf{x}}')\Theta(\hat{\mathbf{x}})\mathbf{F}(\hat{\mathbf{x}},\hat{\mathbf{x}}'')\boldsymbol{\theta}_R(\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}} = \frac{1}{2i}\boldsymbol{\theta}_Q^T(\hat{\mathbf{x}}')\left(\mathbf{F}(\hat{\mathbf{x}}',\hat{\mathbf{x}}'') - \mathbf{F}^\dagger(\hat{\mathbf{x}}'',\hat{\mathbf{x}}')\right)\boldsymbol{\theta}_R(\hat{\mathbf{x}}''). \tag{68}$$

According to Appendix C matrix $\Theta(\hat{\mathbf{x}})$ is given by

$$\Theta(\hat{\mathbf{x}}) = \frac{\omega}{4\pi\rho}\left(\frac{1}{c_P^3}\boldsymbol{\theta}_P(\hat{\mathbf{x}})\boldsymbol{\Gamma}(\hat{\mathbf{x}})\boldsymbol{\theta}_P^T(\hat{\mathbf{x}}) + \frac{1}{c_S^3}\boldsymbol{\theta}_S(\hat{\mathbf{x}})\{\mathbf{I}-\boldsymbol{\Gamma}(\hat{\mathbf{x}})\}\boldsymbol{\theta}_S^T(\hat{\mathbf{x}})\right). \tag{69}$$

Substituting this into Eq. (68), using Eq. (67), taking into account that $\boldsymbol{\theta}_P(\hat{\mathbf{x}})$ and $\boldsymbol{\theta}_S(\hat{\mathbf{x}})$ are real-valued, yields

$$\frac{\omega}{4\pi\rho c_P^3}\oint \mathbf{f}^\dagger_{P,Q}(\hat{\mathbf{x}},\hat{\mathbf{x}}')\boldsymbol{\Gamma}(\hat{\mathbf{x}})\mathbf{f}_{P,R}(\hat{\mathbf{x}},\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}} + \frac{\omega}{4\pi\rho c_S^3}\oint \mathbf{f}^\dagger_{S,Q}(\hat{\mathbf{x}},\hat{\mathbf{x}}')\{\mathbf{I}-\boldsymbol{\Gamma}(\hat{\mathbf{x}})\}\mathbf{f}_{S,R}(\hat{\mathbf{x}},\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}} = \frac{1}{2i}\left(\mathbf{f}_{Q,R}(\hat{\mathbf{x}}',\hat{\mathbf{x}}'') - \mathbf{f}^\dagger_{R,Q}(\hat{\mathbf{x}}'',\hat{\mathbf{x}}')\right), \tag{70}$$

with $\boldsymbol{\Gamma}(\hat{\mathbf{x}})$ again defined in Eq. (62). Equation (70) is the generalized optical theorem for elastodynamic P- and S-waves.[12]

For a piezoelectric scattering domain $\mathbb{D}_s$, the scattering matrix $\mathbf{F}(\hat{\mathbf{x}},-\hat{\mathbf{x}}')$ is subdivided as follows:

$$\mathbf{F}(\hat{\mathbf{x}},-\hat{\mathbf{x}}') = \begin{pmatrix} \mathbf{F}^{\mathrm{EM,EM}}(\hat{\mathbf{x}},-\hat{\mathbf{x}}') & \mathbf{F}^{\mathrm{EM,ED}}(\hat{\mathbf{x}},-\hat{\mathbf{x}}') \\ \mathbf{F}^{\mathrm{ED,EM}}(\hat{\mathbf{x}},-\hat{\mathbf{x}}') & \mathbf{F}^{\mathrm{ED,ED}}(\hat{\mathbf{x}},-\hat{\mathbf{x}}') \end{pmatrix}, \tag{71}$$

where superscripts EM and ED stand for electromagnetic and elastodynamic waves, respectively. The second superscript refers to the type of incident wave, propagating in the $-\hat{\mathbf{x}}'$ direction, whereas the first superscript refers to the type of scattered wave, propagating in the $\hat{\mathbf{x}}$ direction. Substitute this expression into the unified optical theorem [Eq. (50)], together with Eq. (C68) for $\Theta(\hat{\mathbf{x}})$, and rewrite the result in terms of its submatrices. This yields

$$\oint \{\mathbf{F}^{\mathrm{EM,U}}(\hat{\mathbf{x}},\hat{\mathbf{x}}')\}^\dagger\Theta^{\mathrm{EM}}(\hat{\mathbf{x}})\mathbf{F}^{\mathrm{EM,V}}(\hat{\mathbf{x}},\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}} + \oint \{\mathbf{F}^{\mathrm{ED,U}}(\hat{\mathbf{x}},\hat{\mathbf{x}}')\}^\dagger\Theta^{\mathrm{ED}}(\hat{\mathbf{x}})\mathbf{F}^{\mathrm{ED,V}}(\hat{\mathbf{x}},\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}} = \frac{1}{2i}\left(\mathbf{F}^{\mathrm{U,V}}(\hat{\mathbf{x}}',\hat{\mathbf{x}}'') - \{\mathbf{F}^{\mathrm{V,U}}(\hat{\mathbf{x}}'',\hat{\mathbf{x}}')\}^\dagger\right), \tag{72}$$

where each of the superscripts U and V can stand for either EM or ED. Here $\Theta^{\mathrm{EM}}(\hat{\mathbf{x}})$ and $\Theta^{\mathrm{ED}}(\hat{\mathbf{x}})$ are defined by Eqs. (60) and (69), respectively. Introduce a vector $\boldsymbol{\theta}_{\mathrm{Q}}^{\mathrm{U}}(\hat{\mathbf{x}}')$. For $\mathrm{U}=\mathrm{EM}$ this is the same as vector $\boldsymbol{\theta}(\hat{\mathbf{x}}')$ used for electromagnetic waves [e.g., as in Eq. (60)]; in this case subscript Q is a dummy subscript. For U = ED this vector is the same as $\boldsymbol{\theta}_Q(\hat{\mathbf{x}}')$ used for elastodynamic waves [e.g., as in Eq. (69)]; in this case subscript Q can stand for either P or S. Apply $\left\{\boldsymbol{\theta}_Q^{\rm U}(\hat{\mathbf{x}}')\right\}^T$ and $\boldsymbol{\theta}_R^{\rm V}(\hat{\mathbf{x}}'')$ to both sides of Eq. (72), in a similar way as in Eq. (68), and substitute Eqs. (60) and (69). This gives

$$\begin{aligned}
&\frac{(\mu k)^{\rm EM}}{4\pi}\oint\left\{\mathbf{f}_{,Q}^{\rm EM,U}(\hat{\mathbf{x}},\hat{\mathbf{x}}')\right\}^{\dagger}\{\mathbf{I}-\boldsymbol{\Gamma}(\hat{\mathbf{x}})\}\mathbf{f}_{,Q}^{\rm EM,V}(\hat{\mathbf{x}},\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}}\\
&+\frac{\omega}{4\pi\rho c_P^3}\oint\left\{\mathbf{f}_{P,Q}^{\rm ED,U}(\hat{\mathbf{x}},\hat{\mathbf{x}}')\right\}^{\dagger}\boldsymbol{\Gamma}(\hat{\mathbf{x}})\mathbf{f}_{P,R}^{\rm ED,V}(\hat{\mathbf{x}},\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}}\\
&+\frac{\omega}{4\pi\rho c_S^3}\oint\left\{\mathbf{f}_{S,Q}^{\rm ED,U}(\hat{\mathbf{x}},\hat{\mathbf{x}}')\right\}^{\dagger}\{\mathbf{I}-\boldsymbol{\Gamma}(\hat{\mathbf{x}})\}\mathbf{f}_{S,R}^{\rm ED,V}(\hat{\mathbf{x}},\hat{\mathbf{x}}'')\mathrm{d}\Omega_{\hat{\mathbf{x}}}\\
&=\frac{1}{2i}\left(\mathbf{f}_{Q,R}^{\rm U,V}(\hat{\mathbf{x}}',\hat{\mathbf{x}}'')-\left\{\mathbf{f}_{R,Q}^{\rm V,U}(\hat{\mathbf{x}}'',\hat{\mathbf{x}}')\right\}^{\dagger}\right),
\end{aligned}\tag{73}$$

with

$$\mathbf{f}_{Q,R}^{\rm U,V}(\hat{\mathbf{x}}',\hat{\mathbf{x}}'')=\left\{\boldsymbol{\theta}_Q^{\rm U}(\hat{\mathbf{x}}')\right\}^T\mathbf{F}^{\rm U,V}(\hat{\mathbf{x}}',\hat{\mathbf{x}}'')\boldsymbol{\theta}_R^{\rm V}(\hat{\mathbf{x}}''). \tag{74}$$

Equation (73) is the generalized optical theorem for electromagnetic and elastodynamic P- and S-waves, scattered by a piezoelectric contrast in a homogeneous, isotropic embedding.

VII. CONCLUSIONS

Recently, Douma *et al.*[26] derived the generalized optical theorem from reciprocity theorems for acoustic waves in perturbed media. They suggested that their approach could possibly be used to derive a unified optical theorem from a unified Green's function representation.[29] Here it has been shown that this can indeed be done. Equation (50) formulates the unified optical theorem in a compact way. It has been shown in Sec. VI that Eq. (50) encompasses most versions of the optical theorem that have been presented in the literature. Moreover, this unified optical theorem also holds for scattering by anisotropic elastic and piezoelectric scatterers and by bianisotropic (i.e., non-reciprocal) electromagnetic scatterers.

Among the applications of the generalized optical theorem mentioned in the literature are (1) testing numerical modeling schemes for scattering amplitudes,[5] (2) reconstructing the structure of a scatterer from power extinction experiments,[40] and (3) retrieving the scattered part of the Green's function from ambient noise and explaining the spurious events that occur when the noise is not equipartitioned.[23] The unified optical theorem formulated in Eq. (50) provides a starting point for applying these and other methods to the different types of scatterers handled in this paper.

ACKNOWLEDGMENTS

This work was supported by the Netherlands Research Centre for Integrated Solid Earth Science (ISES). H.D. thanks ION Geophysical/GXT Imaging solutions for permission to publish this work. We thank the reviewers for their constructive comments, Evert Slob for the discussions about bianisotropic materials, and Niels Grobbe for his assistance with Appendix A 5 on the piezoelectric wave equation.

APPENDIX A: MATRIX-VECTOR WAVE EQUATIONS

1. Acoustic wave equation

The basic equations for acoustic wave propagation in an inhomogeneous, dissipative, non-flowing fluid are the linearized equation of motion

$$\rho\partial_t v_i + b^v v_i + \partial_i p = f_i \tag{A1}$$

and the linearized stress-strain relation

$$\kappa\partial_t p + b^p p + \partial_i v_i = q. \tag{A2}$$

Lower-case latin subscripts (except t) take on the values 1, 2, and 3 and Einstein's summation convention applies to repeated indices. Here $p = p(\mathbf{x}, t)$ and $v_i = v_i(\mathbf{x}, t)$ represent the acoustic wave field in terms of acoustic pressure and particle velocity, respectively; $\rho = \rho(\mathbf{x})$ and $\kappa = \kappa(\mathbf{x})$ are the medium parameters mass density and compressibility, respectively; $b^v = b^v(\mathbf{x})$ and $b^p = b^p(\mathbf{x})$ are the loss parameters of the medium; finally, $f_i = f_i(\mathbf{x}, t)$ and $q = q(\mathbf{x}, t)$ represent the sources in terms of external volume force and volume injection rate, respectively. These equations can be combined into the general matrix-vector wave Eq. (1), with

$$\mathbf{u}=\begin{pmatrix}p\\v_1\\v_2\\v_3\end{pmatrix},\quad \mathbf{s}=\begin{pmatrix}q\\f_1\\f_2\\f_3\end{pmatrix},\quad \mathbf{A}=\begin{pmatrix}\kappa&0&0&0\\0&\rho&0&0\\0&0&\rho&0\\0&0&0&\rho\end{pmatrix}, \tag{A3}$$

$$\mathbf{B}=\begin{pmatrix}b^p&0&0&0\\0&b^v&0&0\\0&0&b^v&0\\0&0&0&b^v\end{pmatrix},\quad \mathbf{D}_{\mathbf{x}}=\begin{pmatrix}0&\partial_1&\partial_2&\partial_3\\\partial_1&0&0&0\\\partial_2&0&0&0\\\partial_3&0&0&0\end{pmatrix}. \tag{A4}$$

Note that $\mathbf{D}_{\mathbf{x}}$ obeys symmetry relations (2) and (3), with $\mathbf{K}$ defined as

$$\mathbf{K}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}. \tag{A5}$$

Matrices $\mathbf{N}_{\mathbf{x}}$, and $\mathbf{M}(\hat{\mathbf{x}})$, introduced in Eqs. (7) and (34), respectively, are defined as

$$\mathbf{N}_{\mathbf{x}}=\begin{pmatrix}0&n_1&n_2&n_3\\n_1&0&0&0\\n_2&0&0&0\\n_3&0&0&0\end{pmatrix},\quad \mathbf{M}(\hat{\mathbf{x}})=\begin{pmatrix}0&\hat{x}_1&\hat{x}_2&\hat{x}_3\\\hat{x}_1&0&0&0\\\hat{x}_2&0&0&0\\\hat{x}_3&0&0&0\end{pmatrix}. \tag{A6}$$

The frequency-domain matrix $\boldsymbol{\mathcal{A}}$, defined in Eq. (6), is given by

$$\boldsymbol{\mathcal{A}}(\mathbf{x},\omega)=\begin{pmatrix}\kappa(\mathbf{x},\omega)&0&0&0\\0&\rho(\mathbf{x},\omega)&0&0\\0&0&\rho(\mathbf{x},\omega)&0\\0&0&0&\rho(\mathbf{x},\omega)\end{pmatrix}, \tag{A7}$$

with

$$\kappa(\mathbf{x},\omega) = \kappa(\mathbf{x}) - \frac{b^p(\mathbf{x})}{i\omega}, \tag{A8}$$

$$\rho(\mathbf{x},\omega) = \rho(\mathbf{x}) - \frac{b^v(\mathbf{x})}{i\omega}. \tag{A9}$$

Note that $\boldsymbol{\mathcal{A}} = \mathbf{K}\boldsymbol{\mathcal{A}}^T\mathbf{K}$. Combined with Eq. (9) this implies $\boldsymbol{\mathcal{A}}^{(a)} = \boldsymbol{\mathcal{A}}$, meaning that the medium is reciprocal. Energy is conserved when $\boldsymbol{\mathcal{A}}^\dagger = \boldsymbol{\mathcal{A}}$, i.e., when $\Im\{\kappa(\mathbf{x},\omega)\} = \Im\{\rho(\mathbf{x},\omega)\} = 0$, where $\Im$ denotes the imaginary part.

2. Quantum-mechanical wave equation

Schrödinger's wave equation for a particle with mass m in a potential $V = V(\mathbf{x})$ is given by[41,42]

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\partial_i\partial_i\psi + V\psi, \tag{A10}$$

where $\psi = \psi(\mathbf{x}, t)$ is the wave function and $\hbar = h/2\pi$, with h Planck's constant. This equation can be captured in the general matrix-vector wave Eq. (1), with

$$\mathbf{u} = \begin{pmatrix} \psi \\ (\hbar/i)\partial_1\psi \\ (\hbar/i)\partial_2\psi \\ (\hbar/i)\partial_3\psi \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 2m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 2miV/\hbar & 0 & 0 & 0 \\ 0 & -i/\hbar & 0 & 0 \\ 0 & 0 & -i/\hbar & 0 \\ 0 & 0 & 0 & -i/\hbar \end{pmatrix}, \tag{A11}$$

and $\mathbf{s}$ a nul-vector. Furthermore, $\mathbf{D_x}$, $\mathbf{K}$, $\mathbf{N_x}$, and $\mathbf{M}(\hat{\mathbf{x}})$ are defined in Eqs. (A4)–(A6). The frequency-domain matrix $\boldsymbol{\mathcal{A}}$, defined in Eq. (6), is given by

$$\boldsymbol{\mathcal{A}}(\mathbf{x},\omega) = \begin{pmatrix} 2m\left(1 - \dfrac{V(\mathbf{x})}{\hbar\omega}\right) & 0 & 0 & 0 \\ 0 & \dfrac{1}{\hbar\omega} & 0 & 0 \\ 0 & 0 & \dfrac{1}{\hbar\omega} & 0 \\ 0 & 0 & 0 & \dfrac{1}{\hbar\omega} \end{pmatrix}. \tag{A12}$$

Note that $\boldsymbol{\mathcal{A}} = \mathbf{K}\boldsymbol{\mathcal{A}}^T\mathbf{K}$, hence $\boldsymbol{\mathcal{A}}^{(a)} = \boldsymbol{\mathcal{A}}$, meaning that reciprocity is obeyed. Furthermore, $\boldsymbol{\mathcal{A}}^\dagger = \boldsymbol{\mathcal{A}}$, hence, probability is conserved.

3. Electromagnetic wave equation

Maxwell's equations for electromagnetic wave propagation read[38,43]

$$\varepsilon_{ik}\partial_t E_k + \sigma_{ik}E_k - \epsilon_{ijk}\partial_j H_k = -J_i^e, \tag{A13}$$

$$\mu_{km}\partial_t H_m + \epsilon_{klm}\partial_l E_m = -J_k^m, \tag{A14}$$

where $E_k = E_k(\mathbf{x}, t)$ and $H_k = H_k(\mathbf{x}, t)$ are the electric and magnetic field strengths, respectively; $\varepsilon_{ik} = \varepsilon_{ik}(\mathbf{x})$, $\mu_{km} = \mu_{km}(\mathbf{x})$, and $\sigma_{ik} = \sigma_{ik}(\mathbf{x})$ are the anisotropic permittivity, permeability, and conductivity, respectively; $J_i^e = J_i^e(\mathbf{x}, t)$ and $J_k^m = J_k^m(\mathbf{x}, t)$ are source functions in terms of the external electric and magnetic current densities; finally, ϵ_{ijk} is the alternating tensor (or Levi–Civita tensor), with $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$, $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$, and all other components being zero. The permittivity, permeability and conductivity obey the symmetry relations $\varepsilon_{ik} = \varepsilon_{ki}$, $\mu_{km} = \mu_{mk}$, and $\sigma_{ik} = \sigma_{ki}$, respectivey. Equations (A13) and (A14) can be combined into the general matrix-vector Eq. (1), with

$$\mathbf{u} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} -\mathbf{J}^e \\ -\mathbf{J}^m \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \boldsymbol{\varepsilon} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{\mu} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \tag{A15}$$

$$\mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}, \quad \mathbf{J}^e = \begin{pmatrix} J_1^e \\ J_2^e \\ J_3^e \end{pmatrix},$$

$$\mathbf{J}^m = \begin{pmatrix} J_1^m \\ J_2^m \\ J_3^m \end{pmatrix}, \tag{A16}$$

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix},$$

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}, \tag{A17}$$

$$\mathbf{D_x} = \begin{pmatrix} \mathbf{O} & \mathbf{D}_0^T \\ \mathbf{D}_0 & \mathbf{O} \end{pmatrix}, \quad \mathbf{D}_0 = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}, \tag{A18}$$

with $\mathbf{O}$ being the 3×3 null matrix. Note that $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T$, $\boldsymbol{\mu} = \boldsymbol{\mu}^T$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$. $\mathbf{D_x}$ obeys symmetry relations (2) and (3), with $\mathbf{K}$ defined as

$$\mathbf{K} = \begin{pmatrix} -\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \tag{A19}$$

with $\mathbf{I}$ being the 3×3 identity matrix. The frequency-domain matrix $\boldsymbol{\mathcal{A}}$, defined in Eq. (6), is given by

$$\boldsymbol{\mathcal{A}}(\mathbf{x},\omega) = \begin{pmatrix} \boldsymbol{\varepsilon}(\mathbf{x},\omega) & \mathbf{O} \\ \mathbf{O} & \boldsymbol{\mu}(\mathbf{x}) \end{pmatrix}, \tag{A20}$$

with

$$\boldsymbol{\varepsilon}(\mathbf{x},\omega) = \boldsymbol{\varepsilon}(\mathbf{x}) - \frac{\boldsymbol{\sigma}(\mathbf{x})}{i\omega}. \tag{A21}$$

More generally, for bianisotropic materials this matrix becomes a full matrix, according to[35,43,44]

$$\boldsymbol{\mathcal{A}}(\mathbf{x},\omega) = \begin{pmatrix} \boldsymbol{\varepsilon}(\mathbf{x},\omega) & \boldsymbol{\xi}(\mathbf{x},\omega) \\ \boldsymbol{\zeta}(\mathbf{x},\omega) & \boldsymbol{\mu}(\mathbf{x},\omega) \end{pmatrix}. \tag{A22}$$

Note that

$$\boldsymbol{\mathcal{A}}^{(a)} = \mathbf{K}\boldsymbol{\mathcal{A}}^T\mathbf{K} = \begin{pmatrix} \boldsymbol{\varepsilon}^T(\mathbf{x},\omega) & -\boldsymbol{\zeta}^T(\mathbf{x},\omega) \\ -\boldsymbol{\xi}^T(\mathbf{x},\omega) & \boldsymbol{\mu}^T(\mathbf{x},\omega) \end{pmatrix}. \tag{A23}$$

When $\boldsymbol{\zeta} = -\boldsymbol{\xi}^T$ we have $\boldsymbol{\mathcal{A}}^{(a)} = \boldsymbol{\mathcal{A}}$, meaning that the medium is reciprocal.[45] On the other hand, when $\boldsymbol{\zeta} = \boldsymbol{\xi}^T$ the medium is non-reciprocal. Energy is conserved when $\boldsymbol{\mathcal{A}}^\dagger = \boldsymbol{\mathcal{A}}$. In all cases this requires $\Im(\boldsymbol{\varepsilon}) = \Im(\boldsymbol{\mu}) = \mathbf{O}$. In addition, for reciprocal media it requires $\Re(\boldsymbol{\zeta}) = \Re(\boldsymbol{\xi}) = \mathbf{O}$, which occurs in so-called chiral media.[46] On the other hand, for nonreciprocal media obeying $\boldsymbol{\zeta} = \boldsymbol{\xi}^T$ it requires $\Im(\boldsymbol{\zeta}) = \Im(\boldsymbol{\xi}) = \mathbf{O}$, which occurs for example in so-called Faraday media.[47]

4. Elastodynamic wave equation

The linearized equation of motion in a lossless solid reads[38,48,49]

$$\rho\partial_t v_i - \partial_j\tau_{ij} = f_i, \tag{A24}$$

where v_i and τ_{ij} are the particle velocity and stress tensor, respectively, associated to the elastodynamic wave field, ρ is the mass density of the medium and f_i the external volume force. The stress tensor is symmetric, i.e., $\tau_{ij} = \tau_{ji}$. Hooke's linearized stress-strain relation reads

$$-s_{ijkl}\partial_t\tau_{kl} + \left(\partial_i v_j + \partial_j v_i\right)/2 = h_{ij}, \tag{A25}$$

where h_{ij} is the external deformation rate, with $h_{ij} = h_{ji}$, and s_{ijkl} is the compliance tensor, with $s_{ijkl} = s_{jikl} = s_{ijlk} = s_{klij}$. Equations (A24) and (A25) can be combined to yield the general matrix-vector Eq. (1). To this end, rewrite these equations as

$$\rho\partial_t\mathbf{v} - \mathbf{D}_1\boldsymbol{\tau}_1 - \mathbf{D}_2\boldsymbol{\tau}_2 = \mathbf{f} \tag{A26}$$

and

$$-\mathbf{s}_{11}\partial_t\boldsymbol{\tau}_1 - 2\mathbf{s}_{12}\partial_t\boldsymbol{\tau}_2 + \mathbf{D}_1\mathbf{v} = \mathbf{h}_1, \tag{A27}$$

$$-2\mathbf{s}_{21}\partial_t\boldsymbol{\tau}_1 - 4\mathbf{s}_{22}\partial_t\boldsymbol{\tau}_2 + \mathbf{D}_2\mathbf{v} = \mathbf{h}_2, \tag{A28}$$

where

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \boldsymbol{\tau}_1 = \begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \end{pmatrix}, \quad \boldsymbol{\tau}_2 = \begin{pmatrix} \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{pmatrix}, \tag{A29}$$

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad \mathbf{h}_1 = \begin{pmatrix} h_{11} \\ h_{22} \\ h_{33} \end{pmatrix}, \quad \mathbf{h}_2 = \begin{pmatrix} 2h_{23} \\ 2h_{31} \\ 2h_{12} \end{pmatrix}, \tag{A30}$$

$$\mathbf{s}_{11} = \begin{pmatrix} s_{1111} & s_{1122} & s_{1133} \\ s_{2211} & s_{2222} & s_{2233} \\ s_{3311} & s_{3322} & s_{3333} \end{pmatrix},$$

$$\mathbf{s}_{12} = \begin{pmatrix} s_{1123} & s_{1131} & s_{1112} \\ s_{2223} & s_{2231} & s_{2212} \\ s_{3323} & s_{3331} & s_{3312} \end{pmatrix}, \tag{A31}$$

$$\mathbf{s}_{21} = \mathbf{s}_{12}^T, \quad \mathbf{s}_{22} = \begin{pmatrix} s_{2323} & s_{2331} & s_{2312} \\ s_{3123} & s_{3131} & s_{3112} \\ s_{1223} & s_{1231} & s_{1212} \end{pmatrix}, \tag{A32}$$

and

$$\mathbf{D}_1 = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \end{pmatrix}, \quad \mathbf{D}_2 = \begin{pmatrix} 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{pmatrix}. \tag{A33}$$

Equations (A26)–(A28) can be combined into the general matrix-vector Eq. (1), with

$$\mathbf{u} = \begin{pmatrix} \mathbf{v} \\ -\boldsymbol{\tau}_1 \\ -\boldsymbol{\tau}_2 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} \mathbf{f} \\ \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix}, \tag{A34}$$

$$\mathbf{A} = \begin{pmatrix} \rho\mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{s}_{11} & 2\mathbf{s}_{12} \\ \mathbf{O} & 2\mathbf{s}_{21} & 4\mathbf{s}_{22} \end{pmatrix}, \quad \mathbf{D}_\mathbf{x} = \begin{pmatrix} \mathbf{O} & \mathbf{D}_1 & \mathbf{D}_2 \\ \mathbf{D}_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{D}_2 & \mathbf{O} & \mathbf{O} \end{pmatrix}, \tag{A35}$$

and $\mathbf{B}$ a 9×9 null matrix (for the situation of a medium with losses, matrix $\mathbf{B}$ would account for the losses and have a similar structure as matrix $\mathbf{A}$). $\mathbf{D}_\mathbf{x}$ obeys symmetry relations (2) and (3), with $\mathbf{K}$ defined as

$$\mathbf{K} = \begin{pmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -\mathbf{I} \end{pmatrix}. \tag{A36}$$

The frequency-domain matrix $\boldsymbol{\mathcal{A}}$, defined in Eq. (6), is for this case identical to $\mathbf{A}$, defined in Eq. (A35), because a lossless solid is considered.

Note that $\boldsymbol{\mathcal{A}} = \mathbf{K}\boldsymbol{\mathcal{A}}^T\mathbf{K}$, hence $\boldsymbol{\mathcal{A}}^{(a)} = \boldsymbol{\mathcal{A}}$, meaning that reciprocity is obeyed. Furthermore, $\boldsymbol{\mathcal{A}}^\dagger = \boldsymbol{\mathcal{A}}$, hence, energy is conserved.

5. Piezoelectric wave equation

The equations for coupled electromagnetic and elastodynamic waves in a lossless piezoelectric material read[43,50]

$$\varepsilon_{ik}\partial_t E_k - \epsilon_{ijk}\partial_j H_k + d_{ijk}\partial_t\tau_{jk} = -J_i^e, \tag{A37}$$

$$\mu_{km}\partial_t H_m + \epsilon_{klm}\partial_l E_m = -J_k^m, \tag{A38}$$

$$\rho\partial_t v_i - \partial_j\tau_{ij} = f_i, \tag{A39}$$

$$-s_{ijkl}\partial_t\tau_{kl} + \left(\partial_i v_j + \partial_j v_i\right)/2 - d_{ijk}\partial_t E_k = h_{ij}, \tag{A40}$$

where d_{ijk} is the coupling tensor, with $d_{ijk} = d_{jik} = d_{ikj}$. Note that ε_{ik} in Eq. (A37) and s_{ijkl} in Eq. (A40) are parameters measured under constant stress and constant electric field, respectively. Equations (A37)–(A40) can be combined into the general matrix-vector Eq. (1), with

$$\mathbf{u} = \begin{pmatrix} \mathbf{u}^{\mathrm{EM}} \\ \mathbf{u}^{\mathrm{ED}} \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} \mathbf{s}^{\mathrm{EM}} \\ \mathbf{s}^{\mathrm{ED}} \end{pmatrix}, \tag{A41}$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}^{\mathrm{EM}} & \mathbf{A}^{\mathrm{C}} \\ \left\{\mathbf{A}^{\mathrm{C}}\right\}^T & \mathbf{A}^{\mathrm{ED}} \end{pmatrix}, \quad \mathbf{D}_\mathbf{x} = \begin{pmatrix} \mathbf{D}_\mathbf{x}^{\mathrm{EM}} & \mathbf{O} \\ \mathbf{O} & \mathbf{D}_\mathbf{x}^{\mathrm{ED}} \end{pmatrix}, \tag{A42}$$

and $\mathbf{B}$ a 15×15 null matrix. Superscripts EM and ED stand for electromagnetic and elastodynamic, respectively. The expressions for the wave field vectors, source vectors, medium parameter matrices, and differential operators with superscripts EM and ED are given in Appendixes A 3 and A 4, respectively (but here only the lossless reciprocal case is considered). The coupling matrix $\mathbf{A}^{\mathrm{C}}$ is defined as follows:

$$\mathbf{A}^{\mathrm{C}} = \begin{pmatrix} \mathbf{O} & -\mathbf{d}_1 & -2\mathbf{d}_2 \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}, \tag{A43}$$

with

$$\mathbf{d}_1 = \begin{pmatrix} d_{111} & d_{122} & d_{133} \\ d_{211} & d_{222} & d_{233} \\ d_{311} & d_{322} & d_{333} \end{pmatrix}, \quad \mathbf{d}_2 = \begin{pmatrix} d_{123} & d_{131} & d_{112} \\ d_{223} & d_{231} & d_{212} \\ d_{323} & d_{331} & d_{312} \end{pmatrix}. \tag{A44}$$

$\mathbf{D}_{\mathbf{x}}$ obeys symmetry relations (2) and (3), with $\mathbf{K}$ defined as

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}^{\mathrm{EM}} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}^{\mathrm{ED}} \end{pmatrix}, \tag{A45}$$

with $\mathbf{K}^{\mathrm{EM}}$ and $\mathbf{K}^{\mathrm{ED}}$ defined in Appendixes A 3 and A 4, respectively. The frequency-domain matrix $\boldsymbol{\mathcal{A}}$, defined in Eq. (6), is identical to $\mathbf{A}$, defined in Eq. (A42), because a lossless material is considered.

Note that $\boldsymbol{\mathcal{A}} = \mathbf{K}\boldsymbol{\mathcal{A}}^T\mathbf{K}$, hence $\boldsymbol{\mathcal{A}}^{(a)} = \boldsymbol{\mathcal{A}}$, meaning that reciprocity is obeyed. Furthermore, $\boldsymbol{\mathcal{A}}^{\dagger} = \boldsymbol{\mathcal{A}}$, hence, energy is conserved.

APPENDIX B: THE THEOREM OF GAUSS IN MATRIX-VECTOR FORM

For a scalar field $a(\mathbf{x})$, the theorem of Gauss reads

$$\int_{\mathbb{D}} \partial_i a(\mathbf{x}) \mathrm{d}^3\mathbf{x} = \oint_{\partial\mathbb{D}} a(\mathbf{x}) n_i \mathrm{d}^2\mathbf{x}. \tag{B1}$$

Here this theorem is modified for the differential operator matrix $\mathbf{D}_{\mathbf{x}}$ appearing in Eqs. (1) and (5). Let D_{IJ} denote the operator in row I and column J of matrix $\mathbf{D}_{\mathbf{x}}$. The symmetry of $\mathbf{D}_{\mathbf{x}}$ [Eq. (2)] implies $D_{IJ} = D_{JI}$. Define a matrix $\mathbf{N}_{\mathbf{x}}$ which contains the components of the normal vector $\mathbf{n}$ on $\partial\mathbb{D}$, organized in the same way as matrix $\mathbf{D}_{\mathbf{x}}$. Hence, $N_{IJ} = N_{JI}$, where N_{IJ} denotes the element in row I and column J of matrix $\mathbf{N}_{\mathbf{x}}$. Replace the scalar field $a(\mathbf{x})$ by $a_I(\mathbf{x})b_J(\mathbf{x})$ and sum both sides of Eq. (B1) over I and J. This yields

$$\int_{\mathbb{D}} D_{IJ}\{a_I(\mathbf{x})b_J(\mathbf{x})\} \mathrm{d}^3\mathbf{x} = \oint_{\partial\mathbb{D}} a_I(\mathbf{x})b_J(\mathbf{x})N_{IJ} \mathrm{d}^2\mathbf{x}, \tag{B2}$$

where the summation convention applies to repeated capital Latin subscripts, which may run from 1 to 4, 6, 9, or 15, depending on the choice of operator $\mathbf{D}_{\mathbf{x}}$. Applying the product rule for differentiation and using the symmetry property $D_{IJ} = D_{JI}$ yields for the integrand in the left-hand side of Eq. (B2)

$$D_{IJ}(a_I b_J) = a_I D_{IJ} b_J + (D_{JI} a_I) b_J = \mathbf{a}^T \mathbf{D}_{\mathbf{x}} \mathbf{b} + (\mathbf{D}_{\mathbf{x}} \mathbf{a})^T \mathbf{b}, \tag{B3}$$

where $\mathbf{a}$ and $\mathbf{b}$ are vector functions, containing the scalar functions $a_I(\mathbf{x})$ and $b_J(\mathbf{x})$, respectively, Rewriting the integrand in the right-hand side of Eq. (B2) in a similar way, gives the theorem of Gauss in matrix-vector form

$$\int_{\mathbb{D}} \left\{ \mathbf{a}^T \mathbf{D}_{\mathbf{x}} \mathbf{b} + (\mathbf{D}_{\mathbf{x}} \mathbf{a})^T \mathbf{b} \right\} \mathrm{d}^3\mathbf{x} = \oint_{\partial\mathbb{D}} \mathbf{a}^T \mathbf{N}_{\mathbf{x}} \mathbf{b} \mathrm{d}^2\mathbf{x}. \tag{B4}$$

APPENDIX C: GREEN'S MATRICES

1. Acoustic Green's matrix

The frequency-domain Green's matrix $\mathbf{G}(\mathbf{x}, \mathbf{x}')$ is a $L \times L$ matrix, obeying wave equation (15). The element in the kth row and lth column represents the wave field quantity of the kth type observed at $\mathbf{x}$, due to a unit source of the lth type at $\mathbf{x}'$. Here "wave field quantity of the kth type" means the wave field quantity represented by the kth element of wave field vector $\mathbf{u}$. Similarly, "source of the lth type" means the type of source represented by the lth element of source vector $\mathbf{s}$. Hence, for the acoustic situation the Green's matrix can be written as

$$\mathbf{G}(\mathbf{x}, \mathbf{x}') = \begin{pmatrix} G^{p,q} & G^{p,f}_{,1} & G^{p,f}_{,2} & G^{p,f}_{,3} \\ G^{v,q}_1 & G^{v,f}_{1,1} & G^{v,f}_{1,2} & G^{v,f}_{1,3} \\ G^{v,q}_2 & G^{v,f}_{2,1} & G^{v,f}_{2,2} & G^{v,f}_{2,3} \\ G^{v,q}_3 & G^{v,f}_{3,1} & G^{v,f}_{3,2} & G^{v,f}_{3,3} \end{pmatrix} (\mathbf{x}, \mathbf{x}'). \tag{C1}$$

Superscripts p and v refer to the observations of acoustic pressure and particle velocity, respectively, at $\mathbf{x}$, whereas superscripts q and f refer to sources of volume injection rate and external volume force, respectively, at $\mathbf{x}'$. The subscripts refer to the components of the particle velocity and volume force, respectively.

A non-flowing acoustic medium is reciprocal, see Appendix A 1. Hence, symmetry relation (18), with $\mathbf{K}$ defined in Eq. (A5), gives

$$\begin{pmatrix} G^{p,q} & G^{p,f}_{,1} & G^{p,f}_{,2} & G^{p,f}_{,3} \\ G^{v,q}_1 & G^{v,f}_{1,1} & G^{v,f}_{1,2} & G^{v,f}_{1,3} \\ G^{v,q}_2 & G^{v,f}_{2,1} & G^{v,f}_{2,2} & G^{v,f}_{2,3} \\ G^{v,q}_3 & G^{v,f}_{3,1} & G^{v,f}_{3,2} & G^{v,f}_{3,3} \end{pmatrix} (\mathbf{x}, \mathbf{x}')$$
$$= \begin{pmatrix} G^{p,q} & -G^{v,q}_1 & -G^{v,q}_2 & -G^{v,q}_3 \\ -G^{p,f}_{,1} & G^{v,f}_{1,1} & G^{v,f}_{2,1} & G^{v,f}_{3,1} \\ -G^{p,f}_{,2} & G^{v,f}_{1,2} & G^{v,f}_{2,2} & G^{v,f}_{3,2} \\ -G^{p,f}_{,3} & G^{v,f}_{1,3} & G^{v,f}_{2,3} & G^{v,f}_{3,3} \end{pmatrix} (\mathbf{x}', \mathbf{x}). \tag{C2}$$

For convenience $G^{p,q}$ is renamed as G_0. All elements of matrix $\mathbf{G}(\mathbf{x}, \mathbf{x}')$ are now expressed in terms of $G_0(\mathbf{x}, \mathbf{x}')$, for $\mathbf{x} \neq \mathbf{x}'$. Transforming the equation of motion (A1) to the frequency domain, gives for the first column of $\mathbf{G}(\mathbf{x}, \mathbf{x}')$,

$$\begin{pmatrix} 1 \\ \dfrac{\partial_1}{i\omega\rho} \\ \dfrac{\partial_2}{i\omega\rho} \\ \dfrac{\partial_3}{i\omega\rho} \end{pmatrix} G_0(\mathbf{x}, \mathbf{x}'), \tag{C3}$$

with $\rho = \rho(\mathbf{x}, \omega)$ defined in Eq. (A9). Similar expressions hold for the other columns of $\mathbf{G}(\mathbf{x}, \mathbf{x}')$. Based on symmetry relation (C2), the first row of $\mathbf{G}(\mathbf{x}, \mathbf{x}')$ can be expressed as

$$\begin{pmatrix} 1 & -\frac{\partial_1'}{i\omega\rho'} & -\frac{\partial_2'}{i\omega\rho'} & -\frac{\partial_3'}{i\omega\rho'} \end{pmatrix} G_0(\mathbf{x}, \mathbf{x}'), \tag{C4}$$

where ∂_j' denotes differentiation with respect to x_j' and $\rho' = \rho(\mathbf{x}', \omega)$. Combining these two relations gives

$$\mathbf{G}(\mathbf{x},\mathbf{x}') = \begin{pmatrix} 1 \\ \frac{\partial_1}{i\omega\rho} \\ \frac{\partial_2}{i\omega\rho} \\ \frac{\partial_3}{i\omega\rho} \end{pmatrix} \begin{pmatrix} 1 & -\frac{\partial_1'}{i\omega\rho'} & -\frac{\partial_2'}{i\omega\rho'} & -\frac{\partial_3'}{i\omega\rho'} \end{pmatrix} G_0(\mathbf{x},\mathbf{x}'). \tag{C5}$$

From here onward, this Green's matrix is analyzed for a homogeneous lossless background medium. Replace $G_0(\mathbf{x}, \mathbf{x}')$ by the background Green's function

$$\bar{G}_0(\mathbf{y}) = \frac{1}{i\zeta} \frac{\exp(ik|\mathbf{y}|)}{|\mathbf{y}|}, \tag{C6}$$

where

$$\zeta = 4\pi/\omega\rho, \tag{C7}$$

with $\mathbf{y} = \mathbf{x} - \mathbf{x}'$ and $k = \omega/c$, with propagation velocity $c = (\kappa\rho)^{-1/2}$. Here κ and ρ are the compressibility and mass density of the background medium (for notational convenience, bars are omitted on the background medium parameters). In the far field approximation, Eq. (C5) gives

$$\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}') = \boldsymbol{\theta}(\hat{\mathbf{y}})\bar{G}_0(\mathbf{y})\boldsymbol{\theta}^T(\hat{\mathbf{y}}), \tag{C8}$$

where

$$\boldsymbol{\theta}(\hat{\mathbf{y}}) = \begin{pmatrix} 1 \\ \hat{y}_1/\rho c \\ \hat{y}_2/\rho c \\ \hat{y}_2/\rho c \end{pmatrix}, \tag{C9}$$

with $\hat{y}_i = y_i/|\mathbf{y}| = (x_i - x_i')/|\mathbf{x} - \mathbf{x}'|$.

Equations (C6) and (C8) are used to evaluate the term $\bar{\mathbf{G}}^\dagger(\mathbf{x}, \mathbf{0})\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x}, \mathbf{0})$ in Eq. (36). Substitution of Eq. (C8) with $\mathbf{x}' = \mathbf{0}$ gives

$$\bar{\mathbf{G}}^\dagger(\mathbf{x},\mathbf{0})\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x},\mathbf{0}) = \frac{\omega^2\rho^2}{16\pi^2|\mathbf{x}|^2}\boldsymbol{\theta}(\hat{\mathbf{x}})\boldsymbol{\theta}^T(\hat{\mathbf{x}})\mathbf{M}(\hat{\mathbf{x}})\boldsymbol{\theta}(\hat{\mathbf{x}})\boldsymbol{\theta}^T(\hat{\mathbf{x}}). \tag{C10}$$

Using $\mathbf{M}(\hat{\mathbf{x}})$ as defined in Eq. (A6), yields $\boldsymbol{\theta}^T(\hat{\mathbf{x}})\mathbf{M}(\hat{\mathbf{x}})\boldsymbol{\theta}(\hat{\mathbf{x}}) = 2/\rho c$. Hence

$$\bar{\mathbf{G}}^\dagger(\mathbf{x}, \mathbf{0})\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x}, \mathbf{0}) = \frac{2}{\zeta}\frac{\boldsymbol{\Theta}(\hat{\mathbf{x}})}{|\mathbf{x}|^2}, \tag{C11}$$

with

$$\boldsymbol{\Theta}(\hat{\mathbf{x}}) = \frac{k}{4\pi}\boldsymbol{\theta}(\hat{\mathbf{x}})\boldsymbol{\theta}^T(\hat{\mathbf{x}}). \tag{C12}$$

Next, Eqs. (C6) and (C8) are used to establish symmetry relation (46). Consider the Green's function $\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}')$ as defined in Eq. (C8), with $\mathbf{x}$ in the scattering domain $\mathbb{D}_s$ and $\mathbf{x}'$ far from this scattering domain, hence, $|\mathbf{x}| \ll |\mathbf{x}'|$, see Fig. 3. Express $\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}')$ as

$$\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}') = \bar{\mathbf{P}}(\mathbf{x}, \hat{\mathbf{x}}')\bar{\mathbf{G}}(\mathbf{0}, \mathbf{x}'), \tag{C13}$$

where, according to Eq. (C8),

$$\bar{\mathbf{G}}(\mathbf{0}, \mathbf{x}') = \boldsymbol{\theta}(-\hat{\mathbf{x}}')\bar{G}_0(-\mathbf{x}')\boldsymbol{\theta}^T(-\hat{\mathbf{x}}'). \tag{C14}$$

An expression for $\bar{\mathbf{P}}(\mathbf{x}, \hat{\mathbf{x}}')$ is derived by constructing $\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}')$, as defined in Eq. (C8), from $\bar{\mathbf{G}}(\mathbf{0}, \mathbf{x}')$, as defined in Eq. (C14), in three steps.

Step 1: using $(1 \;\; 0 \;\; 0 \;\; 0)\boldsymbol{\theta}(-\hat{\mathbf{x}}') = 1$, eliminate $\boldsymbol{\theta}(-\hat{\mathbf{x}}')$ from Eq. (C14) as follows

$$(1 \;\; 0 \;\; 0 \;\; 0)\bar{\mathbf{G}}(\mathbf{0}, \mathbf{x}') = \bar{G}_0(-\mathbf{x}')\boldsymbol{\theta}^T(-\hat{\mathbf{x}}'). \tag{C15}$$

Step 2: using $\bar{G}_0(\mathbf{y}) \approx \exp(-ik\mathbf{x}\cdot\hat{\mathbf{x}}')\bar{G}_0(-\mathbf{x}')$, applying $\exp(-ik\mathbf{x}\cdot\hat{\mathbf{x}}')$ to the right-hand side of Eq. (C15) gives

$$\exp(-ik\mathbf{x}\cdot\hat{\mathbf{x}}')\bar{G}_0(-\mathbf{x}')\boldsymbol{\theta}^T(-\hat{\mathbf{x}}') \approx \bar{G}_0(\mathbf{y})\boldsymbol{\theta}^T(-\hat{\mathbf{x}}'). \tag{C16}$$

Step 3: $\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}')$ is obtained by applying $\boldsymbol{\theta}(-\hat{\mathbf{x}}')$ to the right-hand side of Eq. (C16) and using $-\hat{\mathbf{x}}' \approx \hat{\mathbf{y}}$ and Eq. (C8). Hence

$$\boldsymbol{\theta}(-\hat{\mathbf{x}}')\bar{G}_0(\mathbf{y})\boldsymbol{\theta}^T(-\hat{\mathbf{x}}') \approx \bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}'). \tag{C17}$$

Combining these three steps, yields

$$\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}') \approx \boldsymbol{\theta}(-\hat{\mathbf{x}}')\exp(-ik\mathbf{x}\cdot\hat{\mathbf{x}}')(1 \;\; 0 \;\; 0 \;\; 0)\bar{\mathbf{G}}(\mathbf{0}, \mathbf{x}'). \tag{C18}$$

Hence

$$\bar{\mathbf{P}}(\mathbf{x}, \hat{\mathbf{x}}') \approx \boldsymbol{\theta}(-\hat{\mathbf{x}}')\exp(-ik\mathbf{x}\cdot\hat{\mathbf{x}}')(1 \;\; 0 \;\; 0 \;\; 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x_1'/\rho c & 0 & 0 & 0 \\ -x_2'/\rho c & 0 & 0 & 0 \\ -x_3'/\rho c & 0 & 0 & 0 \end{pmatrix}\exp(-ik\mathbf{x}\cdot\hat{\mathbf{x}}'). \tag{C19}$$

Note that

$$\bar{\mathbf{P}}(\mathbf{x}, \hat{\mathbf{x}}') = \mathbf{K}\bar{\mathbf{P}}^*(\mathbf{x}, -\hat{\mathbf{x}}')\mathbf{K}, \tag{C20}$$

with $\mathbf{K}$ defined in Eq. (A5), which confirms Eq. (46).

2. Quantum-mechanical Green's matrix

The quantum-mechanical Green's matrix is similar to the acoustic Green's matrix. $\bar{G}_0(\mathbf{y})$ is again given by Eq. (C6), with

$$\zeta = 4\pi\hbar \tag{C21}$$

and $k = \sqrt{2\omega m/\hbar}$. The far-field approximation of the Green's matrix $\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}')$ is again given by Eq. (C8), with

$$\boldsymbol{\theta}(\hat{\mathbf{y}}) = \begin{pmatrix} 1 \\ \hbar k\hat{y}_1 \\ \hbar k\hat{y}_2 \\ \hbar k\hat{y}_3 \end{pmatrix}. \tag{C22}$$

Despite these different definitions, we find that the term $\bar{\mathbf{G}}^\dagger(\mathbf{x}, \mathbf{0})\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x}, \mathbf{0})$ in Eq. (36) can be expressed again by Eqs. (C11) and (C12). Moreover, matrix $\bar{\mathbf{P}}(\mathbf{x}, \hat{\mathbf{x}}')$ obeys again symmetry relation (C20).

3. Electromagnetic Green's matrix

The basic 3×3 far-field electromagnetic Green's matrix in a homogeneous, isotropic, reciprocal, lossless background is given by[38]

$$\bar{\mathbf{G}}_0(\mathbf{y}) = \frac{\mu}{i\zeta}\frac{\exp(ik|\mathbf{y}|)}{|\mathbf{y}|}\{\boldsymbol{\Gamma}(\hat{\mathbf{y}}) - \mathbf{I}\}, \tag{C23}$$

where

$$\zeta = 4\pi/\omega, \tag{C24}$$

$$\boldsymbol{\Gamma}(\hat{\mathbf{y}}) = \begin{pmatrix} \hat{y}_1^2 & \hat{y}_1\hat{y}_2 & \hat{y}_1\hat{y}_3 \\ \hat{y}_2\hat{y}_1 & \hat{y}_2^2 & \hat{y}_2\hat{y}_3 \\ \hat{y}_3\hat{y}_1 & \hat{y}_3\hat{y}_2 & \hat{y}_3^2 \end{pmatrix}, \tag{C25}$$

and $k = \omega/c$, with propagation velocity $c = (\varepsilon\mu)^{-1/2}$. Here ε and μ are the permittivity and permeability of the background. Analogous to the derivation in Appendix C 1 it can be shown that the 6×6 Green's matrix $\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}')$ is, in the far field, related to the basic 3×3 matrix $\bar{\mathbf{G}}_0(\mathbf{y})$, via

$$\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}') = \boldsymbol{\theta}(\hat{\mathbf{y}})\bar{\mathbf{G}}_0(\mathbf{y})\boldsymbol{\theta}^T(\hat{\mathbf{y}}), \tag{C26}$$

with

$$\boldsymbol{\theta}(\hat{\mathbf{y}}) = \begin{pmatrix} \mathbf{I} \\ \frac{1}{\mu c}\mathbf{M}_0(\hat{\mathbf{y}}) \end{pmatrix}, \quad \mathbf{M}_0(\hat{\mathbf{y}}) = \begin{pmatrix} 0 & -\hat{y}_3 & \hat{y}_2 \\ \hat{y}_3 & 0 & -\hat{y}_1 \\ -\hat{y}_2 & \hat{y}_1 & 0 \end{pmatrix}. \tag{C27}$$

Equations (C23) and (C26) are used to evaluate the term $\bar{\mathbf{G}}^\dagger(\mathbf{x}, \mathbf{0})\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x}, \mathbf{0})$ in Eq. (36). Substitution of Eq. (C26) with $\mathbf{x}' = \mathbf{0}$ gives

$$\bar{\mathbf{G}}^\dagger(\mathbf{x}, \mathbf{0})\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x}, 0) = \boldsymbol{\theta}(\hat{\mathbf{x}})\{\bar{\mathbf{G}}_0(\mathbf{x})\}^\dagger\boldsymbol{\theta}^T(\hat{\mathbf{x}})\mathbf{M}(\hat{\mathbf{x}})\boldsymbol{\theta}(\hat{\mathbf{x}}) \times \bar{\mathbf{G}}_0(\mathbf{x})\boldsymbol{\theta}^T(\hat{\mathbf{x}}). \tag{C28}$$

Analogous to the definition of $\mathbf{D}_\mathbf{x}$ in Eq. (A18), it holds that

$$\mathbf{M}(\hat{\mathbf{x}}) = \begin{pmatrix} \mathbf{O} & \mathbf{M}_0^T(\hat{\mathbf{x}}) \\ \mathbf{M}_0(\hat{\mathbf{x}}) & \mathbf{O} \end{pmatrix}, \tag{C29}$$

hence

$$\begin{aligned} \boldsymbol{\theta}^T(\hat{\mathbf{x}})\mathbf{M}(\hat{\mathbf{x}})\boldsymbol{\theta}(\hat{\mathbf{x}}) &= \frac{2}{\mu c}\mathbf{M}_0^T(\hat{\mathbf{x}})\mathbf{M}_0(\hat{\mathbf{x}}) \\ &= \frac{2}{\mu c}\{\mathbf{I} - \boldsymbol{\Gamma}(\hat{\mathbf{x}})\}. \end{aligned} \tag{C30}$$

Substituting this into Eq. (C28), using

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T = \boldsymbol{\Gamma}^2 = \boldsymbol{\Gamma}^3 = \cdots, \tag{C31}$$

gives

$$\bar{\mathbf{G}}^\dagger(\mathbf{x}, \mathbf{0})\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x}, \mathbf{0}) = \frac{2}{\zeta}\frac{\boldsymbol{\Theta}(\hat{\mathbf{x}})}{|\mathbf{x}|^2}, \tag{C32}$$

with

$$\boldsymbol{\Theta}(\hat{\mathbf{x}}) = \frac{\mu k}{4\pi}\boldsymbol{\theta}(\hat{\mathbf{x}})\{\mathbf{I} - \boldsymbol{\Gamma}(\hat{\mathbf{x}})\}\boldsymbol{\theta}^T(\hat{\mathbf{x}}). \tag{C33}$$

Next, the same three steps as in Appendix C 1 are applied to establish symmetry relation (46). Assuming $|\mathbf{x}| \ll |\mathbf{x}'|$ (Fig. 3), express $\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}')$ as

$$\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}') = \bar{\mathbf{P}}(\mathbf{x}, \hat{\mathbf{x}}')\bar{\mathbf{G}}(\mathbf{0}, \mathbf{x}'), \tag{C34}$$

where, analogous to Eq. (C19),

$$\begin{aligned} \bar{\mathbf{P}}(\mathbf{x}, \hat{\mathbf{x}}') &= \boldsymbol{\theta}(-\hat{\mathbf{x}}')\exp(-ik\mathbf{x}\cdot\hat{\mathbf{x}}')(\mathbf{I} \quad \mathbf{O}) \\ &= \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \frac{1}{\mu c}\mathbf{M}_0(-\hat{\mathbf{x}}') & \mathbf{O} \end{pmatrix}\exp(-ik\mathbf{x}\cdot\hat{\mathbf{x}}'). \end{aligned} \tag{C35}$$

Note that

$$\bar{\mathbf{P}}(\mathbf{x}, \hat{\mathbf{x}}') = \mathbf{K}\bar{\mathbf{P}}^*(\mathbf{x}, -\hat{\mathbf{x}}')\mathbf{K}, \tag{C36}$$

with $\mathbf{K}$ defined in Eq. (A19), which confirms Eq. (46).

4. Elastodynamic Green's matrix

The basic 3×3 far-field elastodynamic Green's matrices for P-and S-waves in a homogeneous, isotropic, lossless background medium are given by[49]

$$\bar{\mathbf{G}}_P(\mathbf{y}) = \frac{1}{i\zeta\rho c_P^2}\frac{\exp(ik_P|\mathbf{y}|)}{|\mathbf{y}|}\boldsymbol{\Gamma}(\hat{\mathbf{y}}) \tag{C37}$$

and

$$\bar{\mathbf{G}}_S(\mathbf{y}) = \frac{1}{i\zeta\rho c_S^2}\frac{\exp(ik_S|\mathbf{y}|)}{|\mathbf{y}|}\{\mathbf{I} - \boldsymbol{\Gamma}(\hat{\mathbf{y}})\}, \tag{C38}$$

respectively, where

$$\zeta = 4\pi/\omega, \tag{C39}$$

$\boldsymbol{\Gamma}(\hat{\mathbf{y}})$ defined by Eq. (C25), and $k_{\{P,S\}} = \omega/c_{\{P,S\}}$, with propagation velocities $c_P = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_S = \sqrt{\mu/\rho}$. Here λ, μ, and ρ are the Lamé parameters and mass density of the background medium. Analogous to the derivation in Appendix C 1 it can be shown that the 9×9 Green's matrix $\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}')$ is, in the far field, related to the basic 3×3 matrices $\bar{\mathbf{G}}_P(\mathbf{y})$ and $\bar{\mathbf{G}}_S(\mathbf{y})$, via

$$\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}') = \boldsymbol{\theta}_P(\hat{\mathbf{y}})\bar{\mathbf{G}}_P(\mathbf{y})\boldsymbol{\theta}_P^T(\hat{\mathbf{y}}) + \boldsymbol{\theta}_S(\hat{\mathbf{y}})\bar{\mathbf{G}}_S(\mathbf{y})\boldsymbol{\theta}_S^T(\hat{\mathbf{y}}), \quad \text{(C40)}$$

where

$$\boldsymbol{\theta}_{\{P,S\}}(\hat{\mathbf{y}}) = \begin{pmatrix} \mathbf{I} \\ \frac{1}{c_{\{P,S\}}}\mathbf{c}_{11}\mathbf{M}_1(\hat{\mathbf{y}}) \\ \frac{1}{c_{\{P,S\}}}\mathbf{c}_{22}\mathbf{M}_2(\hat{\mathbf{y}}) \end{pmatrix}, \quad \text{(C41)}$$

with

$$\mathbf{c}_{11} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda \\ \lambda & \lambda+2\mu & \lambda \\ \lambda & \lambda & \lambda+2\mu \end{pmatrix},$$
$$\mathbf{c}_{22} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad \text{(C42)}$$

and

$$\mathbf{M}_1(\hat{\mathbf{y}}) = \begin{pmatrix} \hat{y}_1 & 0 & 0 \\ 0 & \hat{y}_2 & 0 \\ 0 & 0 & \hat{y}_3 \end{pmatrix}, \quad \mathbf{M}_2(\hat{\mathbf{y}}) = \begin{pmatrix} 0 & \hat{y}_3 & \hat{y}_2 \\ \hat{y}_3 & 0 & \hat{y}_1 \\ \hat{y}_2 & \hat{y}_1 & 0 \end{pmatrix}. \quad \text{(C43)}$$

We use Eqs. (C37), (C38), and (C40) to evaluate the term $\bar{\mathbf{G}}^\dagger(\mathbf{x},\mathbf{0})\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x},\mathbf{0})$ in Eq. (36). Substitution of Eq. (C40) with $\mathbf{x}'=\mathbf{0}$ gives

$$\begin{aligned}&\bar{\mathbf{G}}^\dagger(\mathbf{x},\mathbf{0})\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x},\mathbf{0}) \\ &\quad = \{\boldsymbol{\theta}_P(\hat{\mathbf{x}})\bar{\mathbf{G}}_P^\dagger(\mathbf{x})\boldsymbol{\theta}_P^T(\hat{\mathbf{x}}) + \boldsymbol{\theta}_S(\hat{\mathbf{x}})\bar{\mathbf{G}}_S^\dagger(\mathbf{x})\boldsymbol{\theta}_S^T(\hat{\mathbf{x}})\}\mathbf{M}(\hat{\mathbf{x}}) \\ &\quad \times \{\boldsymbol{\theta}_P(\hat{\mathbf{x}})\bar{\mathbf{G}}_P(\mathbf{x})\boldsymbol{\theta}_P^T(\hat{\mathbf{x}}) + \boldsymbol{\theta}_S(\hat{\mathbf{x}})\bar{\mathbf{G}}_S(\mathbf{x})\boldsymbol{\theta}_S^T(\hat{\mathbf{x}})\}.\end{aligned} \quad \text{(C44)}$$

Analogous to the definition of $\mathbf{D}_x$ in Eq. (A35) it holds that

$$\mathbf{M}(\hat{\mathbf{x}}) = \begin{pmatrix} \mathbf{O} & \mathbf{M}_1(\hat{\mathbf{x}}) & \mathbf{M}_2(\hat{\mathbf{x}}) \\ \mathbf{M}_1(\hat{\mathbf{x}}) & \mathbf{O} & \mathbf{O} \\ \mathbf{M}_2(\hat{\mathbf{x}}) & \mathbf{O} & \mathbf{O} \end{pmatrix}. \quad \text{(C45)}$$

Consider the terms $\boldsymbol{\theta}_Q^T(\hat{\mathbf{x}})\mathbf{M}(\hat{\mathbf{x}})\boldsymbol{\theta}_R(\hat{\mathbf{x}})$, where each of the subscripts Q and R can stand for either P or S. Using Eqs. (C41)–(C43) gives

$$\begin{aligned}\boldsymbol{\theta}_Q^T(\hat{\mathbf{x}})\mathbf{M}(\hat{\mathbf{x}})\boldsymbol{\theta}_R(\hat{\mathbf{x}}) &= \left(\frac{1}{c_Q}+\frac{1}{c_R}\right)(\mathbf{M}_1(\hat{\mathbf{x}})\mathbf{c}_{11}\mathbf{M}_1(\hat{\mathbf{x}}) \\ &\quad + \mathbf{M}_2(\hat{\mathbf{x}})\mathbf{c}_{22}\mathbf{M}_2(\hat{\mathbf{x}})) \\ &= \left(\frac{1}{c_Q}+\frac{1}{c_R}\right)((\lambda+\mu)\boldsymbol{\Gamma}(\hat{\mathbf{x}})+\mu\mathbf{I}),\end{aligned} \quad \text{(C46)}$$

with $\boldsymbol{\Gamma}(\hat{\mathbf{x}})$ defined by Eq. (C25). Hence, using Eq. (C31), it is found for the different terms in Eq. (C44) that

$$\bar{\mathbf{G}}_P^\dagger(\mathbf{x})\boldsymbol{\theta}_P^T(\hat{\mathbf{x}})\mathbf{M}(\hat{\mathbf{x}})\boldsymbol{\theta}_P(\hat{\mathbf{x}})\bar{\mathbf{G}}_P(\mathbf{x}) = \frac{\omega^2\boldsymbol{\Gamma}(\hat{\mathbf{x}})}{8\rho c_P^3\pi^2|\mathbf{x}|^2}, \quad \text{(C47)}$$

$$\bar{\mathbf{G}}_P^\dagger(\mathbf{x})\boldsymbol{\theta}_P^T(\hat{\mathbf{x}})\mathbf{M}(\hat{\mathbf{x}})\boldsymbol{\theta}_S(\hat{\mathbf{x}})\bar{\mathbf{G}}_S(\mathbf{x}) = \mathbf{O}, \quad \text{(C48)}$$

$$\bar{\mathbf{G}}_S^\dagger(\mathbf{x})\boldsymbol{\theta}_S^T(\hat{\mathbf{x}})\mathbf{M}(\hat{\mathbf{x}})\boldsymbol{\theta}_P(\hat{\mathbf{x}})\bar{\mathbf{G}}_P(\mathbf{x}) = \mathbf{O}, \quad \text{(C49)}$$

$$\bar{\mathbf{G}}_S^\dagger(\mathbf{x})\boldsymbol{\theta}_S^T(\hat{\mathbf{x}})\mathbf{M}(\hat{\mathbf{x}})\boldsymbol{\theta}_S(\hat{\mathbf{x}})\bar{\mathbf{G}}_S(\mathbf{x}) = \frac{\omega^2\{\mathbf{I}-\boldsymbol{\Gamma}(\hat{\mathbf{x}})\}}{8\rho c_S^3\pi^2|\mathbf{x}|^2}. \quad \text{(C50)}$$

Taking all terms together yields

$$\bar{\mathbf{G}}^\dagger(\mathbf{x},\mathbf{0})\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x},\mathbf{0}) = \frac{2}{\zeta}\frac{\boldsymbol{\Theta}(\hat{\mathbf{x}})}{|\mathbf{x}|^2}, \quad \text{(C51)}$$

with

$$\begin{aligned}\boldsymbol{\Theta}(\hat{\mathbf{x}}) &= \frac{\omega}{4\pi\rho}\left(\frac{1}{c_P^3}\boldsymbol{\theta}_P(\hat{\mathbf{x}})\boldsymbol{\Gamma}(\hat{\mathbf{x}})\boldsymbol{\theta}_P^T(\hat{\mathbf{x}})\right. \\ &\quad \left. + \frac{1}{c_S^3}\boldsymbol{\theta}_S(\hat{\mathbf{x}})\{\mathbf{I}-\boldsymbol{\Gamma}(\hat{\mathbf{x}})\}\boldsymbol{\theta}_S^T(\hat{\mathbf{x}})\right).\end{aligned} \quad \text{(C52)}$$

Next, Eqs. (C37), (C38), and (C40) are used to establish symmetry relation (46). Assuming $|\mathbf{x}| \ll |\mathbf{x}'|$ (Fig. 3), express $\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}')$ as

$$\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}') = \bar{\mathbf{P}}(\mathbf{x},\hat{\mathbf{x}}')\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}'), \quad \text{(C53)}$$

where, according to Eq. (C40),

$$\begin{aligned}\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}') &= \boldsymbol{\theta}_P(-\hat{\mathbf{x}}')\bar{\mathbf{G}}_P(-\mathbf{x}')\boldsymbol{\theta}_P^T(-\hat{\mathbf{x}}') \\ &\quad + \boldsymbol{\theta}_S(-\hat{\mathbf{x}}')\bar{\mathbf{G}}_s(-\mathbf{x}')\boldsymbol{\theta}_S^T(-\hat{\mathbf{x}}').\end{aligned} \quad \text{(C54)}$$

An expression for $\bar{\mathbf{P}}(\mathbf{x},\hat{\mathbf{x}}')$ is derived by constructing $\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}')$, as defined in Eq. (C40), from $\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}')$, as defined in Eq. (C54), in three steps.

Step 1: using $\boldsymbol{\Gamma}(\hat{\mathbf{x}}') = \boldsymbol{\Gamma}(-\hat{\mathbf{x}}')$ as well as Eq. (C31), decompose $\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}')$ into its P- and S-wave constituents, as follows:

$$\begin{pmatrix} \boldsymbol{\Gamma}(\hat{\mathbf{x}}') & \mathbf{O} & \mathbf{O} \\ \mathbf{I}-\boldsymbol{\Gamma}(\hat{\mathbf{x}}') & \mathbf{O} & \mathbf{O} \end{pmatrix}\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}') = \begin{pmatrix} \bar{\mathbf{G}}_P(-\mathbf{x}')\boldsymbol{\theta}_P^T(-\hat{\mathbf{x}}') \\ \bar{\mathbf{G}}_S(-\mathbf{x}')\boldsymbol{\theta}_S^T(-\hat{\mathbf{x}}') \end{pmatrix}. \quad \text{(C55)}$$

Step 2: using $\bar{\mathbf{G}}_{\{P,S\}}(\mathbf{y}) \approx \exp(-ik_{\{P,S\}}\mathbf{x}\cdot\hat{\mathbf{x}}')\bar{\mathbf{G}}_{\{P,S\}}(-\mathbf{x}')$, applying $\exp(-ik_{\{P,S\}}\mathbf{x}\cdot\hat{\mathbf{x}}')$ to the right-hand side of Eq. (C55) gives

$$\begin{aligned}&\begin{pmatrix} \mathbf{I}\exp(-ik_P\mathbf{x}\cdot\hat{\mathbf{x}}') & \mathbf{O} \\ \mathbf{O} & \mathbf{I}\exp(-ik_S\mathbf{x}\cdot\hat{\mathbf{x}}') \end{pmatrix}\begin{pmatrix} \bar{\mathbf{G}}_P(-\mathbf{x}')\boldsymbol{\theta}_P^T(-\hat{\mathbf{x}}') \\ \bar{\mathbf{G}}_S(-\mathbf{x}')\boldsymbol{\theta}_S^T(-\hat{\mathbf{x}}') \end{pmatrix} \\ &\quad = \begin{pmatrix} \bar{\mathbf{G}}_P(\mathbf{y})\boldsymbol{\theta}_P^T(-\hat{\mathbf{x}}') \\ \bar{\mathbf{G}}_S(\mathbf{y})\boldsymbol{\theta}_S^T(-\hat{\mathbf{x}}') \end{pmatrix}.\end{aligned} \quad \text{(C56)}$$

Step 3: compose $\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}')$ from its P- and S-wave constituents by applying $(\boldsymbol{\theta}_P(-\hat{\mathbf{x}}')\;\boldsymbol{\theta}_S(-\hat{\mathbf{x}}'))$ to the right-hand side of Eq. (C56) and using $-\hat{\mathbf{x}}' \approx \hat{\mathbf{y}}$ and Eq. (C40). Hence

$$(\boldsymbol{\theta}_P(-\hat{\mathbf{x}}')\;\boldsymbol{\theta}_S(-\hat{\mathbf{x}}'))\begin{pmatrix} \bar{\mathbf{G}}_P(\mathbf{y})\boldsymbol{\theta}_P^T(-\hat{\mathbf{x}}') \\ \bar{\mathbf{G}}_S(\mathbf{y})\boldsymbol{\theta}_S^T(-\hat{\mathbf{x}}') \end{pmatrix} = \bar{\mathbf{G}}(\mathbf{x},\mathbf{x}'). \quad \text{(C57)}$$

Combining these three steps gives Eq. (C53), with

$$\bar{\mathbf{P}}(\mathbf{x},\hat{\mathbf{x}}') = \begin{pmatrix} \bar{\mathbf{P}}_1(\mathbf{x},\hat{\mathbf{x}}') & \mathbf{O} & \mathbf{O} \end{pmatrix}, \tag{C58}$$

where

$$\bar{\mathbf{P}}_1(\mathbf{x},\hat{\mathbf{x}}') = \boldsymbol{\theta}_P(-\hat{\mathbf{x}}')\boldsymbol{\Gamma}(\hat{\mathbf{x}}')\exp(-ik_P\mathbf{x}\cdot\hat{\mathbf{x}}') + \boldsymbol{\theta}_S(-\hat{\mathbf{x}}')\{\mathbf{I}-\boldsymbol{\Gamma}(\hat{\mathbf{x}}')\}\exp(-ik_S\mathbf{x}\cdot\hat{\mathbf{x}}') \tag{C59}$$

or

$$\bar{\mathbf{P}}(\mathbf{x},\hat{\mathbf{x}}') = \begin{pmatrix} \bar{\mathbf{P}}_{11}(\mathbf{x},\hat{\mathbf{x}}') & \mathbf{O} & \mathbf{O} \\ \bar{\mathbf{P}}_{21}(\mathbf{x},\hat{\mathbf{x}}') & \mathbf{O} & \mathbf{O} \\ \bar{\mathbf{P}}_{31}(\mathbf{x},\hat{\mathbf{x}}') & \mathbf{O} & \mathbf{O} \end{pmatrix}, \tag{C60}$$

where

$$\bar{\mathbf{P}}_{11}(\mathbf{x},\hat{\mathbf{x}}') = \boldsymbol{\Gamma}(\hat{\mathbf{x}}')\exp(-ik_P\mathbf{x}\cdot\hat{\mathbf{x}}') + \{\mathbf{I}-\boldsymbol{\Gamma}(\hat{\mathbf{x}}')\}\exp(-ik_S\mathbf{x}\cdot\hat{\mathbf{x}}'), \tag{C61}$$

$$\bar{\mathbf{P}}_{21}(\mathbf{x},\hat{\mathbf{x}}') = \mathbf{c}_{11}\mathbf{M}_1(-\hat{\mathbf{x}}')(c_P^{-1}\boldsymbol{\Gamma}(\hat{\mathbf{x}}')\exp(-ik_P\mathbf{x}\cdot\hat{\mathbf{x}}') + c_S^{-1}\{\mathbf{I}-\boldsymbol{\Gamma}(\hat{\mathbf{x}}')\}\exp(-ik_S\mathbf{x}\cdot\hat{\mathbf{x}}')), \tag{C62}$$

$$\bar{\mathbf{P}}_{31}(\mathbf{x},\hat{\mathbf{x}}') = \mathbf{c}_{22}\mathbf{M}_2(-\hat{\mathbf{x}}')(c_P^{-1}\boldsymbol{\Gamma}(\hat{\mathbf{x}}')\exp(-ik_P\mathbf{x}\cdot\hat{\mathbf{x}}') + c_S^{-1}\{\mathbf{I}-\boldsymbol{\Gamma}(\hat{\mathbf{x}}')\}\exp(-ik_S\mathbf{x}\cdot\hat{\mathbf{x}}')). \tag{C63}$$

Note that

$$\bar{\mathbf{P}}(\mathbf{x},\hat{\mathbf{x}}') = \mathbf{K}\bar{\mathbf{P}}^*(\mathbf{x},-\hat{\mathbf{x}}')\mathbf{K}, \tag{C64}$$

with $\mathbf{K}$ defined in Eq. (A36), which confirms Eq. (46).

5. Combined electromagnetic and elastodynamic Green's matrix

For a homogeneous, isotropic, lossless background medium, in which electromagnetic and elastodynamic waves propagate independently, the Green's matrices can be combined as follows:

$$\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}') = \begin{pmatrix} \bar{\mathbf{G}}^{\mathrm{EM}}(\mathbf{x},\mathbf{x}') & \mathbf{O} \\ \mathbf{O} & \bar{\mathbf{G}}^{\mathrm{ED}}(\mathbf{x},\mathbf{x}') \end{pmatrix}, \tag{C65}$$

where superscripts EM and ED stand for electromagnetic and elastodynamic, respectively. The expressions for matrices with superscripts EM and ED are given in Appendixes C 3 and C 4, respectively. For the term $\bar{\mathbf{G}}^\dagger(\mathbf{x},\mathbf{0})\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x},\mathbf{0})$ appearing in Eq. (36), with

$$\mathbf{M}(\hat{\mathbf{x}}) = \begin{pmatrix} \mathbf{M}^{\mathrm{EM}}(\hat{\mathbf{x}}) & \mathbf{O} \\ \mathbf{O} & \mathbf{M}^{\mathrm{ED}}(\hat{\mathbf{x}}) \end{pmatrix}, \tag{C66}$$

it is found that

$$\bar{\mathbf{G}}^\dagger(\mathbf{x},\mathbf{0})\mathbf{M}(\hat{\mathbf{x}})\bar{\mathbf{G}}(\mathbf{x},\mathbf{0}) = \frac{2}{\zeta}\frac{\Theta(\hat{\mathbf{x}})}{|\mathbf{x}|^2}, \tag{C67}$$

where $\zeta = 4\pi/\omega$, and

$$\Theta(\hat{\mathbf{x}}) = \begin{pmatrix} \Theta^{\mathrm{EM}}(\hat{\mathbf{x}}) & \mathbf{O} \\ \mathbf{O} & \Theta^{\mathrm{ED}}(\hat{\mathbf{x}}) \end{pmatrix}. \tag{C68}$$

Next, express $\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}')$ as

$$\bar{\mathbf{G}}(\mathbf{x},\mathbf{x}') = \bar{\mathbf{P}}(\mathbf{x},\hat{\mathbf{x}}')\bar{\mathbf{G}}(\mathbf{0},\mathbf{x}'), \tag{C69}$$

where

$$\bar{\mathbf{P}}(\mathbf{x},\hat{\mathbf{x}}') = \begin{pmatrix} \bar{\mathbf{P}}^{\mathrm{EM}}(\mathbf{x},\hat{\mathbf{x}}') & \mathbf{O} \\ \mathbf{O} & \bar{\mathbf{P}}^{\mathrm{ED}}(\mathbf{x},\hat{\mathbf{x}}') \end{pmatrix}. \tag{C70}$$

Note that

$$\bar{\mathbf{P}}(\mathbf{x},\hat{\mathbf{x}}') = \mathbf{K}\bar{\mathbf{P}}^*(\mathbf{x},-\hat{\mathbf{x}}')\mathbf{K}, \tag{C71}$$

with $\mathbf{K}$ defined in Eq. (A45), which confirms Eq. (46).

[1]J. W. Strutt (Lord Rayleigh), "On the transmission of light through an atmosphere containing small particles in suspension, and on the origin of the blue of the sky," Philos. Mag. **47**, 375–384 (1899).

[2]W. Heisenberg, "Die „beobachtbaren Grössen“ in der Theorie der Elementarteilchen (The observable quantities in the theory of elementary particles)," Z. Phys. **120**, 513–538 (1943).

[3]R. Glauber and V. Schomaker, "The theory of electron diffraction," Phys. Rev. **89**, 667–671 (1953).

[4]R. G. Newton, "Optical theorem and beyond," Am. J. Phys. **44**, 639–642 (1976).

[5]P. L. Marston, "Generalized optical theorem for scatterers having inversion symmetry: Applications to acoustic backscattering," J. Acoust. Soc. Am. **109**, 1291–1295 (2001).

[6]P. S. Carney, J. C. Schotland, and E. Wolf, "Generalized optical theorem for reflection, transmission, and extinction of power for scalar fields," Phys. Rev. E **70**, 036611 (2004).

[7]R. Snieder, "The optical theorem for surface waves and the relation with surface-wave attenuation," Geophys. J. **95**, 293–302 (1988).

[8]D. Halliday and A. Curtis, "Seismic interferometry of scattered surface waves in attenuative media," Geophys. J. Int. **178**, 419–446 (2009).

[9]D. Halliday and A. Curtis, "Generalized optical theorem for surface waves and layered media," Phys. Rev. E **79**, 056603 (2009).

[10]T. H. Tan, "Reciprocity relations for scattering of plane, elastic waves," J. Acoust. Soc. Am. **61**, 928–931 (1977).

[11]A. T. de Hoop, "A time-domain energy theorem for the scattering of plane elastic waves," Wave Motion **7**, 569–577 (1985).

[12]L. Lu, Z. Ding, R. S. Zeng, and Z. He, "Retrieval of Green's function and generalized optical theorem for the scattering of complete dyadic fields," J. Acoust. Soc. Am. **129**, 1935–1944 (2011).

[13]D. Torrungrueng, B. Ungan, and J. T. Johnson, "Optical theorem for electromagnetic scattering by a three-dimensional scatterer in the presence of a lossless half space," IEEE Geosci. Remote Sens. Lett. **1**, 131–135 (2004).

[14]D. R. Lytle II, P. S. Carney, J. C. Schotland, and E. Wolf, "Generalized optical theorem for reflection, transmission, and extinction of power for electromagnetic fields," Phys. Rev. E **71**, 056610 (2005).

[15]K. Wapenaar, "Retrieving the elastodynamic Green's function of an arbitrary inhomogeneous medium by cross correlation," Phys. Rev. Lett. **93**, 254301 (2004).

[16]D.-J. van Manen, J. O. A. Robertsson, and A. Curtis, "Modeling of wave propagation in inhomogeneous media," Phys. Rev. Lett. **94**, 164301 (2005).

[17]I. Vasconcelos, R. Snieder, and H. Douma, "Representation theorems and Green's function retrieval for scattering in acoustic media," Phys. Rev. E **80**, 036605 (2009).

[18]M. Campillo and A. Paul, "Long-range correlations in the diffuse seismic coda," Science **299**, 547–549 (2003).

[19]A. Derode, E. Larose, M. Tanter, J. de Rosny, A. Tourin, M. Campillo, and M. Fink, "Recovering the Green's function from field-field correlations in an open scattering medium (L)," J. Acoust. Soc. Am. **113**, 2973–2976 (2003).

[20]N. M. Shapiro and M. Campillo, "Emergence of broadband Rayleigh waves from correlations of the ambient seismic noise," Geophys. Res. Lett. **31**, L07614, doi:10.1029/2004GL019491 (2004).

[21]E. Larose, L. Margerin, A. Derode, B. van Tiggelen, M. Campillo, N. Shapiro, A. Paul, L. Stehly, and M. Tanter, "Correlation of random wave fields: An interdisciplinary review," Geophysics **71**, SI11–SI21 (2006).
[22]D. Draganov, K. Wapenaar, W. Mulder, J. Singer, and A. Verdel, "Retrieval of reflections from seismic background-noise measurements," Geophys. Res. Lett. **34**, L04305, doi:10.1029/2006GL028735 (2007).
[23]R. Snieder, K. van Wijk, M. Haney, and R. Calvert, "Cancellation of spurious arrivals in Green's function extraction and the generalized optical theorem," Phys. Rev. E **78**, 036606 (2008).
[24]R. Snieder, F. J. Sánchez-Sesma, and K. Wapenaar, "Field fluctuations, imaging with backscattered waves, a generalized energy theorem, and the optical theorem," SIAM J. Imaging Sci. **2**, 763–776 (2009).
[25]K. Wapenaar, E. Slob, and R. Snieder, "On seismic interferometry, the generalized optical theorem, and the scattering matrix of a point scatterer," Geophysics **75**, SA27–SA35 (2010).
[26]H. Douma, I. Vasconcelos, and R. Snieder, "The reciprocity theorem for the scattered field is the progenitor of the generalized optical theorem," J. Acoust. Soc. Am. **129**, 2765–2771 (2011).
[27]R. Snieder and C. Fleury, "Cancellation of spurious arrivals in Green's function retrieval of multiple scattered waves," J. Acoust. Soc. Am. **128**, 1598–1605 (2010).
[28]L. Margerin and H. Sato, "Generalized optical theorems for the reconstruction of Green's function of an inhomogeneous elastic medium," J. Acoust. Soc. Am. **130**, 3674–3690 (2011).
[29]K. Wapenaar, E. Slob, and R. Snieder, "Unified Green's function retrieval by cross correlation," Phys. Rev. Lett. **97**, 234301 (2006).
[30]M. V. de Hoop and A. T. de Hoop, "Wave-field reciprocity and optimization in remote sensing," Proc. R. Soc. London, Ser. A **456**, 641–682 (2000).
[31]K. Wapenaar and J. Fokkema, "Reciprocity theorems for diffusion, flow and waves," J. Appl. Mech. **71**, 145–150 (2004).
[32]R. L. Weaver, "Ward identities and the retrieval of Green's functions in the correlations of a diffuse wave field," Wave Motion **45**, 596–604 (2008).
[33]L. M. Lyamshev, "On some integral relationships in acoustics of moving medium," Dokl. Akad. Nauk **138**, 575–578 (1961).
[34]L. M. Brekhovskikh and O. A. Godin, *Acoustics of Layered Media II. Point Sources and Bounded Beams* (Springer, Berlin, 1992), Chap. 4.
[35]E. Slob and K. Wapenaar, "Retrieving the Green's function from cross correlation in a bianisotropic medium," Prog. Electromagn. Res. **93**, 255–274 (2009).
[36]M. Born and E. Wolf, *Principles of Optics* (Pergamon, London, 1965), Chap. 8.
[37]Y. H. Pao and V. Varatharajulu, "Huygens' principle, radiation conditions, and integral formulations for the scattering of elastic waves," J. Acoust. Soc. Am. **59**, 1361–1371 (1976).
[38]A. T. de Hoop, *Handbook of Radiation and Scattering of Waves* (Academic, London, 1995), Chaps. 18, 19, and 26.
[39]H. C. van de Hulst, "On the attenuation of plane waves by obstacles of arbitrary size and form," Physica **15**, 740–746 (1949).
[40]P. S. Carney, E. Wolf, and G. S. Agarwal, "Diffraction tomography using power extinction measurements," J. Opt. Soc. Am. A **16**, 2643–2648 (1999).
[41]A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1961), Vol. I, pp. 59–71.
[42]E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1961), pp. 25–28.
[43]L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, New York, 1960), Chaps. 9 and 11.
[44]E. Post, *Formal Structure of Electromagnetics* (North-Holland, Amsterdam, 1962), pp 1–204.
[45]W. W. Chow, J. Gea-Banacloche, L. M. Pedrotti, V. E. Sanders, W. Schleich, and M. O. Scully, "The ring laser gyro," Rev. Mod. Phys. **57**, 61–104 (1985).
[46]X. M. Yang, J. Y. Chin, Q. Cheng, X. Q. Lin, and T. J. Cui, "Realization and experimental verification of chiral cascaded circuit," IEEE Microw. Wirel. Compon. Lett. **18**, 308–310 (2008).
[47]R. M. Kiehn, G. P. Kiehn, and J. B. Roberds, "Parity and time-reversal symmetry breaking, singular solutions, and Fresnel surfaces," Phys. Rev. A **43**, 5665–5671 (1991).
[48]J. D. Achenbach, *Wave Propagation in Elastic Solids* (North Holland, Amsterdam, 1973), Chap. 3.
[49]K. Aki and P. G. Richards, *Quantitative Seismology* (W. H. Freeman and Company, San Fransisco, 1980), Vol. I, Chaps. 2 and 4.
[50]B. A. Auld, *Acoustic Fields and Waves in Solids* (Wiley-Interscience, New York, 1973), Vol. I, Chap. 8.