

# Reflecting boundary conditions for interferometry by multidimensional deconvolution

# Cornelis Weemstra<sup>a)</sup> and Kees Wapenaar

Department of Geoscience and Engineering, Delft University of Technology, Stevinweg 1, 2628 CN, Delft, the Netherlands

### Karel N. van Dalen

Department of Structural Engineering, Delft University of Technology, Stevinweg 1, 2628 CN, Delft, the Netherlands

(Received 13 June 2017; revised 29 September 2017; accepted 1 October 2017; published online 20 October 2017)

In an acoustical context, interferometry takes advantage of existing (ambient) wavefield recordings by turning receivers into so-called "virtual sources." The medium's response to these virtual sources can be harnessed to image that medium. Most interferometric applications, however, suffer from the fact that the retrieved virtual-source responses deviate from the true medium responses. The accrued artefacts are often predominantly due to a non-isotropic illumination of the medium of interest, and prohibit accurate interferometric imaging. Recently, it has been shown that illumination-related artefacts can be removed by means of a so-called multidimensional deconvolution (MDD) process. However, the current MDD formulation, and hence method, relies on separation of waves traveling inward and outward through the boundary of the medium of interest. As a consequence, it is predominantly useful when receivers are illuminated from one side only. This puts constraints on the applicability of the current MDD formulation to omnidirectional wavefields. In this paper, a modified formulation of the theory underlying interferometry by MDD is presented. This modified formulation eliminates the requirement to separate inward and outward propagating wavefields and, consequently, holds promise for the application of MDD to non-isotropic, omnidirectional wavefields. © *2017 Acoustical Society of America*. https://doi.org/10.1121/1.5007833

[RKS]

Pages: 2242–2257

### I. INTRODUCTION

It has been shown that the Green's function of a medium can be retrieved by simple crosscorrelation of existing recordings. In its simplest form, this can be achieved using wavefield observations by only two receivers: the crosscorrelation process turns one of these receivers into a so-called "virtual source," whose response is retrieved by the other receiver.<sup>1-4</sup> The existing recordings may be due to passive (noise) sources or active sources. Applied to passive wavefields, averaging crosscorrelations over sufficiently long periods of time suffices to retrieve (an estimate of) the medium's Green's function.<sup>5-9</sup> In case of controlled sources, the crosscorrelation process involves an additional summation over the available controlled-source positions.<sup>10,11</sup> Over the last decade, numerous applications have emerged that rely on the explicit relation between the crosscorrelation and the medium's Green's function.9,12,13 Obtaining the virtual-source response by means of simple crosscorrelations will be referred to as "interferometry by crosscorrelation" in this paper.

Only under specific conditions can responses obtained through interferometry by crosscorrelation be related to the medium's Green's function. First, in the absence of sources inside the medium of interest, this medium is required to be lossless. Second, the medium needs to be illuminated with

equal power from all directions.<sup>4</sup> Often these conditions are not fulfilled, and, at best, an estimate of the medium's Green's function is retrieved.<sup>11,14,15</sup> The specific conditions associated with interferometry by crosscorrelation can be explained by the fact that the underlying theory stems from a correlation-type Green's function representation.<sup>4</sup> Starting from a convolution-type Green's function representation, however, the theory underlying interferometry by crosscorrelation can be reformulated in terms of a multidimensional deconvolution (MDD) process. Interferometry by MDD relaxes the conditions inherent to interferometry by crosscorrelation.<sup>16,17</sup> Most notably, MDD acknowledges the threedimensional nature of the wavefield by means of a so-called point-spread function (PSF).<sup>18</sup> This PSF captures irregularities in the illumination pattern. Multidimensionally deconvolving the responses obtained through interferometry by crosscorrelation by the PSF corrects for non-uniformities in the illumination pattern. That is, MDD has the ability to remove artefacts from those responses.

The less severe requirements of MDD regarding medium properties and illumination pattern trade off with greater requirements concerning receiver geometry: whereas a single pair of receivers suffices to estimate the medium Green's function by means of simple crosscorrelation, MDD of this estimate by the PSF requires the virtual source to be part of a "contour" of receivers.<sup>16</sup> This contour can be interpreted as an artificial boundary of the medium that one wants to image, and which does not necessarily coincide with a physical boundary. The PSF is constructed from the

<sup>&</sup>lt;sup>a)</sup>Also at: Seismology Division, Royal Netherlands Meteorological Institute, Utrechtse weg 297, 3731 GA, De Bilt, the Netherlands. Electronic mail: C.weemstra@tudelft.nl

wavefield observations by the receivers along this boundary. Additionally, inward and outward propagating waves need to be separated along this boundary: only waves propagating into the medium that one wants to image should be considered. The latter requirement is a direct consequence of the simplification of the integrand of the convolution-type Green's function representation, which is inherent in the derivation of the conventional MDD formulation. This simplification follows from the assumption that the medium outside the region of interest (the region of interest being the medium whose Green's function one wants to retrieve) is homogeneous. In other words, absorbing boundary conditions apply on the boundary along which the PSF is computed. In this work we show that, alternatively, reflecting boundary conditions can be assumed along (part of) this boundary. This assumption renders the separation of inward and outward propagating waves unnecessary. The retrieved responses, however, contain artificial "reflections" from the boundary closing the medium of interest.

In the following, we first briefly review the conventional simplification of the convolution-type Green's function representation, i.e., the simplification in which absorbing boundaries are assumed. We then derive the alternative MDD formulation assuming reflecting boundary conditions. Using a simple setup, we demonstrate the implications of the reflecting boundary conditions. Subsequently, using a more realistic setup and model, we showcase the advantage of the alternative boundary conditions for the retrieval of virtual-source responses from ambient seismic (surfacewave) noise recorded by ocean-bottom cables. In particular, we show how the modified MDD formulation improves the accuracy of the responses obtained through interferometry by crosscorrelation. Finally, we discuss the results and draw conclusions.

# II. CONVOLUTION-TYPE GREEN'S FUNCTION REPRESENTATION

Let us consider the configuration shown in Fig. 1; a volume  $\mathbb{V}$  is enclosed by a surface  $\mathbb{S}$  with outward pointing normal vector  $\mathbf{n} = (n_1, n_2, n_3)$ . We define a reference Green's function  $\overline{G}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t)$ , which gives the pressure at  $\mathbf{x}_{\mathrm{R}}$ due to an impulsive point source of the rate of change of volume injection rate at  $\mathbf{x}^{19}$  Similar to  $\overline{G}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t)$ , we define a Green's function  $G(\mathbf{x}, \mathbf{x}_{S}, t)$ , which gives the pressure at x due to an impulsive point source of the rate of change of volume injection rate at  $x_S$ . Moreover, we prescribe  $\mathbf{x}_{S}$  to be situated outside S, whereas we choose  $\mathbf{x}_{R}$ inside S. Importantly, the reference Green's function  $G(\mathbf{x}_{R}, \mathbf{x}, t)$  is associated with a reference medium and/or boundary conditions (hence the bar), whereas  $G(\mathbf{x}, \mathbf{x}_{S}, t)$  is associated with the actual medium. Considering additionally the Green's function  $G(\mathbf{x}_{R}, \mathbf{x}_{S}, t)$  in the actual medium, and assuming identical medium parameters for  $\bar{G}(\mathbf{x}_{R}, \mathbf{x}, t)$ ,  $G(\mathbf{x}, \mathbf{x}_{S}, t)$  and  $G(\mathbf{x}_{R}, \mathbf{x}_{S}, t)$  inside S, but a different reference medium outside of S, a convolution-type representation for the Green's function can be derived,<sup>16</sup>



FIG. 1. Configuration for the convolution-type Green's function representation [Eq. (1)]. The rays associated with  $G(\mathbf{x}, \mathbf{x}_S, t)$  and  $G(\mathbf{x}_R, \mathbf{x}_S, t)$  represent full responses, including scattered arrivals due to inhomogeneities inside as well as outside  $\mathbb{S}$ . The reference Green's function  $\overline{G}(\mathbf{x}_R, \mathbf{x}, t)$  represents the full response of the medium in  $\mathbb{V}$  plus possible additional effects associated with the choice of the boundary conditions at  $\mathbb{S}$  and/or different medium parameters outside  $\mathbb{S}$ .

$$G(\mathbf{x}_{\mathrm{R}}, \mathbf{x}_{\mathrm{S}}, t) = \oint_{\mathbb{S}} \frac{1}{\rho(\mathbf{x})} \left( \bar{G}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t) * \nabla G(\mathbf{x}, \mathbf{x}_{\mathrm{S}}, t) - G(\mathbf{x}, \mathbf{x}_{\mathrm{S}}, t) * \nabla \bar{G}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t) \right) \cdot \mathbf{n} \, \mathrm{d}\mathbf{x}.$$
(1)

The spatial derivatives are computed on  $\mathbb{S}$ , i.e., at **x**, and the in-line asterisk \* denotes temporal convolution; mass density is denoted by  $\rho$ . Note that  $[-1/\rho(\mathbf{x})]\nabla G(\mathbf{x}, \mathbf{x}_S, t) \cdot \mathbf{n}$ represents the particle acceleration at **x** perpendicular to  $\mathbb{S}$ due to a monopole source at  $\mathbf{x}_S$  (its time antiderivative hence represents particle velocity perpendicular to  $\mathbb{S}$ ). Instead,  $[-1/\rho(\mathbf{x})]\nabla \overline{G}(\mathbf{x}_R, \mathbf{x}, t) \cdot \mathbf{n}$  represents the pressure at  $\mathbf{x}_R$  due to a dipole source at  $\mathbf{x}$ .<sup>4</sup> The medium in  $\mathbb{V}$  can be arbitrarily heterogeneous, and Eq. (1) holds for media with losses. While  $G(\mathbf{x}, \mathbf{x}_S, t)$  and  $G(\mathbf{x}_R, \mathbf{x}_S, t)$  are related to the observed wavefield, the reference Green's function  $\overline{G}(\mathbf{x}_R, \mathbf{x}, t)$  is not. This allows us to choose convenient boundary conditions for  $\overline{G}(\mathbf{x}_R, \mathbf{x}, t)$  at  $\mathbb{S}$ .

#### **III. ABSORBING BOUNDARY CONDITIONS**

The integrand in Eq. (1) can be simplified assuming absorbing boundary conditions along  $\mathbb{S}$  for  $\overline{G}(\mathbf{x}_{\mathbf{R}}, \mathbf{x}, t)$ . This implies that its reciprocal  $\overline{G}(\mathbf{x}, \mathbf{x}_{\mathbf{R}}, t)$  is outward propagating at  $\mathbf{x}$  on  $\mathbb{S}$ . Furthermore,  $G(\mathbf{x}, \mathbf{x}_{\mathbf{S}}, t)$  can be written as a superposition of inward and outward propagating wavefields at  $\mathbf{x}$ on  $\mathbb{S}$ . Using pseudo-differential operator theory,<sup>20</sup> it can then be shown that the two terms of the integrand in Eq. (1) cancel each other for outward propagating signal of  $G(\mathbf{x}, \mathbf{x}_{\mathbf{S}}, t)$ , whereas they coincide in case inward propagating signal of  $G(\mathbf{x}, \mathbf{x}_{\mathbf{S}}, t)$  is considered. Assuming  $\mathbb{S}$  to be sufficiently smooth and  $\rho$  constant along  $\mathbb{S}$ , Eq. (1) therefore simplifies to<sup>17</sup>

$$G(\mathbf{x}_{\mathrm{R}},\mathbf{x}_{\mathrm{S}},t) = 2 \int_{\mathbb{S}_{\mathrm{rec}}} \bar{G}_{\mathrm{d}}(\mathbf{x}_{\mathrm{R}},\mathbf{x},t) * G^{(\mathrm{in})}(\mathbf{x},\mathbf{x}_{\mathrm{S}},t) \,\mathrm{d}\mathbf{x}.$$
 (2)

The subscript d denotes that  $\overline{G}_d$  is a dipole Green's function:  $\overline{G}_d(\mathbf{x}_R, \mathbf{x}, t) \equiv [-1/\rho] \nabla \overline{G}(\mathbf{x}_R, \mathbf{x}, t) \cdot \mathbf{n}$ . We have replaced  $\mathbb{S}$  with  $\mathbb{S}_{\text{rec}}$ , which comprises that part of  $\mathbb{S}$  on which  $G^{(\text{in})}(\mathbf{x}, \mathbf{x}_S, t) \neq 0$  (and able to be separated from  $G^{(\text{out})}(\mathbf{x}, \mathbf{x}_S, t)$ ): Sommerfeld's radiation condition applies over the half sphere  $\mathbb{S}_0$  that closes  $\mathbb{S}$ . The integral along  $\mathbb{S}_0$  therefore evaluates to zero. This also complies with many practical situations in which the integration surface is limited to an open receiver boundary because the wavefield is simply not recorded along a closed boundary.<sup>21</sup> Figure 2 exemplifies the configuration with an open receiver boundary  $\mathbb{S}_{\text{rec}}$ .

To comply with practice, the Green's functions related to the observed wavefield are convolved with a (transient) source time function  $s(\mathbf{x}_{S}, t)$ , yielding

$$p(\mathbf{x}_{\mathrm{R}},\mathbf{x}_{\mathrm{S}},t) = 2 \int_{\mathbb{S}_{\mathrm{rec}}} \bar{G}_{\mathrm{d}}(\mathbf{x}_{\mathrm{R}},\mathbf{x},t) * p^{(\mathrm{in})}(\mathbf{x},\mathbf{x}_{\mathrm{S}},t) \,\mathrm{d}\mathbf{x}, \quad (3)$$

where  $p(\mathbf{x}_{R}, \mathbf{x}_{S}, t) \equiv G(\mathbf{x}_{R}, \mathbf{x}_{S}, t) * s(\mathbf{x}_{S}, t)$  and  $p^{(in)}(\mathbf{x}, \mathbf{x}_{S}, t) \equiv G^{(in)}(\mathbf{x}, \mathbf{x}_{S}, t) * s(\mathbf{x}_{S}, t)$ . Note that, instead of a point source, the source at  $\mathbf{x}_{S}$  may also be an extended source. In that case,  $p(\mathbf{x}_{R}, \mathbf{x}_{S}, t)$  and  $p^{(in)}(\mathbf{x}, \mathbf{x}_{S}, t)$  represent the responses at  $\mathbf{x}_{R}$  and  $\mathbf{x}$ , respectively, integrated over the extended source. Similarly, the source at  $\mathbf{x}_{S}$  may also be of a different type. For example, Eq. (3) also holds in case at  $\mathbf{x}_{S}$  a spatial differential operator is applied to  $p(\mathbf{x}_{R}, \mathbf{x}_{S}, t)$  and  $p^{(in)}(\mathbf{x}, \mathbf{x}_{S}, t)$ .

In case a multitude of sources exist (on the appropriate side of  $\mathbb{S}_{rec}$ ), Eq. (3) holds for each of these sources individually, effectively yielding a set of equations. Retrieval of  $\bar{G}_d(\mathbf{x}_{\mathbf{R}}, \mathbf{x}, t)$  by solving this set of equations for  $\bar{G}_d(\mathbf{x}_{\mathbf{R}}, \mathbf{x}, t)$ in a least-squares sense is referred to as MDD.<sup>16</sup> We account for the presence of a multitude of sources by denoting the source position of source number k by  $\mathbf{x}_{\mathbf{S}}^{(k)}$ . Equation (3) implies that

$$p(\mathbf{x}_{\mathrm{R}}, \mathbf{x}_{\mathrm{S}}^{(k)}, t) = 2 \int_{\mathbb{S}_{\mathrm{rec}}} \bar{G}_{\mathrm{d}}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t) * p^{(\mathrm{in})}(\mathbf{x}, \mathbf{x}_{\mathrm{S}}^{(k)}, t) \,\mathrm{d}\mathbf{x},$$
(4)



FIG. 2. Configuration associated with absorbing boundary conditions. The meaning of different the symbols is given in Fig. 1.

for all *k*. Introducing the auxiliary location variable  $\mathbf{x}'$  along  $\mathbb{S}_{\text{rec}}$  (Fig. 2), the normal equation is obtained by crosscorrelating both sides of Eq. (4) with  $p^{(\text{in})}(\mathbf{x}', \mathbf{x}_{\text{S}}^{(k)}, t)$  and taking the sum over all sources.<sup>22</sup> The normal equation hence reads

$$C(\mathbf{x}_{\mathrm{R}}, \mathbf{x}', t) = 2 \int_{\mathbb{S}_{\mathrm{rec}}} \bar{G}_{\mathrm{d}}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t) * \Gamma(\mathbf{x}, \mathbf{x}', t) \,\mathrm{d}\mathbf{x},$$
(5)

where

$$C(\mathbf{x}_{\mathrm{R}}, \mathbf{x}', t) \equiv \sum_{k} p(\mathbf{x}_{\mathrm{R}}, \mathbf{x}_{\mathrm{S}}^{(k)}, t) * p^{(\mathrm{in})}(\mathbf{x}', \mathbf{x}_{\mathrm{S}}^{(k)}, -t)$$
(6)

and

$$\Gamma(\mathbf{x}, \mathbf{x}', t) \equiv \sum_{k} p^{(\text{in})}(\mathbf{x}, \mathbf{x}_{\text{S}}^{(k)}, t) * p^{(\text{in})}(\mathbf{x}', \mathbf{x}_{\text{S}}^{(k)}, -t).$$
(7)

Equation (5) shows how the crosscorrelation function  $C(\mathbf{x}_{\rm R}, \mathbf{x}', t)$  (CCF) is proportional to the sought-for dipole Green's function  $\overline{G}_{\rm d}(\mathbf{x}_{\rm R}, \mathbf{x}, t)$ , smeared in space and time by  $\Gamma(\mathbf{x}, \mathbf{x}', t)$ , which is referred to as the PSF. If the sources do not illuminate  $\mathbb{S}_{\rm rec}$  uniformly, the distortion of  $C(\mathbf{x}_{\rm R}, \mathbf{x}', t)$  with respect to  $\overline{G}_{\rm d}$  is quantified by the PSF. MDD involves deconvolving the CCF by the PSF. In practice, this requires discretization of the integral along  $\mathbb{S}_{\rm rec}$ . For a single frequency, Eq. (5) can then be written in matrix notation as<sup>17</sup>

$$\hat{\mathbf{C}} = 2\bar{\mathbf{G}}_{\mathrm{d}}\hat{\mathbf{\Gamma}},\tag{8}$$

where the rows and columns of  $\hat{C}$  correspond to different  $x_R$ and x', respectively;  $\hat{G}_d$  and  $\hat{\Gamma}$  are organized accordingly. The hat indicates that the matrices are defined in the frequency domain. Equation (8) is solved for each discrete frequency individually. Right multiplying Eq. (8) by the stabilized inverse of  $\hat{\Gamma}$  gives

$$\hat{\mathbf{G}}_{d} = \frac{1}{2}\hat{\mathbf{C}}\left(\hat{\mathbf{\Gamma}} + \epsilon^{2}\mathbf{I}\right)^{-1},\tag{9}$$

where **I** denotes the identity matrix and  $\epsilon$  is a small number. Details regarding the inversion can be found in earlier work.<sup>18</sup> Successful inversion for  $\bar{G}_d(\mathbf{x}_R, \mathbf{x}, t)$  has been reported using crosswell seismic reflection data,<sup>23</sup> reflection data in an arctic environment,<sup>24</sup> and ambient seismic surface-wave data.<sup>21,25</sup>

There are settings in which the medium of interest, i.e.,  $\mathbb{V}$ , is surrounded by sources and/or scatterers. The signals originating from these sources and/or scatterers may not have coinciding source time functions. Moreover, their spatial distribution may not be uniform. The irregularity of the resulting illumination pattern implies that the CCF does not accurately estimate the medium's Green's function between  $\mathbf{x}'$  and  $\mathbf{x}_R$ .<sup>14,15,26</sup> At the same time, the (non-uniform) omnidirectional illumination pattern implies that waves traverse the boundary surrounding  $\mathbb{V}$  in both directions, which renders wavefield separation along  $\mathbb{S}$  necessary for application of the MDD scheme described above. Separation of inward and outward propagating waves, however, is not always straightforward and/or possible. In Sec. IV we show that,

for such cases, it may be more appropriate to assume reflecting boundary conditions along  $\mathbb{S}$  in the reference medium.

# **IV. REFLECTING BOUNDARY CONDITIONS**

Recall that the convolution-type representation for the Green's function between  $x_S$  and  $x_R$  [i.e., Eq. (1)] relies on Green's functions that are not necessarily associated with the same medium. The  $G(\mathbf{x}, \mathbf{x}_{S}, t)$  are associated with the actual medium (and hence the observed wavefield), but the  $\overline{G}(\mathbf{x}_{R}, \mathbf{x}, t)$  are associated with a medium that possibly has different boundary conditions at S and/or different medium parameters outside  $\mathbb S.$  (In  $\mathbb V,$  however, the medium parameters are the same for both Green's functions.) An alternative simplification of the integrand in Eq. (1) can therefore be obtained by assuming a reflecting boundary in the reference medium. Two types of reflecting boundary conditions can be considered: (i) zero pressure along  $\mathbb{S}_{\text{rec}}$ , i.e.,  $\overline{G}(\mathbf{x}, \mathbf{x}_{\text{R}}, t) = 0$ , or (ii) zero particle acceleration perpendicular to  $S_{rec}$ , i.e.,  $\nabla \bar{G}(\mathbf{x}, \mathbf{x}_{\mathrm{R}}, t) \cdot \mathbf{n} = 0$ . By invoking source-receiver reciprocity, which implies that  $\bar{G}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t) = \bar{G}(\mathbf{x}, \mathbf{x}_{\mathrm{R}}, t)$  and  $\nabla G(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t) = \nabla G(\mathbf{x}, \mathbf{x}_{\mathrm{R}}, t)$ , the integrand in Eq. (1) can subsequently be simplified. In the next two sections we consider the two types of reflecting boundary conditions separately. In both cases, however, the application of MDD may result in the retrieval of reflections from the boundary of receivers. In Sec. IVC, we highlight the potential of the alternative simplification for the purpose of retrieving virtual-source surface-wave responses from recordings of ambient seismic (surface-wave) noise.

# A. Zero pressure along $\mathbb{S}_{\mathrm{rec}}$

Let us assume that the pressure vanishes along  $S_{\text{rec}}$ , i.e.,  $\bar{G}(\mathbf{x}, \mathbf{x}_{\text{R}}, t) = 0$  along  $S_{\text{rec}}$ ; note that this means that also the particle acceleration tangent to  $S_{\text{rec}}$  vanishes.<sup>27</sup> By invoking source-receiver reciprocity ( $\bar{G}(\mathbf{x}_{\text{R}}, \mathbf{x}, t) = \bar{G}(\mathbf{x}, \mathbf{x}_{\text{R}}, t)$ ) and assuming again the density to be constant along  $S_{\text{rec}}$ , Eq. (1) can then be simplified to

$$G(\mathbf{x}_{\mathrm{R}}, \mathbf{x}_{\mathrm{S}}, t) = \int_{\mathbb{S}_{\mathrm{rec}}} \bar{G}_{\mathrm{d}}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t) * G(\mathbf{x}, \mathbf{x}_{\mathrm{S}}, t) \,\mathrm{d}\mathbf{x}.$$
(10)

This equation is similar to Eq. (2), but has two notable differences. First, instead of the inward propagating wavefield on  $S_{rec}$ , the full wavefield is considered. Second, the right-hand side of Eq. (10) lacks a factor two. Physically, the absence of this factor can be explained by the reflecting nature of  $\mathbb{S}_{rec}$ : the arrivals in  $\bar{G}_{d}(\mathbf{x}_{R}, \mathbf{x}, t)$  that are not associated with reflections from  $S_{rec}$  have simply twice the amplitude of the same arrivals in  $\overline{G}_{d}(\mathbf{x}_{R}, \mathbf{x}, t)$  in Eq. (2). Again,  $\mathbb{S}_{rec}$  only needs to be comprised of those parts of S through which wavefields are propagating into  $\mathbb{V}$ . It suffices to evaluate the integral over x along these parts, because Sommerfeld's radiation condition applies along the remaining parts. Whether  $S_{rec}$  is in practice a closed or an open receiver boundary therefore depends on the illumination pattern. Figure 3 shows an example of a configuration for which reflecting boundaries in the reference medium are a convenient choice.



FIG. 3. Configuration associated with reflecting boundary conditions where the pressure is prescribed to be zero along  $\mathbb{S}_{\text{rec}}$  in the reference medium. The retrieved Green's function  $\overline{G}_{d}$  includes "reflections" from the receiver boundary (note that not all reflections are drawn here). The meaning of the different symbols is given in Fig. 1.

Just as in Sec. III, we consider a source with a source time function  $s(\mathbf{x}_{S}, t)$ . Using this, Eq. (10) can be written as

$$p(\mathbf{x}_{\mathrm{R}}, \mathbf{x}_{\mathrm{S}}, t) = \int_{\mathbb{S}_{\mathrm{rec}}} \bar{G}_{\mathrm{d}}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t) * p(\mathbf{x}, \mathbf{x}_{\mathrm{S}}, t) \,\mathrm{d}\mathbf{x}, \tag{11}$$

where  $p(\mathbf{x}, \mathbf{x}_{S}, t) \equiv G(\mathbf{x}, \mathbf{x}_{S}, t) * s(\mathbf{x}_{S}, t)$ . Assuming many sources exist outside of  $\mathbb{S}_{rec}$  (possibly with different source time functions, source types, and source extents), the normal equation is obtained in a similar way as in Sec. III. That is, we crosscorrelate both sides with the  $p(\mathbf{x}', \mathbf{x}_{S}^{(k)}, t)$  (pressure of full wavefield at  $\mathbf{x}'$  due to source number *k* at source position  $\mathbf{x}_{S}^{(k)}$ ),

$$C(\mathbf{x}_{\mathrm{R}}, \mathbf{x}', t) = \int_{\mathbb{S}_{\mathrm{rec}}} \bar{G}_{\mathrm{d}}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t) * \Gamma(\mathbf{x}, \mathbf{x}', t) \,\mathrm{d}\mathbf{x}, \tag{12}$$

where

$$C(\mathbf{x}_{\mathbf{R}}, \mathbf{x}', t) \equiv \sum_{k} p(\mathbf{x}_{\mathbf{R}}, \mathbf{x}_{\mathbf{S}}^{(k)}, t) * p(\mathbf{x}', \mathbf{x}_{\mathbf{S}}^{(k)}, -t)$$
(13)

and

$$\Gamma(\mathbf{x}, \mathbf{x}', t) \equiv \sum_{k} p(\mathbf{x}, \mathbf{x}_{\mathrm{S}}^{(k)}, t) * p(\mathbf{x}', \mathbf{x}_{\mathrm{S}}^{(k)}, -t).$$
(14)

The expression in Eq. (12) is seemingly similar to the expression in Eq. (5). Both expressions show a CCF that is proportional to a dipole Green's function smeared in space and time by a PSF. Also, in both cases  $\mathbb{S}_{rec}$  only needs to be comprised of those parts of  $\mathbb{S}$  through which waves are propagating into  $\mathbb{V}$ . Nevertheless, notable differences exist between the two relations. First, this involves the nature of

the dipole Green's function. The dipole Green's function in Eq. (5) gives the pressure at  $x_R$  due to a virtual source somewhere on  $\mathbb{S}_{\text{rec}}$  for a medium that coincides with the actual medium inside  $\mathbb{V}$ , but that is homogeneous on and outside of  $\mathbb{S}_{\text{rec}}$ . The dipole Green's function in Eq. (12), however, is associated with a medium that coincides with the actual medium inside  $\mathbb{V}$ , but where  $\mathbb{S}_{rec}$  acts as a free surface.  $\bar{G}_{d}(\mathbf{x}_{R}, \mathbf{x}, t)$  in Eq. (12) therefore contains arrivals due to the "virtual reflector(s)" along  $\mathbb{S}_{\text{rec}}.$  Second, the CCF and the PSF are computed from different wavefields. In case absorbing boundary conditions are assumed, only waves that propagate into  $\mathbb V$  should be used to compute the CCF and PSF [according to Eqs. (6) and (7), respectively]. To the contrary, Eq. (12) also allows outward propagating wavefields to be included in the computation [manifested by Eqs. (13) and (14)]. As a consequence, solving Eq. (12) for  $G_d(\mathbf{x}_R, \mathbf{x}, t)$ does not rely on wavefield separation.

In practice, the multiple (non-uniformly distributed) sources may act simultaneously. We comply with this more general case by considering the total responses instead of the individual source responses. In this case, Eq. (11) becomes

$$p(\mathbf{x}_{\mathrm{R}},t) = \int_{\mathbb{S}_{\mathrm{rec}}} \bar{G}_{\mathrm{d}}(\mathbf{x}_{\mathrm{R}},\mathbf{x},t) * p(\mathbf{x},t) \,\mathrm{d}\mathbf{x},\tag{15}$$

where the pressure at  $\mathbf{x}_{R}$  and  $\mathbf{x}$  is given by

$$p(\mathbf{x}_{\mathrm{R}},t) \equiv \sum_{k} G(\mathbf{x}_{\mathrm{R}},\mathbf{x}_{\mathrm{S}}^{(k)},t) * s(\mathbf{x}_{\mathrm{S}}^{(k)},t)$$
(16)

and

$$p(\mathbf{x},t) \equiv \sum_{k} G(\mathbf{x}, \mathbf{x}_{\mathrm{S}}^{(k)}, t) * s(\mathbf{x}_{\mathrm{S}}^{(k)}, t),$$
(17)

respectively.

Using Eqs. (16) and (17), the CCF and PSF can be defined as  $^{17}$ 

$$C(\mathbf{x}_{\mathrm{R}}, \mathbf{x}', t) \equiv \langle p(\mathbf{x}_{\mathrm{R}}, t) * p(\mathbf{x}', -t) \rangle$$
(18)

and

$$\Gamma(\mathbf{x}, \mathbf{x}', t) \equiv \langle p(\mathbf{x}, t) * p(\mathbf{x}', -t) \rangle,$$
(19)

respectively, where  $\langle \cdot \rangle$  denotes the ensemble average. Clearly, Eq. (12) still holds for these definitions of the CCF and PSF. Note that for the special case of mutually fully uncorrelated (noise) sources, Eqs. (18) and (19) reduce to Eqs. (13) and (14), respectively. This is because in this case  $\langle s(\mathbf{x}_{\rm S}^{(j)}, t) * s(\mathbf{x}_{\rm S}^{(k)}, -t) \rangle = S(\mathbf{x}_{\rm S}^{(k)}, t) \delta_{jk}$ , where  $\delta_{jk}$  is the Kronecker delta and  $S(\mathbf{x}_{\rm S}^{(k)}, t) \equiv s(\mathbf{x}_{\rm S}^{(k)}, t) * s(\mathbf{x}_{\rm S}^{(k)}, -t)$ . In application to ambient seismic/acoustic noise, ensemble averaging is in practice often replaced by time averaging.<sup>9,28</sup> Furthermore, temporal normalization and/or spectral whitening may need to be applied to mitigate the adverse effects arising from the lack of stationarity of the recorded noise field.<sup>29,30</sup>

Inversion of Eq. (12) is similar to the inversion of Eq. (5):  $\bar{G}_{d}(\mathbf{x}_{R}, \mathbf{x}, t)$  is retrieved through Eq. (9), but without the

factor 1/2 at the right-hand side and with the CCF and PSF computed according either to Eqs. (13) and (14), or to Eqs. (18) and (19), respectively. In the latter case, the degree of correlatedness of the (noise) sources determines the stability of the inversion for  $\overline{G}_d(\mathbf{x}_R, \mathbf{x}, t)$ .<sup>31</sup> Sources that are mutually not fully uncorrelated, may lead to difficulties in the inversion.

It should be understood that although the expression in Eq. (11) is completely general, the application of MDD to retrieve  $\bar{G}_d(\mathbf{x}_R, \mathbf{x}, t)$  relies on waves propagating into  $\mathbb{V}$  to generate the reflections. That is to say, in case a virtual reflector is not illuminated from outside  $\mathbb{V}$ , inversion of  $\hat{\Gamma}$  will not be feasible. We demonstrate this in the Appendix by synthesizing the PSF for a simple one-dimensional configuration. Finally, we note that for receiver configurations for which reflecting boundaries are a convenient choice, the assumption of a constant density along  $\mathbb{S}_{rec}$  may well not be very realistic. In that case, the retrieved dipole Green's function is simply scaled differently for different virtual-source positions, i.e.,  $\bar{G}_d(\mathbf{x}_R, \mathbf{x}, t) \equiv [-1/\rho(\mathbf{x})]\nabla \bar{G}(\mathbf{x}_R, \mathbf{x}, t) \cdot \mathbf{n}$ .

We demonstrate the modified MDD scheme for the twodimensional configuration depicted in Fig. 4. The medium considered here is a homogeneous, acoustic half-space below a free surface. The medium of interest, i.e., V, is located between two vertical receiver arrays, and is illuminated by two vertical source arrays along which sources are regularly placed. Note that this model merely serves as a proof of concept; in Sec. IV C, we consider a more realistic illumination pattern and medium. Together, the two receiver arrays comprise  $S_{rec}$ , implying that the pressure is prescribed to be zero in the reference medium. Consequently, Eq. (11) applies along these two lines. Furthermore, the line  $x_3 = 0$  m acts as a free surface in both the actual and the reference medium. Since, by definition, the acoustic pressure vanishes at the free surface, the integrand in Eq. (1) is zero along this line [i.e., both  $\overline{G}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t)$ , which coincides with  $\overline{G}(\mathbf{x}, \mathbf{x}_{\mathrm{R}}, t)$ because of source-receiver reciprocity, and  $G(\mathbf{x}, \mathbf{x}_{S}, t)$  are zero along  $x_3 = 0$  m].

The absence of sources at  $x_3 > 2400$  m ensures that no waves propagate upward through the boundary  $x_3 = 2400 \text{ m}$ into  $\mathbb{V}$ . However, the absence of receivers along  $x_3 = 2400$ m implies that we consider a medium  $\mathbb{V}$  that is not entirely enclosed by  $S_{rec}$ . Moreover, the fact that the two vertical receiver arrays do not extend beyond  $x_3 > 2400$  m (i.e., they have a finite aperture) implies that the integral in Eq. (10)does not represent  $G(\mathbf{x}_{\mathrm{R}},\mathbf{x}_{\mathrm{S}},t)$  entirely accurately for the configuration in Fig. 4. This stems from the fact that  $G(\mathbf{x}_{\mathrm{R}}, \mathbf{x}_{\mathrm{S}}, t)$  is not accurately represented by the integral in Eq. (1) in this case. Only when the receiver boundaries were to extend down to infinity would the convolution-type representation be exact.<sup>32</sup> However, since application of MDD relies on illumination by a multitude of sources, we expect the artefacts resulting from the truncation of the receiver boundary to be relatively minor. This is because, for a specific arrival in a specific  $G(\mathbf{x}_{R}, \mathbf{x}_{S}, t)$  associated with a specific source, a specific point  $\mathbf{x}_{st}$  exists along  $\mathbb{S}_{rec}$  for which the phase of the integrand in Eq. (10) [or Eq. (2) in case of absorbing boundary conditions and single-sided illumination] is stationary. In addition, because a multitude of



FIG. 4. (Color online) Model setup for MDD assuming a reflecting boundary in the reference medium along the two vertical receiver arrays at  $x_1 = 1200$  m and  $x_1 = 3200$  m. The receiver separation is 25 m and they extend down to 2400 m depth (the scales along the  $x_1$  and  $x_3$  axes coincide). Two vertical source arrays, located at  $x_1 = 400$  m and  $x_1 = 4000$  m, respectively, are placed outside of the medium of interest. The interval and maximum depth of the source arrays. For display purposes, only every fourth source and receiver is depicted. The receiver acting as a virtual source is shown in red. The virtual-source response is retrieved at the location of the green receiver. The numbered trajectories indicate the paths associated with the different arrivals in Fig. 5.

sources exist outside of  $\mathbb{V}$ , the arrival in  $\overline{G}_d(\mathbf{x}_R, \mathbf{x}_{st}, t)$  associated with this specific arrival in  $G(\mathbf{x}_R, \mathbf{x}_S, t)$  may additionally be associated with a second arrival in a second Green's function between a second source and  $\mathbf{x}_R$ , but for which the integrand has the same stationary point. In the MDD process, both sources will contribute to the retrieval of  $\overline{G}_d(\mathbf{x}_R, \mathbf{x}_{st}, t)$ . As long as the stationary point does not coincide with the point at which  $\mathbb{S}_{rec}$  is truncated, however, the phase of the integrand will be different for the two different sources at the truncation point. It is therefore that we expect the multidimensional inversion to mitigate the effect of the truncation of the receiver boundary. A detailed analysis, however, is beyond the scope and purpose of this work.

Each source in our model emits a Ricker wavelet with a central frequency of 10 Hz and unit maximum amplitude. For impulsive point sources of the rate of change of volume injection rate and a time dependence  $e^{i\omega t}$  ( $\omega$  denotes angular frequency), the direct arrival in the frequency-domain Green's function, denoted by  $\hat{G}^{(0)}$ , is simply given by a zeroth order Hankel function of the second kind, i.e.,  $\hat{G}^{(0)}(\mathbf{x}_{A}, \mathbf{x}_{B}, \omega) = (-i\rho/4)H_{0}^{(2)}(\omega|\mathbf{x}_{A} - \mathbf{x}_{B}|/c)$  for any  $\mathbf{x}_{A}$  and  $\mathbf{x}_{B}$  [two-dimensional solution to Eq. (20) in Wapenaar and Fokkema<sup>4</sup>]. Of course, the full Green's functions  $G(\mathbf{x}_{R}, \mathbf{x}_{S}, t)$  and  $G(\mathbf{x}, \mathbf{x}_{S}, t)$  contain an additional reflection from the free surface. The velocity  $c = \sqrt{K/\rho}$  is based on the bulk modulus and density of water, i.e.,  $K = 2.2 \times 10^9$  Pa and  $\rho = 1000 \text{ kg/m}^3$ , respectively.

Figure 5 compares the virtual-source response obtained through interferometry by crosscorrelation to the virtual-source response obtained through the application of the MDD formulation derived in this section. The peaks at negative time disappear through the inversion, whereas additional peaks appear at positive time because of the reflecting nature of  $S_{rec}$  [Fig. 5(a)]. Because the regularly placed sources

J. Acoust. Soc. Am. 142 (4), October 2017

result in a (close to) uniform illumination pattern, the CCF and the direct arrival of the dipole Green's function arrive at the same time (arrival 1). The virtual-source response obtained through the application of interferometry by MDD has a higher frequency content than the response obtained through MDD. In part, this can be explained by the fact that interferometry by crosscorrelation results in retrieval of a virtual-source response which, in the frequency-domain, is divided by  $\omega$  [see Eq. (32) in Wapenaar and Fokkema<sup>4</sup>]. In contrast, the dipole character of the virtual-source response retrieved through MDD implies a multiplication by  $\omega$  in that domain. Furthermore, we observe that the amplitude of the



FIG. 5. (Color online) Comparison of interferometry by crosscorrelation with MDD assuming reflecting boundary conditions. In (a), the responses retrieved using interferometry by crosscorrelation (red line) and MDD (green line) are compared; (b) shows the retrieved dipole Green's function over a longer time range. The numbers of the different arrivals in a correspond to the paths depicted in Fig. 4. For comparison, the retrieved dipole Green's function is convolved with the autocorrelation of the sources. Amplitudes are normalized.

free-surface reflection of the MDD response (arrival 2) is lower than the amplitude of the free-surface reflection in the CCF. This can be explained by the inner product with the normal to the receiver boundary implicit in  $\overline{G}_{d}$ .

Arrivals 3 and 4 are the direct result of the alternative MDD formulation. These arrivals are due to reflections from the receiver boundary at  $x_1 = 3200$  m (note the opposite polarity of arrival 3 compared to arrival 1). In addition to these two arrivals, other arrivals exist. For example, consider the arrival associated with the wave emitted by the virtual source, reflected from the receiver boundary at  $x_1 = 3200$  m, the free surface, and the receiver boundary at  $x_1 = 1200$  m, respectively, and finally retrieved by the receiver at  $x_{R}$ . These later arrivals, however, have arrival times that approximately coincide with the horizontally traveling reverberations and hence cannot be distinguished [Fig. 5(b)]. It is clear from Fig. 5(b) that the reflecting boundary condition causes the signal to reverberate within  $\mathbb{V}$ . By muting the (virtual) reverberations of the response, one could extract the (improved) MDD response. For more complex media, however, this may not be trivial. Finally, we note that the truncation at  $x_3 = 2400$  m of the source array along  $x_1 = 400$  m gives rise to spurious arrivals around t = 0.45 s (denoted by T1) and t = 0.65 s (denoted by T2); spurious arrival T1 is associated with the free surface reflection of the truncation arrival. We observe that MDD mostly corrects for the truncation effect, but cannot completely undo it. We also observe that the truncation arrivals reverberate in the reference medium.

### B. Zero particle acceleration perpendicular to $S_{rec}$

Let us now assume that  $\nabla \bar{G}(\mathbf{x}, \mathbf{x}_{R}, t) \cdot \mathbf{n} = 0$  along  $\mathbb{S}_{\text{rec}}$ . This assumption implies that the particle acceleration perpendicular to  $\mathbb{S}_{\text{rec}}$  vanishes. By invoking the reciprocity relation  $\nabla \bar{G}(\mathbf{x}_{R}, \mathbf{x}, t) = \nabla \bar{G}(\mathbf{x}, \mathbf{x}_{R}, t)$ , where in both cases the spatial derivative is computed on  $\mathbb{S}$  (i.e., at  $\mathbf{x}$ ), we find that Eq. (1) reduces to

$$G(\mathbf{x}_{\mathrm{R}}, \mathbf{x}_{\mathrm{S}}, t) = \int_{\mathbb{S}_{\mathrm{rec}}} \frac{1}{\rho(\mathbf{x})} \, \bar{G}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t) * \nabla G(\mathbf{x}, \mathbf{x}_{\mathrm{S}}, t) \cdot \mathbf{n} \, \mathrm{d}\mathbf{x}.$$
(20)

In this case, the wavefield at  $\mathbb{S}_{rec}$  is propagated to  $\mathbf{x}_{R}$  through  $\mathbb{V}$  by the propagator  $\overline{G}(\mathbf{x}_{R}, \mathbf{x}, t)$ . By introducing the Green's functions for particle acceleration<sup>4</sup>

$$\mathbf{G}^{(a)}(\mathbf{x}_{\mathrm{R}},\mathbf{x}_{\mathrm{S}},t) \equiv \frac{-1}{\rho(\mathbf{x}_{\mathrm{R}})} \,\nabla G(\mathbf{x}_{\mathrm{R}},\mathbf{x}_{\mathrm{S}},t),\tag{21}$$

$$\bar{\mathbf{G}}^{(a)}(\mathbf{x}_{\mathrm{R}},\mathbf{x},t) \equiv \frac{-1}{\rho(\mathbf{x}_{\mathrm{R}})} \nabla \bar{G}(\mathbf{x}_{\mathrm{R}},\mathbf{x},t),$$
(22)

and

$$\mathbf{G}^{(\mathrm{a})}(\mathbf{x}, \mathbf{x}_{\mathrm{S}}, t) \equiv \frac{-1}{\rho(\mathbf{x})} \nabla G(\mathbf{x}, \mathbf{x}_{\mathrm{S}}, t), \tag{23}$$

we adhere to the case where particle acceleration is the measured wavefield quantity. Using these Green's functions, multiplication of both sides of Eq. (20) with  $[1/\rho(\mathbf{x}_R)]\nabla_R$ (where the subscript *R* denotes explicitly that the spatial derivative is computed at  $\mathbf{x}_R$ ) gives

$$\begin{split} \mathbf{G}^{(a)}(\mathbf{x}_{\mathrm{R}},\mathbf{x}_{\mathrm{S}},t) &= - \int_{\mathbb{S}_{\mathrm{rec}}} \bar{\mathbf{G}}^{(a)}(\mathbf{x}_{\mathrm{R}},\mathbf{x},t) \\ & * \left[ \mathbf{G}^{(a)}(\mathbf{x},\mathbf{x}_{\mathrm{S}},t) \cdot \mathbf{n} \right] \mathrm{d}\mathbf{x}. \end{split} \tag{24}$$

Consider again a multitude of sources illuminating  $\mathbb{V}$ , each with a different location  $\mathbf{x}^{(k)}$  and source time function  $s(\mathbf{x}_{S}^{(k)}, t)$ . By convolving both sides of Eq. (24) with  $s(\mathbf{x}_{S}^{(k)}, t)$ , we obtain for each of these sources

$$\mathbf{a}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}_{\mathrm{S}}^{(k)}, t) = -\int_{\mathbb{S}_{\mathrm{rec}}} \bar{\mathbf{G}}^{(a)}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t) * \left[\mathbf{a}(\mathbf{x}, \mathbf{x}_{\mathrm{S}}^{(k)}, t) \cdot \mathbf{n}\right] \mathrm{d}\mathbf{x},$$
(25)

where  $\mathbf{a}(\mathbf{x}_{R}, \mathbf{x}_{S}^{(k)}, t) \equiv \mathbf{G}^{(a)}(\mathbf{x}_{R}, \mathbf{x}_{S}^{(k)}, t) * s(\mathbf{x}_{S}^{(k)}, t)$  and  $\mathbf{a}(\mathbf{x}, \mathbf{x}_{S}^{(k)}, t) \equiv \mathbf{G}^{(a)}(\mathbf{x}, \mathbf{x}_{S}^{(k)}, t) * s(\mathbf{x}_{S}^{(k)}, t)$ . Just as for Eqs. (3) and (11), the source at  $\mathbf{x}_{S}^{(k)}$  may also be an extended source (and/or of a different type). In that case,  $\mathbf{a}(\mathbf{x}_{R}, \mathbf{x}_{S}^{(k)}, t)$  and  $\mathbf{a}(\mathbf{x}, \mathbf{x}_{S}^{(k)}, t)$  represent the responses at  $\mathbf{x}_{R}$  and  $\mathbf{x}$ , respectively, integrated over the extended source (and/or due to the different source type).

Retrieval of  $\overline{\mathbf{G}}^{(a)}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t)$  involves the application of MDD. Once again, this is achieved by solving the set of equations associated with the multitude of sources in a least-squares sense. In this case, the normal equation is obtained by crosscorrelating both sides with  $[\mathbf{a}(\mathbf{x}', \mathbf{x}_{\mathrm{S}}^{(k)}, t) \cdot \mathbf{n}]$  (inner product between normal vector and particle acceleration of the full wavefield at  $\mathbf{x}'$  due to source number *k* at source position  $\mathbf{x}_{\mathrm{S}}^{(k)}$ ),

$$\mathbf{C}(\mathbf{x}_{\mathrm{R}},\mathbf{x}',t) = \int_{\mathbb{S}_{\mathrm{rec}}} \bar{\mathbf{G}}^{(\mathrm{a})}(\mathbf{x}_{\mathrm{R}},\mathbf{x},t) * \Gamma(\mathbf{x},\mathbf{x}',t) \,\mathrm{d}\mathbf{x}, \qquad (26)$$

where

$$\mathbf{C}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}', t) \equiv \sum_{k} \mathbf{a}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}_{\mathrm{S}}^{(k)}, t) * \left[ \mathbf{a}(\mathbf{x}', \mathbf{x}_{\mathrm{S}}^{(k)}, -t) \cdot \mathbf{n} \right]$$
(27)

and

$$\Gamma(\mathbf{x}, \mathbf{x}', t) \equiv \sum_{k} \left[ \mathbf{a}(\mathbf{x}, \mathbf{x}_{\mathrm{S}}^{(k)}, t) \cdot \mathbf{n} \right] * \left[ \mathbf{a}(\mathbf{x}', \mathbf{x}_{\mathrm{S}}^{(k)}, -t) \cdot \mathbf{n} \right].$$
(28)

Equation (26) shows that CCF is proportional to  $\bar{\mathbf{G}}^{(a)}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}, t)$  smeared in space and time by  $\Gamma(\mathbf{x}, \mathbf{x}', t)$ . Note that this PSF is the same for each of the three components  $x_1, x_2$ , and  $x_3$ . Just as in Sec. IV A, successful application of MDD relies on (non-isotropic) wavefields propagating into as well as out of  $\mathbb{V}$  through  $\mathbb{S}_{\mathrm{rec}}$ .

# C. Potential for the retrieval of ambient seismic noise surface-wave responses

Settings exist in which sources and/or scatterers surround the medium of interest, but illuminate it with varying



FIG. 6. (Color online) Configuration for the simulation of single-mode surface waves. Only every fifth boundary receiver is depicted (the location of every 20th source is explicitly indicated). The responses from these boundary receivers (i.e., the virtual sources) are retrieved at the location of the green receiver ( $x_R$ ). Sources are irregularly distributed around the receiver array and vary in amplitude (the diameter of a circle circumscribing a star scales linearly with that source's amplitude). The amplitude as a function of frequency is presented in Fig. 7 (top). A single isotropic point scatterer (black dot) is located 1500 m east and 200 m north of  $x_R$ .

intensity. The non-uniform illumination precludes accurate Green's function estimation by the CCF, whereas, at the same time, its omnidirectional nature requires separation of inward and outward propagating waves to allow application of conventional (absorbing boundary) MDD. For such wave-fields, the MDD formulation resulting from the reflecting boundary conditions is of particular interest. A notable example is the ambient seismic (surface-wave) wavefield, which, over the past decade, has been exploited in numerous studies.<sup>6,21,28,33,34</sup> In this section, we showcase the advantage of the alternative boundary conditions for the application of MDD to ambient seismic noise recorded by ocean-bottom cables. Their aperture and dense spatial sampling<sup>35,36</sup> make these deployments particularly well suited for the application of MDD.

In case of seismic waves, the formulation underlying MDD results from a simplification of the elastodynamic convolution-type representation theorem for the Green's function between  $x_S$  and  $x_R$ .<sup>21,37</sup> Just as the acoustic convolution-type representation theorem [i.e., Eq. (1)], the elastodynamic representation theorem can be simplified assuming absorbing or reflecting boundary conditions. The ambient seismic field at the Earth's surface, however, is often dominated by single-mode (dispersive) surface waves.<sup>28,38</sup> In addition, along the surface of a laterally homogeneous and isotropic earth, the behavior of this type of waves is very similar to scalar body waves in a two-dimensional homogeneous lossy medium, but with  $c = c(\omega)$ .<sup>39</sup> That is, the frequency-domain Green's function describing the particle velocity at the surface of such an

earth is proportional to the zeroth-order Hankel function described in Sec. IV A. $^{40}$ 

The configuration of the receiver array, as well as the location and relative strength of the modeled sources, is shown in Fig. 6. The closed receiver boundary has an east–



FIG. 7. The frequency-dependent amplitude (top) and phase velocity (bottom) used to simulate the single-mode surface-wave wavefield associated with the configuration in Fig. 6.



FIG. 8. (Color online) Responses of virtual sources 1-150 retrieved at  $x_R$ ; only every fifth virtual-source response is shown. Responses retrieved through interferometry by crosscorrelation are depicted in red, whereas responses retrieved through MDD are depicted in green. In both cases, the retrieved responses are plotted on top of the directly modeled responses (dashed black line). The latter responses are modeled in the actual medium, and hence do not contain virtual reflections from the rectangular receiver boundary. For comparison, these directly modeled responses are also depicted individually in the leftmost (interferometry by crosscorrelation) and rightmost (MDD) panels. Note that these differ because of the dipole character of the retrieved MDD responses (inner product with the receiver boundary). The retrieved and directly modeled dipole Green's functions are convolved with the autocorrelation of the sources for comparison. The values on the vertical axis refer to the boundary receiver numbers (Fig. 6); the corners of the boundary are indicated for reference. The dashed boxes indicate the traces shown in Fig. 11.

west and north-south extent of 10 and 5 km, respectively. This resembles the aperture of contemporary ocean-bottom deployments.<sup>33,35</sup> The receiver boundary is illuminated by 500 randomly placed sources, whose spatial placement is governed by a non-uniform probability distribution; for example, more sources are (expected to be) placed west of the receiver boundary than south. In accordance with the theory of Sec. IVA, no sources are prescribed inside the receiver boundary. Furthermore, the sources vary in amplitude: the function describing a source's amplitude as a function of frequency (Fig. 7; top) is scaled by a factor that varies per source. The homogeneous medium has been perturbed by a single isotropic point scatterer.<sup>41</sup> The strength of this point scatterer, which is capped by the optical theorem, has been given the maximum amplitude possible in two dimensions, i.e., its scattering amplitude is given by -4i.<sup>26</sup>

In the frequency domain, the Green's function describing the vertical particle-velocity associated with a singlemode surface wave propagating along the surface of a laterally homogeneous and isotropic earth behaves according to<sup>25,40</sup>

$$\hat{G}(\mathbf{x}_{\mathrm{R}}, \mathbf{x}_{\mathrm{S}}, \omega) \propto H_0^{(2)}(k|\mathbf{x}_{\mathrm{R}} - \mathbf{x}_{\mathrm{S}}|),$$
(29)

where k is the wavenumber, and  $\mathbf{x}_{\rm S}$  and  $\mathbf{x}_{\rm R}$  are the source and receiver location, respectively. We assume a dissipative medium, which implies that k is complex-valued with its real and imaginary part coinciding with  $\omega/c(\omega)$  and  $-\alpha$ , respectively, i.e.,  $k = \omega/c(\omega) - i\alpha(\omega)$ . Here,  $c(\omega)$  and  $\alpha(\omega)$ denote the (frequency dependent) phase velocity and attenuation coefficient, respectively. Because we assume the medium to be only slightly dissipative ( $\alpha$  is small in the sense that  $\alpha \ll \omega/c$ ), the complex wavenumber can be approximated by  $(\omega/c)\sqrt{1 - (2i\alpha c/\omega)}$ .<sup>26</sup> The Hankel function may in that case be approximated by  $H_0^{(2)}(\omega|\mathbf{x}_{\rm R} - \mathbf{x}_{\rm S}|/c(\omega))e^{-\alpha|\mathbf{x}_{\rm R}-\mathbf{x}_{\rm S}|}$ .<sup>39</sup> We mimic surface-wave dispersion (an increase of phase velocity with depth) by letting the phase-



FIG. 9. (Color online) Same as Fig. 8, but for virtual source numbers 150-300 (northern and western receiver boundary in Fig. 6).

velocity  $c(\omega)$  decrease with increasing angular frequency according to Fig. 7 (bottom); the attenuation coefficient  $\alpha$  is set to a constant value of  $7.5 \times 10^{-5}$  m<sup>-1</sup> for all frequencies. Note that both phase velocity and frequency-dependent amplitude (Fig. 7) of the sources are modeled using reasonable values for Scholte waves traveling along the sea bed.<sup>42</sup>

We first consider the noise sources (random phases in the frequency domain) to act sequentially, which implies that the CCF and PSF are computed by summing crosscorrelations over source locations [i.e., as in Eqs. (13) and (14), respectively]. Figures 8 and 9 present the virtual-source responses of every fifth virtual source along the rectangular receiver array. These responses are retrieved at  $x_R$ . The responses retrieved using interferometry by crosscorrelation and MDD, as well as the directly modeled responses, are all normalized by the maximum amplitude of the response of the 38th virtual source, which is the boundary receiver that is closest to  $x_R$ . Relative amplitude differences between different virtualsource responses are therefore maintained. Importantly, the directly modeled responses are the responses associated with the actual medium, instead of the reference medium. That is, these responses do not contain virtual reflections associated with the rectangular receiver boundary.

The responses retrieved through interferometry by crosscorrelation (i.e., the CCF) contain significantly more



FIG. 10. (Color online) Responses of virtual source number 228. The virtual source response retrieved through interferometry by crosscorrelation is depicted at the bottom by the red solid line, whereas the response retrieved through MDD is depicted at the top by the green solid line. In both cases, the retrieved responses are plotted on top of the directly modeled responses in the actual medium (dashed black line). Note that these differ because of the dipole character of the retrieved MDD responses (inner product with the receiver boundary). The retrieved and directly modeled dipole Green's functions are convolved with the autocorrelation of the sources for comparison. Note the energy arriving at around 18 s in the MDD response, which is virtually reflected from the opposite receiver boundary.



FIG. 11. (Color online) Isotropically scattered arrivals in the virtual-source responses from boundary receivers 34–43. Responses retrieved through interferometry by crosscorrelation are depicted in red (left); responses retrieved through MDD are depicted in green (right). In both cases the retrieved responses are plotted on top of the directly modeled responses in the actual medium (dashed black line). For comparison, the retrieved and directly modeled dipole Green's functions are convolved with the autocorrelation of the sources.

and stronger spurious arrivals. For example, significant spurious energy is present for virtual-source numbers 223-268. In contrast, responses obtained through the application of MDD are practically free of spurious arrivals prior to the direct arrival. Upon comparison of the responses of a single virtual source (Fig. 10), this becomes particularly clear. Additionally, the amplitudes of the responses retrieved through the application of interferometry by crosscorrelation exhibit a considerable misfit with respect to the amplitudes of the directly modeled responses. This amplitude misfit disappears after deconvolution of the CCF by the PSF. Note that the amplitudes of the responses retrieved though interferometry by crosscorrelation and MDD differ differently for different virtual sources because of the dipole character of the MDD response. Both the amplitude corrections and the suppression of spurious energy demonstrate how the application of MDD removes the adverse effects of azimuthal and spatial variations in the illumination pattern.

Apart from the direct surface-wave arrival, the virtualsource responses contain an arrival scattered from the isotropic point scatterer. Additionally, however, the virtual-source responses retrieved through the application of MDD contain many other arrivals trailing the direct surface wave. These arrivals are due to the receiver boundary acting as a virtual reflector and are well observable in Figs. 8 and 9. The virtual reflection from the opposite receiver boundary is particularly conspicuous in Fig. 10. In principle, the actual medium response could be recovered by muting these virtual arrivals. In practice, many strong heterogeneities in  $\mathbb{V}$ , i.e., in the actual medium, and/or crooked receiver boundaries may complicate the separation in time of the actual medium response and the arrivals associated with the reflecting receiver boundaries. In Fig. 11, we compare the arrivals scattered from the isotropic point scatterer. We observe that both timing and amplitude of the MDD responses are more accurate. The additional arrival around t = 15 s can be attributed to the reflection from the receiver boundary of the isotropically scattered virtual-source signal.

Using the configuration in Fig. 6, we conduct a second numerical experiment where we consider simultaneously acting, mutually uncorrelated noise sources. Consequently, each receiver's recording is a superposition of the noise originating from the 500 noise sources. A total of 576 noise windows (or realizations) have been modeled, each with a duration of 2.5 min. In total, this adds up to one day of ambient seismic surface-wave noise. Figure 12 gives an example of 2.5 min of noise recorded by the receiver at  $x_R$ . In Fig. 13, the virtual-source responses retrieved through interferometry by crosscorrelation are compared to the virtual-source responses retrieved through the application of MDD (assuming reflecting boundary conditions) for



FIG. 12. An example of 2.5 min of ambient seismic surface-wave noise at  $x_R$ . The noise is a superposition of the noise generated by the 500 noise sources depicted in Fig. 6. The frequency-dependent amplitude of each source is dictated by Fig. 7 (top), but scaled by a source-specific factor. It is propagated to  $x_R$  using the Green's function in Eq. (29), including an arrival due to the isotropic point scatterer.



FIG. 13. (Color online) Virtual-source responses from boundary receivers 212–216. Responses retrieved through interferometry by crosscorrelation are depicted in red (bottom); responses retrieved through MDD are depicted in green (top). The responses are retrieved from wavefields of simultaneously acting, mutually uncorrelated noise sources. For comparison, the retrieved dipole Green's functions (MDD responses) are convolved with the autocorrelation of the sources.

virtual-source numbers 212-216. The fact that virtual source 214 is vertically above  $\mathbf{x}_{R}$  implies that the arrival time of the direct surface waves (around 9s) should be nearly constant for these five virtual sources. The fact that this does not appear to be the case for the responses retrieved through interferometry by crosscorrelation (Fig. 13; red responses), can be attributed to the non-uniform illumination pattern. Application of MDD corrects the arrival time of the direct surface-wave arrivals resulting in arrival times that are practically constant for these five virtual sources (Fig. 13; green responses). Moreover, just as for the sequential sources, we find that deconvolution by the PSF suppresses spurious energy at early times and creates virtual reflections at later times. In this case, however, the spurious energy is not entirely removed, which is probably due to the fact that the simultaneously acting noise sources cause the CCF and the PSF to contain more spurious energy than was the case for the sequentially acting noise sources.

#### V. DISCUSSION AND CONCLUSIONS

The acoustic Green's function between a source outside a medium of interest and a receiver inside that medium can be represented by an integral over the boundary of that medium, where the integrand is a sum of two terms, both temporal convolutions [Eq. (1)]. In the first term, the spatial derivative (at the boundary) of the Green's function between the source and the boundary is convolved with the Green's function between the boundary and the receiver. In the second term, the Green's function between the source and the boundary is convolved with the spatial derivative (at the boundary) of the Green's function between the boundary and the receiver. Because of the convolutions, this representation for the Green's function between the source and the receiver is generally referred to as a "convolution-type representation."<sup>17</sup> Importantly, however, the Green's functions describing the wave propagation between the boundary and the receiver can be defined in a different medium than the Green's functions that describe the waves propagating from the source to the boundary. In this work, the latter Green's functions are associated with the actual medium, and hence with the Green's function between the physical source and the receiver, whereas the former Green's functions are defined between a virtual source and a receiver in a reference medium.

In this work, we let the reference medium coincide with the actual medium inside the medium of interest, but investigate different conditions of the reference medium on and outside of that boundary. Specifically, we consider the reference medium to be of absorbing and of reflecting nature on that boundary. In both cases, a formulation results which allows the retrieval of the Green's functions between the boundary and the receiver by means of a MDD process. In practice, this implies that the wavefield is required to be captured along the boundary of the medium of interest. The MDD process turns the boundary receivers into so-called virtual sources, whose responses are corrected for illuminationrelated artefacts and account for dissipation in the medium of interest. This is an improvement with respect to virtualsource responses retrieved through the application of interferometry by crosscorrelation, because these responses have been shown to suffer from irregularities in the illumination pattern.<sup>11,14,15</sup>

Conventionally, absorbing boundary conditions are assumed in the reference medium.<sup>16,17</sup> Because the resulting MDD formulation relies on wavefields propagating exclusively into the medium of interest, it requires omnidirectional wavefields to be separated in inward propagating and outward propagating wavefields along the receiver boundary. Assuming reflecting boundary conditions renders separation of wavefields unnecessary. However, the retrieved Green's functions contain virtual reflections from the receiver boundary. In other words, the MDD process exploits the (non-uniform) illumination to turn the receiver boundary into a virtual reflector.

Alternative boundary conditions of the medium associated with the deconvolved wavefield have previously been used for the one-dimensional case of a building response.<sup>43</sup>

Similar to our analysis, these authors consider reflecting and absorbing boundary conditions. In their case, however, the boundary conditions follow from the definition of the deconvolution operator, whereas they are prescribed explicitly in this study. As such, they also do not employ the term "reference medium." An example application of changing the boundary conditions is, in their context, the estimation of intrinsic attenuation.<sup>44</sup>

The term virtual reflector has been used previously.<sup>45</sup> The virtual reflector referred to in this study, however, is rather different from the virtual reflector introduced by those authors. First, we consider sources randomly distributed outside the medium of interest, whose signal is recorded along a receiver boundary encompassing that medium. In contrast, those authors consider a receiver boundary that records the response to a source inside the medium of interest (alternatively, using source-receiver reciprocity, the receiver boundary can be replaced by a source boundary, and the source inside the medium of interest by a receiver). Second, their method yields the response due to a virtual-source inside the medium of interest, instead of on the boundary of that medium. Third, their response lacks the direct arrival and other arrivals associated with internal scattering in the medium of interest.

In Sec. IV C, we have shown the potential of the newly derived MDD formulation for the retrieval of surface-wave responses from recordings of ambient seismic noise. We acknowledge, however, that for such applications, specific preprocessing of the field data will be required.<sup>46,47</sup> In general, frequent interruption of the stream of ambient seismic noise by earthquakes and other (anthropogenic) events complicate the application of interferometric methods.<sup>30</sup> Additionally, the assumption of a single-mode surface wave overwhelming other surface-wave modes may in practice not always be valid,<sup>48</sup> requiring different surface-wave modes to be separated prior to application of MDD.<sup>37</sup>

The responses retrieved through the application of MDD assuming reflecting boundary conditions could be particularly well suited for the purpose of full-waveform inversion (FWI). The computational costs of FWI schemes are known to increase due to the necessity to extend the computational domain outside the domain of interest.<sup>49</sup> This domain extension is needed to fulfill the absorbing boundary conditions in the theory underlying full-waveform formulations.<sup>50</sup> Clearly, FWI of the virtual-source responses containing virtual reflections will not require such an absorbing layer at the periphery of the numerical domain.

The term coda is conventionally used to describe the relatively late-arriving multiply scattered waves in a seismogram. Due to the reflections from the receiver boundaries, the newly retrieved virtual-source responses will contain artificial, but meaningful, coda. Similar to natural coda waves, these artificial coda waves will have traversed the medium of interest several times and in different directions, and will therefore be more sensitive to structural changes in that medium.<sup>51</sup> Formulations (coda-wave sensitivity kernels) could therefore be developed that allow the artificial coda to be exploited for the purpose of time-lapse monitoring.<sup>52–54</sup> A potential oceanographic application (of the exploitation) of the artificial coda would be the application of MDD to so-called autonomous vertical line arrays.<sup>9,55</sup>

In Sec. IV, we mentioned the possibility to mute the virtual reflections in order to retrieve the actual medium response. Of course, in more complex media, it will not be straightforward to distinguish between virtual reflections and arrivals due to internal scattering in the actual medium. In fact, the virtual reflections and the coda of the actual medium response may well overlap in time. Surface-related multiple elimination (SRME) algorithms<sup>56</sup> may be adapted for the purpose of removing the virtual reflections. Of course, the fact the amplitudes of different arrivals in the retrieved virtualsource responses are scaled by different factors (depending on the angles at which the ray paths associated with the respective arrivals depart from the receiver boundary) needs to be taken into account in this case. The application (or inclusion) of SRME algorithms in the derived (reflecting boundary) MDD formulation is beyond the scope of this work, however, and will be the subject of future work.

### ACKNOWLEDGMENTS

We thank Pablo David Garcia Lopez for his original comment, which led us to investigate the matter.

# APPENDIX: THE PSF IN A ONE-DIMENSIONAL WAVEFIELD

The expression in Eq. (11) in the main text is completely general. The application of MDD to retrieve  $\bar{G}_d(\mathbf{x}_R, \mathbf{x}, t)$ , however, relies on waves propagating into  $\mathbb{V}$  to generate the reflections. In other words, in case a virtual reflector is not illuminated from outside  $\mathbb{V}$ , inversion of  $\hat{\Gamma}$  will not be feasible. We demonstrate this in this appendix by analytically evaluating Eq. (12) for a one-dimensional homogeneous medium. We therefore consider the following (frequency-domain) wave equation:

$$\frac{1}{\rho}\frac{\partial^2 G(x, x_{\rm S}, \omega)}{\partial x^2} + \frac{\omega^2}{\rho c^2}\hat{G}(x, x_{\rm S}, \omega) = -\delta(x - x_{\rm S}),\tag{A1}$$

where, as in the main text, the Green's function  $\hat{G}(x, x_{\rm S}, \omega)$  gives the pressure at x due to an impulsive point source of the rate of change of volume injection rate at  $x_{\rm S}$ . The Green's function for this wave equation reads<sup>57</sup>

$$\hat{G}(x, x_{\rm S}, \omega) = \frac{-i\rho c}{2\omega} e^{-i(\omega|x-x_{\rm S}|/c)},\tag{A2}$$

where a Fourier decomposition with time dependence  $e^{i\omega t}$  has been performed.

Let us consider two sources, located at  $x_{\rm S}^{(1)}$  and  $x_{\rm S}^{(2)}$ , whose power spectra are given by  $\hat{S}^{(1)} \equiv \hat{s}(x^{(1)}, \omega)\hat{s}^*(x^{(1)}, \omega)$  and  $\hat{S}^{(2)} \equiv \hat{s}(x^{(2)}, \omega)\hat{s}^*(x^{(2)}, \omega)$ , respectively (in general,  $\hat{z}^*$ denotes complex conjugation of the variable  $\hat{z}$ ). The multiplications  $\hat{s}\hat{s}^*$  correspond to autocorrelations in the time domain. Substitution of the Green's functions associated with these two sources in the frequency-domain counterparts of Eqs. (13) and (14) gives



FIG. 14. One-dimensional configuration for which the PSF is examined.

$$\hat{C}(x_{\rm R}, x', \omega) = \sum_{k=1}^{2} \hat{G}\left(x_{\rm R}, x_{\rm S}^{(k)}, \omega\right) \hat{G}^{*}\left(x', x_{\rm S}^{(k)}, \omega\right) \hat{S}^{(k)}(\omega)$$

$$= \frac{\rho^{2} c^{2}}{4\omega^{2}} \left(\hat{S}^{(1)} e^{i\omega\left(|x'-x_{\rm S}^{(1)}|-|x_{\rm R}-x_{\rm S}^{(1)}|\right)/c} + \hat{S}^{(2)} e^{i\omega\left(|x'-x_{\rm S}^{(2)}|-|x_{\rm R}-x_{\rm S}^{(2)}|\right)/c}\right)$$
(A3)

and

$$\begin{split} \hat{\Gamma}(x, x', \omega) &= \sum_{k=1}^{2} \hat{G}\left(x, x_{\rm S}^{(k)}, \omega\right) \hat{G}^{*}\left(x', x_{\rm S}^{(k)}, \omega\right) \hat{S}^{(k)}(\omega) \\ &= \frac{\rho^{2} c^{2}}{4\omega^{2}} \left( \hat{S}^{(1)} e^{i\omega \left(|x' - x_{\rm S}^{(1)}| - |x - x_{\rm S}^{(1)}|\right)/c} \right. \\ &\quad + \hat{S}^{(2)} e^{i\omega \left(|x' - x_{\rm S}^{(2)}| - |x - x_{\rm S}^{(2)}|\right)/c} \right), \end{split}$$
(A4)

respectively.

In this appendix, we synthesize the PSF for the configuration in Fig. 14. The one-dimensional medium of interest  $\mathbb V$  is located between receivers  $x^{(1)}$  and  $x^{(2)}$ , which, together, form  $\mathbb{S}_{\text{rec}}$ . In other words,  $x^{(1)}$  and  $x^{(2)}$  are the locations for which the variables x and x' in Eqs. (A3) and (A4) are evaluated. For the exponents in Eq. (A3), the configuration in Fig. 14 implies that

$$\begin{aligned} |x^{(1)} - x_{\rm S}^{(1)}| - |x_{\rm R} - x_{\rm S}^{(1)}| &= x^{(1)} - x_{\rm R}, \\ |x^{(1)} - x_{\rm S}^{(2)}| - |x_{\rm R} - x_{\rm S}^{(2)}| &= x_{\rm R} - x^{(1)}, \\ |x^{(2)} - x_{\rm S}^{(1)}| - |x_{\rm R} - x_{\rm S}^{(1)}| &= x^{(2)} - x_{\rm R}, \\ |x^{(2)} - x_{\rm S}^{(2)}| - |x_{\rm R} - x_{\rm S}^{(2)}| &= x_{\rm R} - x^{(2)}, \end{aligned}$$
(A5)

and, for the exponents in Eq. (A4), that

$$\begin{aligned} |x^{(1)} - x_{\rm S}^{(1)}| &- |x^{(2)} - x_{\rm S}^{(1)}| = x^{(1)} - x^{(2)}, \\ |x^{(1)} - x_{\rm S}^{(2)}| &- |x^{(2)} - x_{\rm S}^{(2)}| = x^{(2)} - x^{(1)}, \\ |x^{(2)} - x_{\rm S}^{(1)}| &- |x^{(1)} - x_{\rm S}^{(1)}| = x^{(2)} - x^{(1)}, \\ |x^{(2)} - x_{\rm S}^{(2)}| &- |x^{(1)} - x_{\rm S}^{(2)}| = x^{(1)} - x^{(2)}. \end{aligned}$$
(A6)

Note that we have only considered  $x \neq x'$  for the exponents in Eq. (A4), because the exponents always evaluate to zero for x = x' (and hence the exponential terms to one).

Just as Eq. (5) can be written in matrix notation in the frequency domain [Eq. (8)], the frequency-domain counterpart of Eq. (12) can be written as

~

$$\hat{\mathbf{C}} = \hat{\bar{\mathbf{G}}}_{\mathrm{d}}\hat{\boldsymbol{\Gamma}}.\tag{A7}$$

In one dimension, the rows and columns of  $\hat{C}$  correspond to different  $x_{\rm R}$  and x', respectively;  $\hat{\mathbf{G}}_{\rm d}$  and  $\hat{\boldsymbol{\Gamma}}$  are organized accordingly. In particular, for our configuration, the variables x and x' in Eqs. (A3) and (A4) are evaluated over two locations only  $(x^{(1)} \text{ and } x^{(2)})$  and hence  $\hat{\Gamma}$  is 2 × 2 matrix. Because we furthermore consider a single receiver location  $x_{\rm R}$ for the retrieval of the virtual source responses (see Fig. 14),  $\hat{\mathbf{C}}$  and  $\bar{\mathbf{G}}_d$  are both 1 × 2 matrices. The matrix  $\bar{\mathbf{G}}_d$  reads

$$\hat{\bar{\mathbf{G}}}_{d} = (\hat{\bar{G}}_{d}(x_{\mathbf{R}}, x^{(1)}, \omega) \quad \hat{\bar{G}}_{d}(x_{\mathbf{R}}, x^{(2)}, \omega)).$$
 (A8)

Using the relations in Eqs. (A5) and (A6), Eqs. (A3) and (A4) yield the following expressions for  $\hat{C}$  and  $\hat{\Gamma}$ , respectively:

$$\hat{\mathbf{C}} = \frac{\rho^2 c^2}{4\omega^2} \left( \hat{S}^{(1)} e^{i\omega(x^{(1)} - x_{\mathbf{R}})/c} + \hat{S}^{(2)} e^{i\omega(x_{\mathbf{R}} - x^{(1)})/c} - \hat{S}^{(1)} e^{i\omega(x^{(2)} - x_{\mathbf{R}})/c} + \hat{S}^{(2)} e^{i\omega(x_{\mathbf{R}} - x^{(2)})/c} \right)$$
(A9)

1

and

$$\hat{\Gamma} = \frac{\rho^2 c^2}{4\omega^2} \begin{pmatrix} \hat{S}^{(1)} + \hat{S}^{(2)} & \hat{S}^{(1)} e^{i\omega(x^{(1)} - x^{(2)})/c} \\ \hat{S}^{(1)} e^{i\omega(x^{(1)} - x^{(2)})/c} + \hat{S}^{(2)} e^{i\omega(x^{(2)} - x^{(1)})/c} & \hat{S}^{(1)} + \hat{S}^{(2)} \end{pmatrix}.$$
(A10)

I

The determinant of  $\hat{\Gamma}$ , denoted by  $|\hat{\Gamma}|$ , reads

$$|\hat{\Gamma}| = 2\hat{S}^{(1)}\hat{S}^{(2)} (1 - \Re \left[ e^{i\omega 2(x^{(1)} - x^{(2)})/c} \right]), \tag{A11}$$

where the operator  $\Re[\cdots]$  maps its complex argument into its real part. Successful application of interferometry by MDD relies on the ability to invert  $\hat{\Gamma}$ . Equation (A11), however, reveals that there are two cases for which the determinant of  $\hat{\Gamma}$  may evaluate to zero. First, using the fact that  $\Re[e^{i\omega 2(x^{(1)}-x^{(2)})/c}] = \cos[2\omega(x^{(1)}-x^{(2)})/c]$ , we observe that  $\hat{\Gamma}$  is singular for specific "spurious" frequencies  $\omega = \omega_n \equiv n\pi c/(x^{(1)}-x^{(2)})$ , where *n* is a positive integer. For dissipative media, however,  $\Gamma$  will not become singular. This can easily be shown by repeating the analysis above using a Green's function that accounts for dissipation, i.e., by multiplying the right-hand side of Eq. (A2) by an exponentially decaying term  $e^{-\alpha|x-x_S|}$ .

Second, and more important, we observe that  $\hat{\Gamma}$  becomes singular in case either  $\hat{S}^{(1)}$  or  $\hat{S}^{(2)}$  (or both) is zero. This demonstrates that successful inversion of  $\hat{\Gamma}$  requires illumination from both sides for our one-dimensional setup. The power of the illumination from one side does not need to coincide with the power of the illumination from the other side though. In fact, just as deconvolution by the PSF corrects virtual-source responses for artefacts due to variations in power of the sources illuminating a two- or three-dimensional medium from one side,<sup>16,17</sup> deconvolution by the PSF in Eq. (A10) corrects the virtual-source response for artefacts due to differences between  $\hat{S}^{(1)}$  and  $\hat{S}^{(2)}$  in our one-dimensional model.

- <sup>2</sup>A. Derode, E. Larose, M. Tanter, J. de Rosny, A. Tourin, M. Campillo, and M. Fink, "Recovering the Green's function from field-field correlations in an open scattering medium," J. Acoust. Soc. Am. **113**, 2973–2976 (2003).
- <sup>3</sup>A. Bakulin and R. Calvert, "Virtual source: New method for imaging and 4D below complex overburden," SEG Technical Program Expanded Abstracts **23**, 2477–2480 (2004).
- <sup>4</sup>K. Wapenaar and J. Fokkema, "Green's function representations for seismic interferometry," Geophysics 71, SI33–SI46 (2006).
- <sup>5</sup>R. Weaver and O. Lobkis, "On the emergence of the Green's function in the correlations of a diffuse field: pulse-echo using thermal phonons," Ultrasonics 40, 435–439 (2002).
- <sup>6</sup>N. M. Shapiro and M. Campillo, "Emergence of broadband Rayleigh waves from correlations of the ambient seismic noise," Geophys. Res. Lett. **31**, L07614, https://doi.org/10.1029/2004GL019491 (2004).
- <sup>7</sup>K. G. Sabra, S. Conti, P. Roux, and W. A. Kuperman, "Passive in vivo elastography from skeletal muscle noise," Appl. Phys. Lett. **90**, 194101 (2007).
- <sup>8</sup>M. M. Haney, "Infrasonic ambient noise interferometry from correlations of microbaroms," Geophys. Res. Lett. **36**, L19808, https://doi.org/ 10.1029/2009GL040179 (2009).
- <sup>9</sup>O. A. Godin, N. A. Zabotin, and V. V. Goncharov, "Ocean tomography with acoustic daylight," Geophys. Res. Lett. **37**, L13605, https://doi.org/ 10.1029/2010GL043623 (2010).
- <sup>10</sup>A. Bakulin and R. Calvert, "The virtual source method: Theory and case study," Geophysics **71**, SI139–SI150 (2006).
- <sup>11</sup>B. Froment, M. Campillo, P. Roux, P. Gouédard, A. Verdel, and R. L. Weaver, "Estimation of the effect of nonisotropically distributed energy on the apparent arrival time in correlations," Geophysics 75, SA85–SA93 (2010).
- <sup>12</sup>N. M. Shapiro, M. Campillo, L. Stehly, and M. H. Ritzwoller, "High-resolution surface-wave tomography from ambient seismic noise," Science **307**, 1615–1618 (2005).
- <sup>13</sup>J. Verbeke, L. Boschi, L. Stehly, E. Kissling, and A. Michelini, "High-resolution Rayleigh-wave velocity maps of central Europe from a dense ambient-noise data set," Geophys. J. Int. 188, 1173–1187 (2012).
- <sup>14</sup>V. C. Tsai, "On establishing the accuracy of noise tomography travel-time measurements in a realistic medium," Geophys. J. Int. **178**, 1555–1564 (2009).

- <sup>15</sup>R. Weaver, B. Froment, and M. Campillo, "On the correlation of nonisotropically distributed ballistic scalar diffuse waves," J. Acoust. Soc. Am. **126**, 1817–1826 (2009).
- <sup>16</sup>K. Wapenaar and J. van der Neut, "A representation for Green's function retrieval by multidimensional deconvolution," J. Acoust. Soc. Am. **128**, EL366–EL371 (2010).
- <sup>17</sup>K. Wapenaar, J. van der Neut, E. Ruigrok, D. Draganov, J. Hunziker, E. Slob, J. Thorbecke, and R. Snieder, "Seismic interferometry by crosscorrelation and by multidimensional deconvolution: A systematic comparison," Geophys. J. Int. 185, 1335–1364 (2011).
- <sup>18</sup>J. van der Neut, "Interferometric redatuming by multidimensional deconvolution," Ph.D. thesis, Delft University of Technology, Delft, the Netherlands, 2012.
- <sup>19</sup>C. P. A. Wapenaar and A. J. Berkhout, *Elastic Wave Field Extrapolation: Redatuming of Single- and Multi-Component Seismic Data*, Vol. 2 (Elsevier Science, Amsterdam, the Netherlands, 1989).
- <sup>20</sup>L. Fishman, "One-way wave propagation methods in direct and inverse scalar wave propagation modeling," Radio Sci. 28, 865–876, https:// doi.org/10.1029/93RS01632 (1993).
- <sup>21</sup>C. Weemstra, D. Draganov, E. N. Ruigrok, J. Hunziker, M. Gomez, and K. Wapenaar, "Application of seismic interferometry by multidimensional deconvolution to ambient seismic noise recorded in Malargüe, Argentina," Geophys. J. Int. 208, 693–714 (2017).
- <sup>22</sup>J. van der Neut, E. Ruigrok, D. Draganov, and K. Wapenaar, "Retrieving the earth's reflection response by multi-dimensional deconvolution of ambient seismic noise," in *Proceedings of the 72nd EAGE Conference & Exhibition*, Barcelona, Spain (June 14–17, 2010).
- <sup>23</sup>S. Minato, T. Matsuoka, T. Tsuji, D. Draganov, J. Hunziker, and K. Wapenaar, "Seismic interferometry using multidimensional deconvolution and crosscorrelation for crosswell sesimic reflection data without borehole sources," Geophysics **76**, SA19–SA34 (2011).
- <sup>24</sup>F. Poletto and C. Bellezza, "Multidimensional deconvolution of seismicinterferometry Arctic data," SEG Technical Program Expanded Abstracts **31**, 1–5 (2012).
- <sup>25</sup>K. N. van Dalen, T. D. Mikesell, E. N. Ruigrok, and K. Wapenaar, "Retrieving surface waves from ambient seismic noise using seismic interferometry by multidimensional deconvolution," J. Geophys. Res.: Solid Earth **120**, 944–961, https://doi.org/10.1002/2014JB011262 (2015).
- <sup>26</sup>C. Weemstra, R. Snieder, and L. Boschi, "On the estimation of attenuation from the ambient seismic field: Inferences from distributions of isotropic point scatterers," Geophys. J. Int. **203**, 1054–1071 (2015).
- <sup>27</sup>R. Snieder, "Extracting the Green's function of attenuating heterogeneous acoustic media from uncorrelated waves," J. Acoust. Soc. Am. 121, 2637–2643 (2007).
- <sup>28</sup>C. Weemstra, L. Boschi, A. Goertz, and B. Artman, "Seismic attenuation from recordings of ambient noise," Geophysics 78, Q1–Q14 (2013).
- <sup>29</sup>G. D. Bensen, M. H. Ritzwoller, M. P. Barmin, A. L. Levshin, F. Lin, M. P. Moschetti, N. M. Shapiro, and Y. Yang, "Processing seismic ambient noise data to obtain reliable broad-band surface wave dispersion measurements," Geophys. J. Int. 169, 1239–1260 (2007).
- <sup>30</sup>J. Groos, S. Bussat, and J. Ritter, "Performance of different processing schemes in seismic noise cross-correlations," Geophys. J. Int. 188, 498–512 (2012).
- <sup>31</sup>K. Wapenaar, J. van der Neut, and J. Thorbecke, "On the relation between seismic interferometry and the simultaneous-source method," Geophys. Prospect. **60**, 802–823 (2012).
- <sup>32</sup>C. P. A. Wapenaar, "The infinite aperture paradox," J. Seis. Explor. 1, 325–336 (1992).
- <sup>33</sup>S. A. L. de Ridder and B. L. Biondi, "Daily reservoir-scale subsurface monitoring using ambient seismic noise," Geophys. Res. Lett. 40, 1–6, https://doi.org/10.1002/grl.50594 (2013).
- <sup>34</sup>F. Korostelev, C. Weemstra, S. Leroy, L. Boschi, D. Keir, Y. Ren, I. Molinari, A. Ahmed, G. W. Stuart, F. Rolandone, K. Khanbari, J. O. S. Hammond, J. M. Kendall, C. Doubre, I. A. Ganad, B. Goitom, and A. Ayele, "Magmatism on rift flanks: Insights from ambient noise phase velocity in afar region," Geophys. Res. Lett. 42, 2179–2188, https://doi.org/10.1002/2015GL063259 (2015).
- <sup>35</sup>A. Goertz, J. Richardson, J. Faragher, C. Remington, P. Morton, P. Barros, and C. Theodoro, "Microseismicity recorded by a fiberoptic ocean-bottom PRM installation offshore Brazil," in *Proceedings of the Third EAGE Workshop on Permanent Reservoir Monitoring 2014—Proactive Reservoir Management: PRM and Beyond*, Oslo, Norway (March 17–19, 2014).

<sup>&</sup>lt;sup>1</sup>O. I. Lobkis and R. L. Weaver, "On the emergence of the Green's function in the correlations of a diffuse field," J. Acoust. Soc. Am. **110**, 3011–3017 (2001).

- <sup>36</sup>S. A. L. de Ridder, B. L. Biondi, and R. G. Clapp, "Time-lapse seismic noise correlation tomography at Valhall," Geophys. Res. Lett. 41, 6116–6122, https://doi.org/10.1002/2014GL061156 (2014).
- <sup>37</sup>K. N. van Dalen, K. Wapenaar, and D. F. Halliday, "Surface wave retrieval in layered media using seismic interferometry by multidimensional deconvolution," Geophys. J. Int. **196**, 230–242 (2014).
- <sup>38</sup>F.-C. Lin, D. Li, R. W. Clayton, and D. Hollis, "High-resolution 3D shallow crustal structure in Long Beach, California: Application of ambient noise tomography on a dense seismic array," Geophysics 78, Q45–Q56 (2013).
- <sup>39</sup>V. C. Tsai, "Understanding the amplitudes of noise correlation measurements," J. Geophys. Res. **116**, B09311, https://doi.org/10.1029/ 2011JB008483 (2011).
- <sup>40</sup>K. Aki and P. G. Richards, *Quantitative Seismology*, 2nd ed. (University Science Books, Sausalito, CA, 2002), p. 700.
- <sup>41</sup>R. G. Newton, Scattering Theory of Waves and Particles (Dover Publications Inc., Mineola, NY, 2002), p. 749.
- <sup>42</sup>S. de Ridder and J. Dellinger, "Ambient seismic noise eikonal tomography for near-surface imaging at Valhall," The Leading Edge **30**, 506–512 (2011).
- <sup>43</sup>R. Snieder, J. Sheiman, and R. Calvert, "Equivalence of the virtual-source method and wave-field deconvolution in seismic interferometry," Phys. Rev. E 73, 066620 (2006).
- <sup>44</sup>C. Newton and R. Snieder, "Estimating intrinsic attenuation of a building using deconvolution interferometry and time reversal," Bull. Seismol. Soc. Am. **102**, 2200–2208 (2012).
- <sup>45</sup>F. Poletto and K. Wapenaar, "Virtual reflector representation theorem (acoustic medium)," J. Acoust. Soc. Am. **125**, EL111–EL116 (2009).
- <sup>46</sup>S. M. Hanasoge and M. Branicki, "Interpreting cross-correlations of onebit filtered seismic noise," Geophys. J. Int. **195**, 1811–1830 (2013).

- <sup>47</sup>C. Weemstra, W. Westra, R. Snieder, and L. Boschi, "On estimating attenuation from the amplitude of the spectrally whitened ambient seismic field," Geophys. J. Int. **197**, 1770–1788 (2014).
- <sup>48</sup>W. P. Kimman, X. Campman, and J. Trampert, "Characteristics of seismic noise: Fundamental and higher mode energy observed in the northeast of the Netherlands," Bull. Seismol. Soc. Am. **102**, 1388–1399 (2012).
- <sup>49</sup>J. Virieux and S. Operto, "An overview of full-waveform inversion in exploration geophysics," Geophysics 74, WCC127–WCC152 (2009).
- <sup>50</sup>K. J. Marfurt, "Accuracy of finite-difference and finite-element modeling of the scalar and elastic wave equations," Geophysics 49, 533–549 (1984).
- <sup>51</sup>R. Snieder, A. Grêt, H. Douma, and J. Scales, "Coda wave interferometry for estimating nonlinear behavior in seismic velocity," Science 295, 2253–2255 (2002).
- <sup>52</sup>C. Pacheco and R. Snieder, "Time-lapse travel time change of multiply scattered acoustic waves," J. Acoust. Soc. Am. 118, 1300–1310 (2005).
- <sup>53</sup>A. Obermann, T. Planes, E. Larose, C. Sens-Schonfelder, and M. Campillo, "Depth sensitivity of seismic coda waves to velocity perturbations in an elastic heterogeneous medium," Geophys. J. Int. **194**, 372–382 (2013).
- <sup>54</sup>C. Kanu and R. Snieder, "Numerical computation of the sensitivity kernel for monitoring weak changes with multiply scattered acoustic waves," Geophys. J. Int. 203, 1923–1936 (2015).
- <sup>55</sup>P. F. Worcester and R. C. Spindel, "North Pacific Acoustic Laboratory," J. Acoust. Soc. Am. **117**, 1499–1510 (2005).
- <sup>56</sup>D. J. Verschuur, A. J. Berkhout, and C. P. A. Wapenaar, "Adaptive surface-related multiple elimination," Geophysics 57, 1166–1177 (1992).
- <sup>57</sup>R. Snieder, A Guided Tour of Mathematical Methods for the Physical Sciences (Cambridge University Press, New York, 2004), p. 507.