Reciprocity theorems for two-way and one-way wave vectors: A comparison

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For acoustic applications in which there is a ''preferred direction of propagation'' (the axial direction) it is useful to arrange the two-way and one-way wave equations into the same matrix-vector formalism. In this formalism, axial variations of the wave vector are expressed in terms of lateral variations of the same wave vector. The two-way wave vector contains the field quantities pressure and velocity (axial component only), whereas the one-way wave vector contains waves propagating in the positive and negative axial direction. By exploiting the equivalent form of the two-way and one-way matrix-vector equations, it appears to be possible to derive two-way and one-way reciprocity theorems that have an equivalent form but a different interpretation. The main differences appear in the boundary integrals for unbounded media, in the contrast terms, and (for the correlation-type theorems) in the handling of evanescent waves. © *1996 Acoustical Society of America.*

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INTRODUCTION

In several areas of acoustic research it is common use to introduce a ''preferred direction of propagation'' and to exploit this preference in the construction of solutions of the acoustic wave equation. Whereas in underwater acoustics the preferred propagation direction is *horizontal*, in seismic applications the *vertical* axis is generally chosen as the preferred direction. Throughout this paper the Cartesian position coordinates are denoted by the vector $\mathbf{x}=(x_1,x_2,x_3)$ and the x_3 axis is chosen parallel to the preferred propagation direction. Hence, for underwater acoustics x_3 denotes ''range,'' whereas it denotes ''depth'' for seismic applications. For simplicity, in the following we speak for both applications of the *axial* coordinate x_3 and the *lateral* coordinates x_1 and x_2.

The aim of this paper is to derive and compare *reciprocity* theorems for two-way and one-way wave vectors in configurations with a direction of preference. In general, a reciprocity theorem interrelates the quantities that characterize two admissible physical states that could occur in one and the same domain (de Hoop, 1988). One can distinguish between convolution-type and correlation-type reciprocity theorems (Bojarski, 1983). Generally speaking, these two types of reciprocity theorems find their applications in forward and inverse problems, respectively. We start this paper with a brief review of both types of scalar reciprocity theorems.

In order to take account of the direction of preference, it is useful to express the axial variations of some wave field quantities in terms of the lateral variations of the same quantities. First we introduce what we will call the ''two-way wave vector,'' which contains the acoustic pressure and the axial component of the particle velocity, and we derive reciprocity theorems for this wave vector. When rewritten in scalar form, they appear to be very similar to the abovementioned scalar reciprocity theorems, as might be expected. The vector form, however, is better suited for a comparison with the reciprocity theorems for, what we will call, the ''one-way wave vector.''

This one-way wave vector is introduced next; it contains waves propagating in the positive and negative axial direction. Of course, when the medium parameters vary in the axial direction, these waves are coupled. The reciprocity theorems for the one-way wave vector are derived along the same lines as the reciprocity theorems for the two-way wave vector. Not surprisingly, they appear to have the same form. Finally we discuss some similarities and differences between these two classes of reciprocity theorems.

I. REVIEW OF RECIPROCITY THEOREMS

In this section we briefly review the scalar form of the acoustic reciprocity theorems of the convolution type and of the correlation type. We closely follow de Hoop (1988) and Fokkema and van den Berg (1993). The former author derives reciprocity theorems in the time domain; the latter authors in the time domain, the Laplace domain, and the frequency domain. Here we only consider the frequency domain.

A. Basic acoustic equations

In this subsection we give the basic equations for an acoustic wave field in an inhomogeneous lossless fluid medium. Throughout this paper we assume that the medium parameters are ''sufficiently smooth'' functions of position and time invariant. We define the Fourier transform with respect to time (t) of a real function as

$$U(\omega)=\int_{-\infty}^{\infty}u(t)\exp(-j\omega t)\,dt \tag{1}$$

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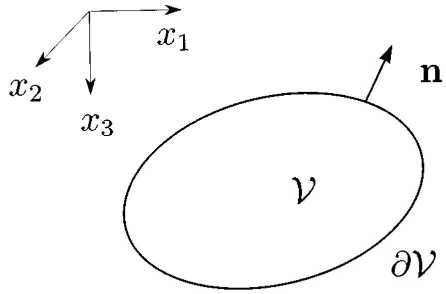

FIG. 1. Configuration for Rayleigh's reciprocity theorem.

and its inverse as

$$u(t)=\frac{1}{\pi}\,\mathrm{Re}\left(\int_0^\infty U(\omega)\exp(j\omega t)d\omega\right), \tag{2}$$

where j is the imaginary unit and ω denotes the angular frequency. Note that we consider positive frequencies only. In the remainder of this paper all functions are in the frequency domain; the ω dependency is not explicitly denoted.

In the space-frequency domain, the equations that govern linear acoustic wave propagation read

$$\partial_k P + j\omega\varrho V_k = F_k \tag{3}$$

and

$$\partial_k V_k + j\omega\kappa P = Q, \tag{4}$$

where P is the acoustic pressure, V_k is the particle velocity, ϱ is the volume density of mass, κ is the compressibility, F_k is the volume source density of volume force, and Q is the volume source density of volume injection rate. The Latin subscripts take on the values 1–3 and the summation convention applies to repeated subscripts.

B. Reciprocity theorem of the convolution type

We introduce two acoustic states (i.e., wave fields, medium parameters, and sources) that will be distinguished by the subscripts A and B. For these two states we consider the interaction quantity $\partial_k\{P_A V_{k,B} - V_{k,A} P_B\}$. Applying the product rule for differentiation, substituting Eqs. (3) and (4) for states A and B, integrating the result over a volume $\mathscr{V}$ with boundary $\partial\mathscr{V}$ and outward pointing normal vector $\mathbf{n}=(n_1,n_2,n_3)$ (see Fig. 1), and applying the theorem of Gauss yields

$$\begin{aligned}\int_{\partial\mathscr{V}}&\{P_A V_{k,B} - V_{k,A} P_B\} n_k\, d^2\mathbf{x}\\ &= -j\omega\int_{\mathscr{V}}\{P_A(\kappa_B-\kappa_A)P_B - V_{k,A}(\varrho_B-\varrho_A)V_{k,B}\}d^3\mathbf{x}\\ &\quad+\int_{\mathscr{V}}\{P_A Q_B - V_{k,A}F_{k,B} + F_{k,A}V_{k,B} - Q_A P_B\}d^3\mathbf{x}.\end{aligned} \tag{5}$$

Equation (5) is Rayleigh's reciprocity theorem of the convolution type (Rayleigh, 1878). We speak of ''convolution type'' since the products in the frequency domain ($P_A V_{k,B}$, etc.) correspond to convolutions in the time domain.

We conclude this subsection by considering some special cases.

Unbounded media: Consider the situation in which the medium at and outside $\partial\mathscr{V}$ is unbounded, homogeneous, and source-free in both states. Assume that the wave fields in both states are causally related to the sources in $\mathscr{V}$. Then, if $\varrho_A=\varrho_B$ and $\kappa_A=\kappa_B$ at and outside $\partial\mathscr{V}$, the boundary integral on the left-hand side of Eq. (5) vanishes (Bleistein, 1984; Fokkema and van den Berg, 1993).

Field reciprocity: Assume that the above-mentioned conditions are fulfilled and that $\varrho_A=\varrho_B$ and $\kappa_A=\kappa_B$ in $\mathscr{V}$ as well. Then the first volume integral on the right-hand side of Eq. (5) vanishes. Furthermore, consider point sources in states A and B at $\mathbf{x}_A\in\mathscr{V}$ and $\mathbf{x}_B\in\mathscr{V}$, respectively, according to

$$Q_A(\mathbf{x})=q_A\delta(\mathbf{x}-\mathbf{x}_A),\quad F_{k,A}(\mathbf{x})=f_{k,A}\delta(\mathbf{x}-\mathbf{x}_A) \tag{6}$$

and

$$Q_B(\mathbf{x})=q_B\delta(\mathbf{x}-\mathbf{x}_B),\quad F_{k,B}(\mathbf{x})=f_{k,B}\delta(\mathbf{x}-\mathbf{x}_B), \tag{7}$$

where $\delta(\mathbf{x})=\delta(x_1)\,\delta(x_2)\,\delta(x_3)$. Equation (5) thus yields

$$P_A(\mathbf{x}_B)q_B - V_{k,A}(\mathbf{x}_B)f_{k,B} = -f_{k,A}V_{k,B}(\mathbf{x}_A) + q_A P_B(\mathbf{x}_A). \tag{8}$$

For the special case that $q_A=q_B$ and $f_{k,A}=f_{k,B}=0$ this reduces to

$$P_A(\mathbf{x}_B)=P_B(\mathbf{x}_A). \tag{9}$$

C. Reciprocity theorem of the correlation type

We consider the interaction quantity $\partial_k\{P_A^* V_{k,B} + V_{k,A}^* P_B\}$, where * denotes complex conjugation. Following the same procedure as in the previous subsection, we obtain

$$\begin{aligned}\int_{\partial\mathscr{V}}&\{P_A^* V_{k,B} + V_{k,A}^* P_B\} n_k d^2\mathbf{x}\\ &= -j\omega\int_{\mathscr{V}}\{P_A^*(\kappa_B-\kappa_A)P_B + V_{k,A}^*(\varrho_B-\varrho_A)V_{k,B}\}d^3\mathbf{x}\\ &\quad+\int_{\mathscr{V}}\{P_A^* Q_B + V_{k,A}^* F_{k,B} + F_{k,A}^* V_{k,B} + Q_A^* P_B\}d^3\mathbf{x}.\end{aligned} \tag{10}$$

Equation (10) is the reciprocity theorem of the correlation type. We speak of ''correlation type'' since the products in the frequency domain ($P_A^* V_{k,B}$, etc.) correspond to correlations in the time domain.

Power conservation: When states A and B are identical, Eq. (10) yields (omitting the subscripts A and B)

$$\begin{aligned}\int_{\partial\mathscr{V}}&\{P^* V_k + V_k^* P\} n_k\, d^2\mathbf{x}\\ &= \int_{\mathscr{V}}\{P^* Q + V_k^* F_k + F_k^* V_k + Q^* P\}d^3\mathbf{x}.\end{aligned} \tag{11}$$

Equation (11) formulates the conservation of acoustic power. For this reason, Eq. (10) is often referred to as the power reciprocity theorem.

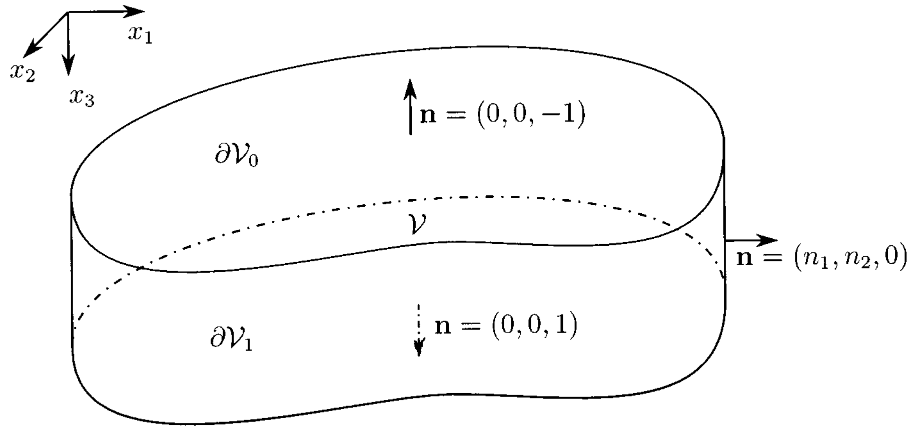

FIG. 2. Modified configuration for reciprocity theorems for situations with a direction of preference.

D. Modified configuration

In the remainder of this paper we consider situations in which the direction of preference is taken along the x_3 axis. A natural choice for the integration volume $\mathscr{V}$ is shown in Fig. 2. Its boundary $\partial\mathscr{V}$ consists of two planar surfaces perpendicular to the x_3 axis and a cylindrical surface with its axis parallel to the x_3 axis. When convenient, we will rewrite the volume integrals as

$$\int_{\mathscr{V}}\{\cdot\}d^3\mathbf{x}=\int_a^b dx_3\int_{\mathscr{D}}\{\cdot\}d^2\mathbf{x}_L\,, \tag{12}$$

where a and b are the x_3 coordinates of the planar surfaces, $\mathscr{D}$ is a cross section of $\mathscr{V}$ normal to the x_3 axis, and $\mathbf{x}_L$ denotes the lateral coordinates, according to $\mathbf{x}_L=(x_1,x_2)$. The combination of the two planar surfaces will be denoted by $\partial\mathscr{V}_0$; the cylindrical surface by $\partial\mathscr{V}_1$. The outward-pointing normal vector on $\partial\mathscr{V}_0$ reads $\mathbf{n}=(0,0,n_3)$, with $n_3=-1$ and $n_3=+1$, respectively; on $\partial\mathscr{V}_1$ it reads $\mathbf{n}=(n_1,n_2,0)$.

For underwater acoustics the x_3 axis is horizontal and boundary conditions apply to (a part of) $\partial\mathscr{V}_1$. For seismic situations the x_3 axis is pointing downward and the radius of $\partial\mathscr{V}_1$ is infinite; boundary conditions may apply to (a part of) $\partial\mathscr{V}_0$. Both finite and infinite, surfaces $\partial\mathscr{V}_1$ will be considered.

II. TWO-WAY WAVE EQUATION IN MATRIX-VECTOR FORM

Given the fact that the direction of preference is taken along the x_3 axis, it is useful to reorganize Eqs. (1) and (2) such that the axial variations of the wave field are expressed in terms of the lateral variations of the same wave field. To this end we separate the axial derivatives $\partial_3 P$ and $\partial_3 V_3$ from the lateral derivatives and we eliminate V_1 and V_2. The resulting two equations for P and V_3 are combined into one matrix-vector equation, according to

$$\partial_3\mathbf{Q}=\hat{\mathbf{A}}\mathbf{Q}+\mathbf{D}. \tag{13}$$

The circumflex denotes and *operator* containing the lateral differentiation operators ∂_1 and ∂_2. We refer to Eq. (13) as the two-way wave equation. The two-way wave vector $\mathbf{Q}$ and the two-way source vector $\mathbf{D}$ are defined as

$$\mathbf{Q}=\begin{pmatrix} P \\ V_3 \end{pmatrix},\quad \mathbf{D}=\begin{pmatrix} F_3 \\ Q-\dfrac{1}{j\omega}\,\partial_\alpha\left(\dfrac{1}{\varrho}\,F_\alpha\right) \end{pmatrix}. \tag{14}$$

Greek subscripts take on the values 1 and 2. The two-way operator matrix $\hat{\mathbf{A}}$ is defined as

$$\hat{\mathbf{A}}=\begin{pmatrix} 0 & -j\omega\varrho \\ -j\omega\hat{\mathscr{H}} & 0 \end{pmatrix}, \tag{15}$$

where

$$\hat{\mathscr{H}}=\kappa+\frac{1}{\omega^2}\,\partial_\alpha\left(\frac{1}{\varrho}\,\partial_\alpha\cdot\right). \tag{16}$$

III. ADJOINT TWO-WAY OPERATOR MATRIX

In the derivation of the reciprocity theorems for the two-way wave vector $\mathbf{Q}$ we will make use of the adjoint of the two-way operator matrix $\hat{\mathbf{A}}$. In this section we first introduce the adjoint of operator $\hat{\mathscr{H}}$ and, subsequently, we use this result to find the adjoint two-way operator matrix.

We define the inner product for two-dimensional square-integrable complex-valued functions as follows:

$$\langle f,g\rangle=\int_{\mathscr{D}}f^*(\mathbf{x}_L)g(\mathbf{x}_L)d^2\mathbf{x}_L\,, \tag{17}$$

where the domain $\mathscr{D}$ is a subset of $\mathbb{R}^2$, according to $\mathscr{D}\subseteq\mathbb{R}^2$.

We introduce the adjoint operator $\hat{\mathscr{H}}^\dagger$ via

$$\langle f,\hat{\mathscr{H}}g\rangle=\langle\hat{\mathscr{H}}^\dagger f,g\rangle. \tag{18}$$

First we consider the situation that $\mathscr{D}$ is bounded. From Eqs. (16) and (17) and the theorem of Gauss we thus find for the left-hand side of Eq. (18)

$$\langle f,\hat{\mathscr{H}}g\rangle=\int_{\mathscr{D}}\left(\kappa f^*g-\frac{1}{\varrho\,\omega^2}\,(\partial_\alpha f^*)(\partial_\alpha g)\right)d^2\mathbf{x}_L + \frac{1}{\omega^2}\int_{\partial\mathscr{D}}\frac{1}{\rho}\,(f^*\partial_\alpha g)n_\alpha\,d\mathbf{x}_L\,, \tag{19}$$

or

$$\langle f,\hat{\mathscr{H}}g\rangle=\langle\hat{\mathscr{H}}^*f,g\rangle+\frac{1}{\omega^2}\int_{\partial\mathscr{D}}\frac{1}{\rho}\,(f^*\partial_\alpha g-g\partial_\alpha f^*)n_\alpha\;d\mathbf{x}_L\,, \tag{20}$$

where $\partial\mathscr{D}$ is the boundary of $\mathscr{D}$, with outward-pointing normal vector $\mathbf{n}_L=(n_1,n_2)$. Hence, when f and g satisfy homogeneous Dirichlet or Neumann boundary conditions, according to

$$f=g=0\quad\text{or}\quad n_\alpha\partial_\alpha f=n_\alpha\partial_\alpha g=0\quad\text{on}\quad\partial\mathscr{D}, \tag{21}$$

we find from Eqs. (18) and (20)

$$\hat{\mathscr{H}}^\dagger=\hat{\mathscr{H}}^*, \tag{22}$$

or, since we consider a lossless medium (i.e., since κ and ϱ are real functions),

$$\hat{\mathscr{H}}^\dagger=\hat{\mathscr{H}}. \tag{23}$$

The latter result implies that $\hat{\mathscr{H}}$ is *self-adjoint* when it is defined on the space of functions that satisfy homogeneous Dirichlet or Neumann boundary conditions on $\partial\mathscr{D}$.

When $\mathscr{D}$ is unbounded, $\hat{\mathscr{H}}$ is again self-adjoint when it is defined on a space of functions with ''sufficient decay'' at infinity (i.e., a properly chosen Sobolev space).

Analogous to Eq. (17), we define the inner product for *vector* functions as follows:

$$\langle\mathbf{f},\mathbf{g}\rangle=\int_{\mathscr{D}}\mathbf{f}^H(\mathbf{x}_L)\mathbf{g}(\mathbf{x}_L)d^2\mathbf{x}_L\,, \tag{24}$$

where ^{H} denotes complex conjugation and transposition. Analogous to Eq. (18), we introduce the adjoint of the two-way operator matrix $\hat{\mathbf{A}}$, as defined in Eq. (15), via

$$\langle\mathbf{f},\hat{\mathbf{A}}\mathbf{g}\rangle=\langle\hat{\mathbf{A}}^\dagger\mathbf{f},\mathbf{g}\rangle. \tag{25}$$

From the results above, we thus find that the adjoint two-way operator matrix reads

$$\hat{\mathbf{A}}^\dagger=\hat{\mathbf{A}}^H, \tag{26}$$

assuming that $\hat{\mathbf{A}}$ is defined on a space of vector functions that satisfy homogeneous Dirichlet or Neumann boundary conditions on $\partial\mathscr{D}$ when $\mathscr{D}$ is bounded, or that have sufficient decay at infinity when $\mathscr{D}$ is unbounded.

IV. RECIPROCITY THEOREMS FOR THE TWO-WAY WAVE VECTOR

A. Two-way reciprocity theorem of the convolution type

We derive the reciprocity theorem of the convolution type for the two-way wave vector $\mathbf{Q}$. The two different states will be distinguished by the subscripts A and B. We consider the interaction between the acoustic pressure in one state and the *axial* component of the particle velocity in the other state and vice versa. To be more specific, we consider the interaction quantity

$$\partial_3\{P_AV_{3,B}-V_{3,A}P_B\}. \tag{27}$$

To simplify the notation, we rewrite this interaction quantity as

$$\partial_3\{\mathbf{Q}_A^T\mathbf{N}\mathbf{Q}_B\}, \tag{28}$$

where ^{T} denotes transposition and

$$\mathbf{N}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{29}$$

Applying the product rule for differentiation, substituting the two-way wave equation (13) for states A and B, integrating the result over the volume $\mathscr{V}$ with boundary $\partial\mathscr{V}_0\cup\partial\mathscr{V}_1$, as introduced in Fig. 2, and applying the theorem of Gauss yields the following two-way reciprocity theorem of the convolution type

$$\begin{aligned}\int_{\partial\mathscr{V}_0}\mathbf{Q}_A^T\mathbf{N}\mathbf{Q}_Bn_3\;d^2\mathbf{x}_L &=\int_{\mathscr{V}}\{\mathbf{Q}_A^T\mathbf{N}\hat{\mathbf{A}}_B\mathbf{Q}_B+(\hat{\mathbf{A}}_A\mathbf{Q}_A)^T\mathbf{N}\mathbf{Q}_B\}d^3\mathbf{x}\\ &\quad+\int_{\mathscr{V}}\{\mathbf{Q}_A^T\mathbf{N}\mathbf{D}_B+\mathbf{D}_A^T\mathbf{N}\mathbf{Q}_B\}d^3\mathbf{x}.\end{aligned} \tag{30}$$

Unlike in Eq. (5), it is not directly clear that the first volume integral on the right-hand side of Eq. (30) represents a ''contrast term'' that vanishes when the medium parameters in both states are identical. On account of equations (12) and (24)–(26), however, we may write for the term containing the transposed operator matrix

$$\int_a^b dx_3\int_{\mathscr{D}}(\hat{\mathbf{A}}_A\mathbf{Q}_A)^T\mathbf{N}\mathbf{Q}_Bd^2\mathbf{x}_L=\int_a^b dx_3\int_{\mathscr{D}}\mathbf{Q}_A^T(\hat{\mathbf{A}}_A^T\mathbf{N}\mathbf{Q}_B)d^2\mathbf{x}_L\,, \tag{31}$$

assuming that $\mathbf{Q}_A$ and $\mathbf{Q}_B$ satisfy homogeneous Dirichlet or Neumann boundary conditions on $\partial\mathscr{D}$ when $\mathscr{D}$ is bounded or that they have sufficient decay at infinity when $\mathscr{D}$ is unbounded. Note that $\partial\mathscr{D}$ is a cross section of $\partial\mathscr{V}_1$ for any x_3 between a and b. Hence, the above-mentioned homogeneous Dirichlet or Neumann boundary conditions are fulfilled when $\partial\mathscr{V}_1$ is a free boundary or a rigid boundary (bear in mind that V_3 is the *tangential* velocity component at the cylindrical surface).

On account of Eq. (15) we have

$$\hat{\mathbf{A}}^T\mathbf{N}=-\mathbf{N}\hat{\mathbf{A}}. \tag{32}$$

Using (31) as well as (32) in Eq. (30) gives

$$\int_{\partial\mathscr{V}_0}\mathbf{Q}_A^T\mathbf{N}\mathbf{Q}_Bn_3\;d^2\mathbf{x}_L=\int_{\mathscr{V}}\mathbf{Q}_A^T\mathbf{N}\hat{\boldsymbol{\Delta}}\mathbf{Q}_B\;d^3\mathbf{x}+\int_{\mathscr{V}}\{\mathbf{Q}_A^T\mathbf{N}\mathbf{D}_B+\mathbf{D}_A^T\mathbf{N}\mathbf{Q}_B\}d^3\mathbf{x}, \tag{33}$$

where the contrast operator $\hat{\boldsymbol{\Delta}}$ is defined by

$$\hat{\boldsymbol{\Delta}}=\hat{\mathbf{A}}_B-\hat{\mathbf{A}}_A\,. \tag{34}$$

This vector form of the two-way reciprocity theorem of the convolution type will be compared later on with its one-way counterpart. To conclude this subsection, we rewrite this theorem in scalar form by substituting Eqs. (14), (15), (29), and (34). We thus obtain

$$\int_{\partial\mathscr{V}_0}\{P_A V_{3,B}-V_{3,A}P_B\}n_3\;d^2\mathbf{x}_L$$
$$=-j\omega\int_{\mathscr{V}}[P_A(\hat{\mathscr{K}}_B-\hat{\mathscr{K}}_A)P_B-V_{3,A}(\varrho_B-\varrho_A)V_{3,B}]d^3\mathbf{x}$$
$$+\int_{\mathscr{V}}\left\{P_A\left[Q_B-\frac{1}{j\omega}\,\partial_\alpha\left(\frac{1}{\varrho_B}\,F_{\alpha,B}\right)\right]-V_{3,A}F_{3,B}\right.$$
$$\left.+F_{3,A}V_{3,B}-\left[Q_A-\frac{1}{j\omega}\,\partial_\alpha\left(\frac{1}{\varrho_A}\,F_{\alpha,A}\right)\right]P_B\right\}d^3\mathbf{x}. \tag{35}$$

Note that this reciprocity theorem has a similar form as Eq. (5). The main difference is that the contrast function $(\kappa_B-\kappa_A)$ has been replaced by a contrast operator $(\hat{\mathscr{K}}_B-\hat{\mathscr{K}}_A)$ and that the lateral velocity components have been eliminated. In Appendix B it is shown that the axial velocity component V_3 can be eliminated as well by means of a ''Dirichlet-to-Neumann operator.''

B. Two-way reciprocity theorem of the correlation type

We derive the reciprocity theorem of the correlation type for the two-way wave vector $\mathbf{Q}$. This time we consider the interaction quantity

$$\partial_3\{P_A^* V_{3,B}+V_{3,A}^* P_B\}. \tag{36}$$

To simplify the notation, we rewrite this interaction quantity as

$$\partial_3\{\mathbf{Q}_A^H\mathbf{K}\mathbf{Q}_B\}, \tag{37}$$

where

$$\mathbf{K}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{38}$$

Following a similar procedure as in the previous subsection, using the integral property

$$\int_a^b dx_3\int_{\mathscr{D}}(\hat{\mathbf{A}}_A\mathbf{Q}_A)^H\mathbf{K}\mathbf{Q}_B\;d^2\mathbf{x}_L$$
$$=\int_a^b dx_3\int_{\mathscr{D}}\mathbf{Q}_A^H(\hat{\mathbf{A}}_A^H\mathbf{K}\mathbf{Q}_B)d^2\mathbf{x}_L \tag{39}$$

and the symmetry relation

$$\hat{\mathbf{A}}^H\mathbf{K}=-\mathbf{K}\hat{\mathbf{A}}, \tag{40}$$

yields

$$\int_{\partial\mathscr{V}_0}\mathbf{Q}_A^H\mathbf{K}\mathbf{Q}_B n_3\;d^2\mathbf{x}_L$$
$$=\int_{\mathscr{V}}\mathbf{Q}_A^H\mathbf{K}\hat{\mathbf{\Delta}}\mathbf{Q}_B\;d^3\mathbf{x}+\int_{\mathscr{V}}\{\mathbf{Q}_A^H\mathbf{K}\mathbf{D}_B+\mathbf{D}_A^H\mathbf{K}\mathbf{Q}_B\}d^3\mathbf{x}, \tag{41}$$

where the contrast operator $\hat{\mathbf{\Delta}}$ is again given by Eq. (34). This vector form of the two-way reciprocity theorem of the correlation type will be compared later on with its one-way counterpart.

V. ONE-WAY WAVE EQUATION IN MATRIX-VECTOR FORM

In this section we decompose the acoustic two-way wave equation into a system of coupled equations for the one-way wave fields P^+ and P^-, propagating in the positive and negative axial direction, respectively. This decomposition is not uniquely defined (see Brekhovskikh, 1960, or Corones *et al.*, 1983). Here we follow an approach analogous to the decomposition approach in laterally invariant media (see Ursin, 1983, for an overview). We introduce a *one-way wave vector* $\mathbf{P}$ and a *one-way source vector* $\mathbf{S}$, according to

$$\mathbf{P}=\begin{pmatrix}P^+\\ P^-\end{pmatrix}\quad\text{and}\quad\mathbf{S}=\begin{pmatrix}S^+\\ S^-\end{pmatrix}. \tag{42}$$

In an axially invariant medium (i.e., a medium in which the medium parameters do not depend on the axial coordinate x_3), P^+ and P^- propagate independently, hence, for this situation $\mathbf{P}$ satisfies an equation of the same form as Eq. (13), with the antidiagonal operator matrix $\hat{\mathbf{A}}$ replaced by a diagonal operator matrix $-j\omega\hat{\mathbf{\Lambda}}$, where $\hat{\mathbf{\Lambda}}$ will be called the *axial slowness* operator matrix. Our aim is to find an equation for $\mathbf{P}$ of the form of Eq. (13) for an arbitrary inhomogeneous medium in such a way that the operator matrix reduces again to $-j\omega\hat{\mathbf{\Lambda}}$ when the axial variations of the medium parameters vanish.

We introduce operator matrices $\hat{\mathbf{\Lambda}}$, $\hat{\mathbf{L}}$, and $\hat{\mathbf{L}}^{-1}$ that satisfy the relation

$$\hat{\mathbf{A}}=-j\omega\hat{\mathbf{L}}\hat{\mathbf{\Lambda}}\hat{\mathbf{L}}^{-1}, \tag{43}$$

in such a way that $\hat{\mathbf{\Lambda}}$ is diagonal. For an extensive list of references on the theoretical and numerical aspects of this decomposition, see Fishman *et al.* (1987). Some relevant recent references are Wapenaar and Berkhout (1989), de Hoop (1992, 1996), Corones *et al.* (1992), and Fishman (1993). Upon substitution of Eq. (43), together with

$$\mathbf{Q}=\hat{\mathbf{L}}\mathbf{P}\quad\text{and}\quad\mathbf{D}=\hat{\mathbf{L}}\mathbf{S} \tag{44}$$

into Eq. (13), we obtain after some straightforward manipulations the following system of coupled one-way wave equations:

$$\partial_3\mathbf{P}=\hat{\mathbf{B}}\mathbf{P}+\mathbf{S}, \tag{45}$$

where the one-way operator matrix $\hat{\mathbf{B}}$ is defined as

$$\hat{\mathbf{B}}=-j\omega\hat{\mathbf{\Lambda}}+\hat{\mathbf{\Theta}}, \tag{46}$$

with the coupling operator matrix $\hat{\mathbf{\Theta}}$ defined as

$$\hat{\mathbf{\Theta}}=-\hat{\mathbf{L}}^{-1}\,\partial_3\mathbf{L}. \tag{47}$$

In the following we refer to Eq. (45) simply as the one-way wave equation. Note that in an axially invariant medium the one-way operator matrix $\hat{\mathbf{B}}$ indeed reduces to $-j\omega\hat{\mathbf{\Lambda}}$, yielding decoupled equations $\partial_3 P^{\pm}=\mp j\omega\hat{\Lambda}P^{\pm}$, with $\hat{\Lambda}$ defined below. (During the revision of this paper Dr. Fishman pointed out that an exact equation of the same form can be constructed for the total wavefield in inhomogeneous media, i.e., $\partial_3 P=\hat{\mathscr{Q}}P$, where $\hat{\mathscr{Q}}$ is called the ''Dirichlet-to-Neumann operator.'' This is briefly discussed in Appendix B.)

Next, we analyze the decomposition introduced in Eq. (43). For convenience we recast $\hat{\mathbf{A}}$, as defined in Eq. (15), into a slightly different form, according to

$$\hat{\mathbf{A}}=\begin{pmatrix} 0 & -j\omega\varrho \\ \frac{1}{j\omega\varrho^{1/2}}(\hat{\mathscr{H}}_2\varrho^{-1/2}\cdot) & 0 \end{pmatrix}, \tag{48}$$

where

$$\hat{\mathscr{H}}_2=\omega^2\varrho^{1/2}(\hat{\mathscr{K}}\varrho^{1/2}\cdot), \tag{49}$$

or, upon substitution of Eq. (16),

$$\hat{\mathscr{H}}_2=\left(\frac{\omega}{c'}\right)^2+\partial_\alpha\partial_\alpha, \tag{50}$$

with

$$\left(\frac{\omega}{c'}\right)^2=\left(\frac{\omega}{c}\right)^2-\frac{3(\partial_\alpha\varrho)(\partial_\alpha\varrho)}{4\varrho^2}+\frac{(\partial_\alpha\partial_\alpha\varrho)}{2\varrho}, \tag{51}$$

and $c=(\kappa\varrho)^{-1/2}$ (Wapenaar and Berkhout, 1989, Appendix B; de Hoop, 1992). Note that $\hat{\mathscr{H}}_2$, as defined in Eq. (50), represents the *Helmholtz operator*, with c' being a modified propagation velocity, according to Eq. (51).

Due to the antidiagonal structure of $\hat{\mathbf{A}}$, as defined in Eq. (48), the operator matrices $\hat{\mathbf{\Lambda}}$, $\hat{\mathbf{L}}$, and $\hat{\mathbf{L}}^{-1}$ have the following structure:

$$\hat{\mathbf{\Lambda}}=\begin{pmatrix} \hat{\Lambda} & 0 \\ 0 & -\hat{\Lambda} \end{pmatrix} \tag{52}$$

and

$$\hat{\mathbf{L}}=\begin{pmatrix} \hat{L}_1 & \hat{L}_1 \\ \hat{L}_2 & -\hat{L}_2 \end{pmatrix}, \quad \hat{\mathbf{L}}^{-1}=\frac{1}{2}\begin{pmatrix} \hat{L}_1^{-1} & \hat{L}_2^{-1} \\ \hat{L}_1^{-1} & -\hat{L}_2^{-1} \end{pmatrix}, \tag{53}$$

where the scalar operators $\hat{\Lambda}$, $\hat{L}_1$, and $\hat{L}_2$ satisfy the following relations:

$$-j\omega\varrho=-j\omega\hat{L}_1\hat{\Lambda}\hat{L}_2^{-1} \tag{54}$$

and

$$\frac{1}{j\omega\varrho^{1/2}}(\hat{\mathscr{H}}_2\varrho^{-1/2}\cdot)=-j\omega\hat{L}_2\hat{\Lambda}\hat{L}_1^{-1}. \tag{55}$$

It can now be verified by substitution that for $\hat{\Lambda}$, $\hat{L}_1$, $\hat{L}_1^{-1}$, $\hat{L}_2$, and $\hat{L}_2^{-1}$ we may write

$$\hat{\Lambda}=\omega^{-1}\hat{\mathscr{H}}_1, \tag{56}$$

$$\hat{L}_1=\left(\frac{\omega\varrho}{2}\right)^{1/2}\hat{\mathscr{H}}_1^{-1/2}, \quad \hat{L}_1^{-1}=\left(\frac{\omega}{2}\right)^{-1/2}(\hat{\mathscr{H}}_1^{1/2}\varrho^{-1/2}\cdot), \tag{57}$$

$$\hat{L}_2=(2\omega\varrho)^{-1/2}\hat{\mathscr{H}}_1^{1/2}, \quad \hat{L}_2^{-1}=(2\omega)^{1/2}(\hat{\mathscr{H}}_1^{-1/2}\varrho^{1/2}\cdot), \tag{58}$$

where the *square-root operator* $\hat{\mathscr{H}}_1$ is related to the Helmholtz operator $\hat{\mathscr{H}}_2$ according to

$$\hat{\mathscr{H}}_2=\hat{\mathscr{H}}_1\hat{\mathscr{H}}_1. \tag{59}$$

Unlike operator $\hat{\mathscr{H}}_2$, the operators $\hat{\mathscr{H}}_1$, $\hat{\Lambda}$, $\hat{L}_1$, and $\hat{L}_2$ cannot be written as polynomials in ∂_α. Therefore these operators are so-called *pseudo-differential* operators (Kumano-go, 1974; Fishman, 1992). The reader who is interested in exact and uniform approximate constructions of these operators as well as a number of relevant estimates is referred to Fishman (1992), Fishman *et al.* (1996), and de Hoop (1992, 1996).

VI. ADJOINT ONE-WAY OPERATOR MATRIX

In the derivation of the reciprocity theorems for the one-way wave vector **P** we will make use of the adjoint of the one-way operator matrix $\hat{\mathbf{B}}$. In this section we use the adjoint of the square-root operator $\hat{\mathscr{H}}_1$ to find the adjoint one-way operator matrix. We will assume that $\hat{\mathscr{H}}_1$ and $\hat{\mathbf{B}}$ are defined on the same space as $\hat{\mathscr{H}}_2$ and $\hat{\mathbf{A}}$, respectively.

In Appendix A we derive

$$\hat{\mathscr{H}}_1^\dagger=\hat{\mathscr{H}}_1^*. \tag{60}$$

In a similar way, the following relations can be found for $\hat{L}_1^\dagger$ and $\hat{L}_2^\dagger$:

$$\hat{L}_1^\dagger=\tfrac{1}{2}(\hat{L}_2^{-1})^* \tag{61}$$

and

$$\hat{L}_2^\dagger=\tfrac{1}{2}(\hat{L}_1^{-1})^*. \tag{62}$$

Note that, unlike the Helmholtz operator $\hat{\mathscr{H}}_2$, the square-root operator $\hat{\mathscr{H}}_1$ is not self-adjoint. However, in this and the next section we will show that the properties (60)–(62) are sufficient for deriving reciprocity theorems for the one-way wave vector.

Using Eqs. (52), (56), and (60) we obtain

$$\hat{\mathbf{\Lambda}}^\dagger=\hat{\mathbf{\Lambda}}^H. \tag{63}$$

Hence, for a medium in which the parameters do not depend on the axial coordinate, we find from Eqs. (46) and (63)

$$\hat{\mathbf{B}}^\dagger=\hat{\mathbf{B}}^H. \tag{64}$$

Based on the analogy with Eq. (26) (i.e., $\hat{\mathbf{A}}^\dagger=\hat{\mathbf{A}}^H$) we may expect that for this situation the one-way reciprocity theorems are easily found. For arbitrarily inhomogeneous media, however, the situation is more complicated, since $\hat{\mathbf{\Theta}}^\dagger\neq\hat{\mathbf{\Theta}}^H$ and thus $\hat{\mathbf{B}}^\dagger\neq\hat{\mathbf{B}}^H$.

A careful reexamination of the derivation of the two-way reciprocity theorem of the convolution-type (33) shows that this theorem does not depend on the symmetry relations (26) and (32) separately, but only on their combination, i.e., on

$$\hat{\mathbf{A}}^\dagger=-\mathbf{N}\hat{\mathbf{A}}^*\mathbf{N}^{-1}. \tag{65}$$

We will now show that we can find a similar relation for the one-way operator matrix $\hat{\mathbf{B}}$ for arbitrarily inhomogeneous media. From the structure of $\hat{\mathbf{\Lambda}}$, $\hat{\mathbf{L}}$, and $\hat{\mathbf{L}}^{-1}$, as defined in Eqs. (52) and (53), as well as from Eqs. (60)–(62) we find

$$\hat{\mathbf{\Lambda}}^\dagger=-\mathbf{N}\hat{\mathbf{\Lambda}}^*\mathbf{N}^{-1} \tag{66}$$

and

$$\hat{\mathbf{L}}^\dagger=-\mathbf{N}(\hat{\mathbf{L}}^{-1})^*\mathbf{N}^{-1}, \tag{67}$$

or, equivalently,

$$(\hat{\mathbf{L}}^{-1})^{\dagger}=-\mathbf{N}\hat{\mathbf{L}}^{*}\mathbf{N}^{-1}, \tag{68}$$

with $\mathbf{N}$ defined in Eq. (29). For the coupling operator matrix $\hat{\boldsymbol{\Theta}}$, defined in Eq. (47), we thus find

$$\hat{\boldsymbol{\Theta}}^{\dagger}=-(\partial_3\hat{\mathbf{L}})^{\dagger}(\hat{\mathbf{L}}^{-1})^{\dagger}=-\mathbf{N}(\partial_3\hat{\mathbf{L}}^{-1})^{*}\hat{\mathbf{L}}^{*}\mathbf{N}^{-1}. \tag{69}$$

Using the property

$$\mathbf{O}=\partial_3(\hat{\mathbf{L}}^{-1}\hat{\mathbf{L}})=(\partial_3\hat{\mathbf{L}}^{-1})\hat{\mathbf{L}}+\hat{\mathbf{L}}^{-1}\partial_3\hat{\mathbf{L}}, \tag{70}$$

where $\mathbf{O}$ is the null matrix, this gives

$$\hat{\boldsymbol{\Theta}}^{\dagger}=\mathbf{N}(\hat{\mathbf{L}}^{-1}\partial_3\hat{\mathbf{L}})^{*}\mathbf{N}^{-1}=-\mathbf{N}\hat{\boldsymbol{\Theta}}^{*}\mathbf{N}^{-1}. \tag{71}$$

From Eqs. (66) and (71) we thus find for the one-way operator matrix $\hat{\mathbf{B}}$, as defined in Eq. (46),

$$\hat{\mathbf{B}}^{\dagger}=-\mathbf{N}\hat{\mathbf{B}}^{*}\mathbf{N}^{-1}. \tag{72}$$

Note that this relation is indeed analogous to the symmetry property (65) of the two-way operator matrix $\hat{\mathbf{A}}$. In the next section we will use Eq. (72) in the derivation of the one-way reciprocity theorem of the convolution type.

A reexamination of the derivation of the two-way reciprocity theorem of the correlation type (41) shows that this theorem depends on the combination of the symmetry relations (26) and (40), i.e., on

$$\hat{\mathbf{A}}^{\dagger}=-\mathbf{K}\hat{\mathbf{A}}\mathbf{K}^{-1}. \tag{73}$$

A similar relation for $\hat{\mathbf{B}}$ cannot be given, even for the situation in which the medium parameters do not depend on the axial coordinate. This implies that we cannot find an exact one-way reciprocity theorem of the correlation type. As an alternative, consider the following approximate symmetry relations:

$$\hat{\mathscr{H}}_1^{\dagger}\approx\hat{\mathscr{H}}_1, \tag{74}$$

$$\hat{L}_1^{\dagger}\approx\tfrac{1}{2}(\hat{L}_2^{-1}), \tag{75}$$

and

$$\hat{L}_2^{\dagger}\approx\tfrac{1}{2}(\hat{L}_1^{-1}). \tag{76}$$

The approximation signs denote that evanescent waves are ignored (see Appendix A). Using these relations we find in a similar way as above the following approximate symmetry relations;

$$\hat{\boldsymbol{\Lambda}}^{\dagger}\approx\mathbf{J}\hat{\boldsymbol{\Lambda}}\mathbf{J}^{-1}, \tag{77}$$

$$\hat{\mathbf{L}}^{\dagger}\approx\mathbf{J}\hat{\mathbf{L}}^{-1}\mathbf{K}^{-1}, \tag{78}$$

$$\hat{\boldsymbol{\Theta}}^{\dagger}\approx-\mathbf{J}\hat{\boldsymbol{\Theta}}\mathbf{J}^{-1}, \tag{79}$$

and

$$\hat{\mathbf{B}}^{\dagger}\approx-\mathbf{J}\hat{\mathbf{B}}\mathbf{J}^{-1}, \tag{80}$$

where $\mathbf{K}$ is defined in Eq. (38) and

$$\mathbf{J}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{81}$$

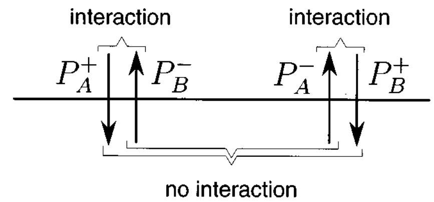

FIG. 3. Both terms of the interaction quantity for the one-way reciprocity theorem of the convolution type contain waves that propagate in opposite directions.

In the next section we will use Eq. (80) in the derivation of an approximate one-way reciprocity theorem of the correlation type.

VII. RECIPROCITY THEOREMS FOR THE ONE-WAY WAVE VECTOR

A. One-way reciprocity theorem of the convolution type

We derive the reciprocity theorem of the convolution type for the one-way wave vector $\mathbf{P}$. The two different states will be distinguished by the subscripts A and B. Looking at the analogous symmetry properties of the two-way and one-way operator matrices (i.e., $\hat{\mathbf{A}}^{\dagger}=-\mathbf{N}\hat{\mathbf{A}}^{*}\mathbf{N}^{-1}$ vs $\hat{\mathbf{B}}^{\dagger}=-\mathbf{N}\hat{\mathbf{B}}^{*}\mathbf{N}^{-1}$), an obvious choice for the interaction quantity is, analogous to Eq. (28),

$$\partial_3\{\mathbf{P}_A^T\mathbf{N}\mathbf{P}_B\} \tag{82}$$

or

$$\partial_3\{P_A^+P_B^- - P_A^-P_B^+\}. \tag{83}$$

Apparently, we consider the interaction between oppositely propagating waves (see Fig. 3). Applying the product rule for differentiation, substituting the one-way wave equation (45) for states A and B, integrating the result over the volume $\mathscr{V}$ with boundary $\partial\mathscr{V}_0\cup\partial\mathscr{V}_1$, as introduced in Fig. 2, applying the theorem of Gauss, and using symmetry relation (72) yields the following one-way reciprocity theorem of the convolution type:

$$\int_{\partial\mathscr{V}_0}\mathbf{P}_A^T\mathbf{N}\mathbf{P}_B n_3\, d^2\mathbf{x}_L$$

$$=\int_{\mathscr{V}}\mathbf{P}_A^T\mathbf{N}\hat{\boldsymbol{\Delta}}\mathbf{P}_B d^3\mathbf{x}+\int_{\mathscr{V}}\{\mathbf{P}_A^T\mathbf{N}\mathbf{S}_B+\mathbf{S}_A^T\mathbf{N}\mathbf{P}_B\}d^3\mathbf{x}, \tag{84}$$

where the contrast operator $\hat{\boldsymbol{\Delta}}$ is given by

$$\hat{\boldsymbol{\Delta}}=\hat{\mathbf{B}}_B-\hat{\mathbf{B}}_A. \tag{85}$$

Since we used symmetry relation (72), it is implicitly assumed that $\mathbf{P}_A$ and $\mathbf{P}_B$ satisfy homogeneous Dirichlet or Neumann boundary conditions on $\partial\mathscr{V}_1$ when $\mathscr{V}$ is bounded or that they have sufficient decay at infinity when the radius of $\partial\mathscr{V}_1$ is infinite. These conditions are fulfilled for $\mathbf{P}_A$ and $\mathbf{P}_B$ when they are fulfilled for $\mathbf{Q}_A$ and $\mathbf{Q}_B$.

Several linear and nonlinear representations can be derived from the one-way reciprocity theorem (84) by choosing for states A and B a Green's state and the actual state, respectively. In particular a 3-D ''generalized primary repre-

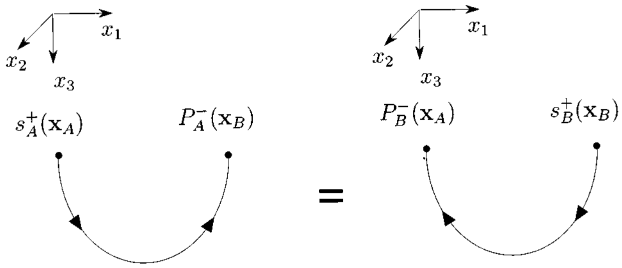

FIG. 4. Field reciprocity for one-way sources and receivers.

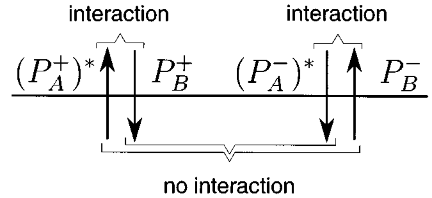

FIG. 5. Both terms of the interaction quantity for the one-way reciprocity theorem of the correlation type contain waves that propagate in opposite directions.

sentation'' can be derived which forms the point of departure for the derivation of (seismic) imaging techniques for finely layered media (Wapenaar, 1996).

We conclude this subsection by analyzing reciprocity theorem (84) for some special cases.

Unbounded media: Consider the situation in which the medium at and outside $\partial\mathscr{V}_0$ is unbounded, homogeneous, and source-free in both states. Assume that the wave fields in both states are causally related to the sources in $\mathscr{V}$. Then in both states the wave fields are outgoing at $\partial\mathscr{V}_0$ (i.e., $P_A^+=P_B^+=0$ for $x_3=a$ and $P_A^-=P_B^-=0$ for $x_3=b$) and it is easily seen that $\mathbf{P}_A^T\mathbf{N}\mathbf{P}_B=P_A^+P_B^- - P_A^-P_B^+=0$ at $\partial\mathscr{V}_0$, so the boundary integral on the left-hand side of Eq. (84) vanishes. Apparently it is *not* required that the medium parameters at and outside $\partial\mathscr{V}_0$ are identical in both states, unlike the conditions for the vanishing boundary integral in the two-way reciprocity theorem. We come back to this later on.

Field reciprocity: Assume that the above-mentioned conditions are fulfilled and that $\varrho_A=\varrho_B$ and $\kappa_A=\kappa_B$ inside as well as outside $\mathscr{V}$. Then the first volume integral on the right-hand side of Eq. (84) vanishes. Furthermore, consider point sources in states A and B at $\mathbf{x}_A\in\mathscr{V}$ and $\mathbf{x}_B\in\mathscr{V}$, respectively, according to

$$\mathbf{S}_A(\mathbf{x})=\mathbf{s}_A\delta(\mathbf{x}-\mathbf{x}_A)=\begin{pmatrix} s_A^+ \\ s_A^- \end{pmatrix}\delta(\mathbf{x}-\mathbf{x}_A) \tag{86}$$

and

$$\mathbf{S}_B(\mathbf{x})=\mathbf{s}_B\delta(\mathbf{x}-\mathbf{x}_B)=\begin{pmatrix} s_B^+ \\ s_B^- \end{pmatrix}\delta(\mathbf{x}-\mathbf{x}_B). \tag{87}$$

Equation (84) thus yields

$$\mathbf{P}_A^T(\mathbf{x}_B)\mathbf{N}\mathbf{s}_B=-\mathbf{s}_A^T\mathbf{N}\mathbf{P}_B(\mathbf{x}_A) \tag{88}$$

or

$$P_A^+(\mathbf{x}_B)s_B^- - P_A^-(\mathbf{x}_B)s_B^+ = -s_A^+P_B^-(\mathbf{x}_A)+s_A^-P_B^+(\mathbf{x}_A). \tag{89}$$

For the special case that $s_A^+=s_B^+$ and $s_A^-=s_B^-=0$ this reduces to

$$P_A^-(\mathbf{x}_B)=P_B^-(\mathbf{x}_A) \tag{90}$$

(see Fig. 4).

B. One-way reciprocity theorem of the correlation type

We derive the reciprocity theorem of the correlation type for the one-way wave vector $\mathbf{P}$. Looking at the analogous symmetry properties of the two-way and one-way operator matrices (i.e., $\hat{\mathbf{A}}^\dagger=-\mathbf{K}\hat{\mathbf{A}}\mathbf{K}^{-1}$ vs $\hat{\mathbf{B}}^\dagger\approx-\mathbf{J}\hat{\mathbf{B}}\mathbf{J}^{-1}$), an obvious choice for the interaction quantity is, analogous to Eq. (37),

$$\partial_3\{\mathbf{P}_A^H\mathbf{J}\mathbf{P}_B\} \tag{91}$$

or

$$\partial_3\{(P_A^+)^*P_B^+ - (P_A^-)^*P_B^-\}. \tag{92}$$

Apparently, again we consider the interaction between oppositely propagating waves [bear in mind that complex conjugation changes the propagation direction (see Fig. 5)]. Following a similar procedure as in the previous subsection yields the following one-way reciprocity theorem of the correlation type

$$\int_{\partial\mathscr{V}_0}\mathbf{P}_A^H\mathbf{J}\mathbf{P}_B n_3\, d^2\mathbf{x}_L$$
$$\approx\int_{\mathscr{V}}\mathbf{P}_A^H\mathbf{J}\hat{\boldsymbol{\Delta}}\mathbf{P}_B\, d^3\mathbf{x}+\int_{\mathscr{V}}\{\mathbf{P}_A^H\mathbf{J}\mathbf{S}_B+\mathbf{S}_A^H\mathbf{J}\mathbf{P}_B\}d^3\mathbf{x}, \tag{93}$$

where the contrast operator $\hat{\boldsymbol{\Delta}}$ is again given by Eq. (85). The approximation sign denotes that evanescent waves are ignored.

In a paper on one-way representations (Wapenaar, 1996) we use reciprocity theorem (93) to derive approximate but *stable* inverse propagators for one-way wave fields in arbitrarily inhomogeneous media. The stability of these inverse propagators is due to the fact that the ''erroneously handled'' evanescent waves are suppressed instead of amplified. These inverse propagators find their application in (seismic) imaging techniques for moderately inhomogeneous and finely layered media. The main approximation due to the negligence of the evanescent waves is a limitation of the maximum obtainable lateral resolution (Berkhout, 1984; Wapenaar *et al.*, 1989).

VIII. COMPARISON

In the previous sections we have derived the reciprocity theorems for the two-way and one-way wave vectors independent of each other. We will now investigate whether it is possible to derive the one-way from the two-way reciprocity theorems. According to Eq. (44) we have $\mathbf{Q}_A=\hat{\mathbf{L}}_A\mathbf{P}_A$ and $\mathbf{Q}_B=\hat{\mathbf{L}}_B\mathbf{P}_B$. Substitution on the left-hand side of the two-way reciprocity theorem of the convolution-type (33) gives

$$\int_{\partial\mathcal{V}_0} \mathbf{Q}_A^T \mathbf{N} \mathbf{Q}_B n_3 \; d^2\mathbf{x}_L = \int_{\partial\mathcal{V}_0} (\hat{\mathbf{L}}_A \mathbf{P}_A)^T \mathbf{N} \hat{\mathbf{L}}_B \mathbf{P}_B n_3 \; d^2\mathbf{x}_L , \tag{94}$$

or, using Eq. (67),

$$\int_{\partial\mathcal{V}_0} \mathbf{Q}_A^T \mathbf{N} \mathbf{Q}_B n_3 \; d^2\mathbf{x}_L = -\int_{\partial\mathcal{V}_0} \mathbf{P}_A^T \mathbf{N} \hat{\mathbf{L}}_A^{-1} \hat{\mathbf{L}}_B \mathbf{P}_B n_3 \; d^2\mathbf{x}_L . \tag{95}$$

Similarly, using Eqs. (43), (44), and (67) in the two terms on the right-hand side of (33) yields

$$\int_{\mathcal{V}} \mathbf{Q}_A^T \mathbf{N} \{ \hat{\mathbf{A}}_B - \hat{\mathbf{A}}_A \} \mathbf{Q}_B \; d^3\mathbf{x} = j\omega \int_{\mathcal{V}} \mathbf{P}_A^T \mathbf{N} \{ \hat{\mathbf{L}}_A^{-1} \hat{\mathbf{L}}_B \hat{\mathbf{\Lambda}}_B - \hat{\mathbf{\Lambda}}_A \hat{\mathbf{L}}_A^{-1} \hat{\mathbf{L}}_B \} \mathbf{P}_B \; d^3\mathbf{x} \tag{96}$$

and

$$\int_{\mathcal{V}} \{ \mathbf{Q}_A^T \mathbf{N} \mathbf{D}_B + \mathbf{D}_A^T \mathbf{N} \mathbf{Q}_B \} d^3\mathbf{x} = -\int_{\mathcal{V}} \{ \mathbf{P}_A^T \mathbf{N} \hat{\mathbf{L}}_A^{-1} \hat{\mathbf{L}}_B \mathbf{S}_B + \mathbf{S}_A^T \mathbf{N} \hat{\mathbf{L}}_A^{-1} \hat{\mathbf{L}}_B \mathbf{P}_B \} d^3\mathbf{x}, \tag{97}$$

respectively. Note that the terms $\hat{\mathbf{L}}_A^{-1} \hat{\mathbf{L}}_B$ vanish when the medium parameters in both states are identical. Hence, for this situation it is seen from Eqs. (95)–(97) that there is a ''one-to-one'' correspondence between the individual terms in the two-way and one-way reciprocity theorems. In essence, this is how we originally derived one-way representations of the convolution type and of the correlation type (Berkhout and Wapenaar, 1989; Wapenaar *et al.*, 1989).

Apparently, for the more general situation in which the medium parameters in the two states are not identical, this one-to-one correspondence is absent. This explains why the conditions for the vanishing boundary integral are different for the two-way and one-way reciprocity theorems.

Relations similar to (95)–(97) can be found for the reciprocity theorems of the correlation type.

Obviously, the terms in the one-way reciprocity theorems (84) and (93) have a more attractive form than those on the right-hand sides of Eqs. (95)–(97). In particular, the contrast term $\hat{\mathbf{\Delta}} = \hat{\mathbf{B}}_B - \hat{\mathbf{B}}_A = -j\omega \hat{\mathbf{\Lambda}}_B + \hat{\mathbf{\Theta}}_B + j\omega \hat{\mathbf{\Lambda}}_A - \hat{\mathbf{\Theta}}_A$ has the favorable property that it can be fully expressed in terms of the coupling operator of state B by choosing $\hat{\mathbf{\Lambda}}_A = \hat{\mathbf{\Lambda}}_B$ and $\hat{\mathbf{\Theta}}_A = \mathbf{O}$, leaving $\hat{\mathbf{\Delta}} = \hat{\mathbf{\Theta}}_B$. This property can be exploited in the derivation of various representations for the one-way wave vector $\mathbf{P}$ (Wapenaar, 1996).

IX. CONCLUSIONS

We have analyzed the acoustic two-way and one-way wave equations for applications in which there is a ''preferred direction of propagation.'' In particular we have derived two-way and one-way reciprocity theorems of the convolution type and of the correlation type. Whereas the two-way reciprocity theorems specify the relation between the physical wave field quantities pressure and velocity in two different acoustic states, the one-way reciprocity theorems give a similar relation between the waves propagating in the positive and negative axial direction in both states.

We have made the following observations:

Unbounded media: The boundary integral in the two-way reciprocity theorem of the convolution type vanishes when at and outside $\partial\mathcal{V}$ the medium is unbounded, homogeneous, and source-free in both states and $\kappa_A = \kappa_B$ and $e_A = e_B$; the boundary integral in the one-way reciprocity theorem of the convolution type vanishes also when $\kappa_A \neq \kappa_B$ and $e_A \neq e_B$ at and outside $\partial\mathcal{V}$.

Contrast terms: The contrast terms in the two-way and one-way reciprocity theorems are defined in terms of operators that vanish when the medium parameters in both states are identical. Moreover, the contrast term in the one-way reciprocity theorem can be reformulated in terms of the coupling operator of one of the two states. This property can be exploited in the derivation of various representations for the one-way wave vector $\mathbf{P}$. These representations are the point of departure for the derivation of several imaging techniques.

Approximations: The two-way reciprocity theorems of the convolution and correlation type as well as the one-way reciprocity theorem of the convolution type are exact. The one-way reciprocity theorem of the correlation type ignores evanescent waves. This ''erroneous handling'' of evanescent waves facilitates the derivation of stable inverse wave field propagators for arbitrary inhomogeneous media. These inverse propagators find their application in imaging techniques. The main approximation due to the negligence of the evanescent waves is a limitation of the maximum obtainable lateral resolution.

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APPENDIX A: ADJOINT SQUARE-ROOT OPERATOR

In this Appendix we derive the adjoint square-root operator $\hat{\mathcal{H}}_1^\dagger$. We only discuss the main steps; for a more elaborate derivation see Grimbergen *et al.* (1997) or Wapenaar and Grimbergen (1996).

Throughout this Appendix we assume that the Helmholtz operator $\hat{\mathcal{H}}_2$ is defined on an appropriate space, so that it is self-adjoint (the conditions for the self-adjointness of $\hat{\mathcal{H}}_2$ are the same as those for $\hat{\mathcal{H}}$). The square-root operator $\hat{\mathcal{H}}_1$ will be defined on the same space, but it will appear not to be self-adjoint.

We introduce the *kernel* $\mathcal{H}_2$ of the operator $\hat{\mathcal{H}}_2$ via

$$\{\hat{\mathcal{H}}_2 g\}(\mathbf{x}_L) = \int_{\mathcal{D}} \mathcal{H}_2(\mathbf{x}_L ; \mathbf{x}_L') g(\mathbf{x}_L') d^2\mathbf{x}_L' . \tag{A1}$$

For notational convenience we drop the x_3 dependency throughout this Appendix.

As a result of the spectral theorem for self-adjoint operators (Reed and Simon, 1972, 1979; Weidman, 1980), the kernel $\mathcal{H}_2(\mathbf{x}_L ; \mathbf{x}_L')$ can be expanded in terms of a complete set of *real-valued* orthonormal eigenfunctions ϕ and eigenvalues λ, according to

$$\mathscr{H}_2(\mathbf{x}_L;\mathbf{x}_L')=\sum_i \phi^{(i)}(\mathbf{x}_L)\lambda_i\phi^{(i)}(\mathbf{x}_L'), \tag{A2}$$

when $\mathscr{D}$ is bounded (see Lu and McLaughlin, 1996, for a similar expression), or

$$\mathscr{H}_2(\mathbf{x}_L;\mathbf{x}_L')=\int_{\mathbb{R}^2}\phi(\mathbf{x}_L,\boldsymbol{\kappa})\lambda(\boldsymbol{\kappa})\phi(\mathbf{x}_L',\boldsymbol{\kappa})d^2\boldsymbol{\kappa}$$
$$+\sum_i \phi^{(i)}(\mathbf{x}_L)\lambda_i\phi^{(i)}(\mathbf{x}_L'), \tag{A3}$$

when $\mathscr{D}$ is unbounded. These expressions and other expressions for kernels given below should be understood in the sense of generalized functions. Positive and negative λs correspond to propagating and evanescent eigenmodes, respectively.

Analogous to Eq. (A1), we introduce the kernel of the square-root operator $\hat{\mathscr{H}}_1$ via

$$\{\hat{\mathscr{H}}_1 g\}(\mathbf{x}_L)=\int_{\mathscr{D}}\mathscr{H}_1(\mathbf{x}_L;\mathbf{x}_L')g(\mathbf{x}_L')d^2\mathbf{x}_L'. \tag{A4}$$

Note that, on account of Eq. (59), $\mathscr{H}_2$ and $\mathscr{H}_1$ are related to each other according to

$$\mathscr{H}_2(\mathbf{x}_L;\mathbf{x}_L')=\int_{\mathscr{D}}\mathscr{H}_1(\mathbf{x}_L;\mathbf{x}_L'')\mathscr{H}_1(\mathbf{x}_L'';\mathbf{x}_L')d^2\mathbf{x}_L''. \tag{A5}$$

From Eqs. (A2), (A3), and (A5) and the orthonormality property of the eigenfunctions ϕ it now follows that for $\mathscr{H}_1(\mathbf{x}_L;\mathbf{x}_L')$ we may write

$$\mathscr{H}_1(\mathbf{x}_L;\mathbf{x}_L')=\sum_i \phi^{(i)}(\mathbf{x}_L)\lambda_i^{1/2}\phi^{(i)}(\mathbf{x}_L'), \tag{A6}$$

when $\mathscr{D}$ is bounded, or

$$\mathscr{H}_1(\mathbf{x}_L;\mathbf{x}_L')=\int_{\mathbb{R}^2}\phi(\mathbf{x}_L,\boldsymbol{\kappa})\lambda^{1/2}(\boldsymbol{\kappa})\phi(\mathbf{x}_L',\boldsymbol{\kappa})d^2\boldsymbol{\kappa}$$
$$+\sum_i \phi^{(i)}(\mathbf{x}_L)\lambda_i^{1/2}\phi^{(i)}(\mathbf{x}_L'), \tag{A7}$$

when $\mathscr{D}$ is unbounded. Unlike $\mathscr{H}_2(\mathbf{x}_L;\mathbf{x}_L')$, the kernel $\mathscr{H}_1(\mathbf{x}_L;\mathbf{x}_L')$ is not purely real valued, since $\lambda^{1/2}$ becomes imaginary for negative λ, i.e., for evanescent eigenmodes. As a consequence, the square-root operator $\hat{\mathscr{H}}_1$ is not self-adjoint.

From Eqs. (A2), (A3), (A6), and (A7) we find the following symmetry relations for the kernels of $\hat{\mathscr{H}}_2$ and $\hat{\mathscr{H}}_1$:

$$\mathscr{H}_2(\mathbf{x}_L';\mathbf{x}_L)=\mathscr{H}_2(\mathbf{x}_L;\mathbf{x}_L') \tag{A8}$$

and

$$\mathscr{H}_1(\mathbf{x}_L';\mathbf{x}_L)=\mathscr{H}_1(\mathbf{x}_L;\mathbf{x}_L'). \tag{A9}$$

Using the latter symmetry property, the adjoint square-root operator $\hat{\mathscr{H}}_1^\dagger$ is found as follows:

$$\begin{aligned}\langle f,\hat{\mathscr{H}}_1 g\rangle&=\int_{\mathscr{D}}f^*(\mathbf{x}_L)\{\hat{\mathscr{H}}_1 g\}(\mathbf{x}_L)d^2\mathbf{x}_L\\&=\int_{\mathscr{D}}f^*(\mathbf{x}_L)\left(\int_{\mathscr{D}}\mathscr{H}_1(\mathbf{x}_L;\mathbf{x}_L')g(\mathbf{x}_L')d^2\mathbf{x}_L'\right)d^2\mathbf{x}_L\\&=\int_{\mathscr{D}}\left(\int_{\mathscr{D}}\mathscr{H}_1(\mathbf{x}_L';\mathbf{x}_L)f^*(\mathbf{x}_L)d^2\mathbf{x}_L\right)g(\mathbf{x}_L')d^2\mathbf{x}_L'\\&=\int_{\mathscr{D}}[\{\hat{\mathscr{H}}_1 f^*\}(\mathbf{x}_L')]g(\mathbf{x}_L')d^2\mathbf{x}_L'=\langle\hat{\mathscr{H}}_1^* f,g\rangle\end{aligned} \tag{A10}$$

or

$$\hat{\mathscr{H}}_1^\dagger=\hat{\mathscr{H}}_1^*. \tag{A11}$$

This expression plays a key role in the derivation of the one-way reciprocity theorem of the convolution type.

Note that, when evanescent waves are ignored, we may write

$$\mathscr{H}_1(\mathbf{x}_L';\mathbf{x}_L)\approx\mathscr{H}_1^*(\mathbf{x}_L;\mathbf{x}_L') \tag{A12}$$

and, consequently,

$$\hat{\mathscr{H}}_1^\dagger\approx\hat{\mathscr{H}}_1. \tag{A13}$$

This expression is used in the derivation of an approximate one-way reciprocity theorem of the correlation type.

APPENDIX B: RECIPROCITY, INVOLVING A DIRICHLET-TO-NEUMANN OPERATOR

This Appendix has been added during the revision. Recently interesting work has been done on reformulating the Helmholtz equation as a first-order equation on the total wave field alone, according to $\partial_3 P=\hat{\mathscr{Q}}P$, where $\hat{\mathscr{Q}}$ is called the ''Dirichlet-to-Neumann operator'' (Fishman *et al.*, 1996; Haines and de Hoop, 1996; Lu and McLaughlin, 1996). In this Appendix we modify this operator for variable density (and variable compressibility) media and we use the results to reformulate reciprocity theorem (35) in terms of the total acoustic pressure alone.

For a source-free region, we introduce a modified Dirichlet-to-Neumann operator $\hat{\mathscr{M}}$ via

$$\partial_3 P=-j\omega\varrho\hat{\mathscr{M}}P \tag{B1}$$

or

$$V_3=\hat{\mathscr{M}}P. \tag{B2}$$

Note that $\hat{\mathscr{M}}$ is just a scaled version of $\hat{\mathscr{Q}}$, the scaling factor being $-(j\omega\varrho)^{-1}$. It will turn out below that, for variable density media, $\hat{\mathscr{M}}$ obeys a simple symmetry relation, which is essential for the reformulation of the reciprocity theorem.

We derive an equation for $\hat{\mathscr{M}}$ by differentiating both sides of Eq. (B2) with respect to x_3, according to

$$\partial_3 V_3=(\partial_3\hat{\mathscr{M}})P+\hat{\mathscr{M}}\,\partial_3 P. \tag{B3}$$

Using $\partial_3 V_3=-j\omega\hat{\mathscr{K}}P$, with $\hat{\mathscr{K}}$ defined in Eq. (16), and substituting Eq. (B1) on the right-hand side of Eq. (B3), we obtain

$$-j\omega\hat{\mathscr{K}}P=(\partial_3\hat{\mathscr{M}}-j\omega\hat{\mathscr{M}}\varrho\hat{\mathscr{M}})P, \tag{B4}$$

or

$$\partial_3\hat{\mathscr{M}}=-j\omega\hat{\mathscr{K}}+j\omega\hat{\mathscr{M}}\varrho\hat{\mathscr{M}}. \tag{B5}$$

This nonlinear differential equation for $\hat{\mathscr{M}}$ has the form of a Ricatti equation. It is usually supplemented with suitably chosen initial conditions, so that its solution $\hat{\mathscr{M}}$, substituted in Eq. (B1), yields a system that is well-posed for marching (Fishman *et al.*, 1996).

In order to derive a symmetry relation for $\hat{\mathscr{M}}$ let us first consider the special situation of a medium in which the parameters do not depend on x_3. Then the left-hand side in Eq. (B5) becomes zero and we thus obtain

$$\hat{\mathscr{M}}\varrho\hat{\mathscr{M}}=\hat{\mathscr{K}}, \tag{B6}$$

or, using Eq. (49),

$$\omega^2\varrho^{1/2}\hat{\mathscr{M}}\varrho\hat{\mathscr{M}}\varrho^{1/2}=\hat{\mathscr{H}}_2, \tag{B7}$$

or, using Eq. (59),

$$\omega\varrho^{1/2}\hat{\mathscr{M}}\varrho^{1/2}=\pm\hat{\mathscr{H}}_1. \tag{B8}$$

Obviously for this situation $\hat{\mathscr{M}}$ obeys a symmetry relation of the form of Eq. (A11):

$$\hat{\mathscr{M}}^\dagger=\hat{\mathscr{M}}^*. \tag{B9}$$

For arbitrary inhomogeneous media, Eq. (B5) can be solved along the lines suggested by Lu and McLaughlin (1996). It thus follows that $\hat{\mathscr{M}}$ can be fully expressed in terms of operators that all obey symmetry relations of the form of Eq. (A11), which means that Eq. (B9) also holds for arbitrary inhomogeneous media.

We will now use Eq. (B2) to eliminate V_3 from reciprocity theorem (35). We define point sources of volume injection rate according to

$$Q_A(\mathbf{x})=q_A\delta(\mathbf{x}-\mathbf{x}_A),\quad Q_B(\mathbf{x})=q_B\delta(\mathbf{x}-\mathbf{x}_B) \tag{B10}$$

and we assume $F_{k,A}=F_{k,B}=0$. The source points $\mathbf{x}_A$ and/or $\mathbf{x}_B$ may be situated inside or outside $\partial\mathscr{V}_0$. It depends, among others, on the position of these source points how Eq. (B5) is solved for both states A and B (for more details, see Fishman *et al.*, 1996). Substituting Eqs. (B2) and (B10) into reciprocity theorem (35), using symmetry relation (B9), yields

$$\begin{aligned}\int_{\partial\mathscr{V}_0}&P_A\{\hat{\mathscr{M}}_B-\hat{\mathscr{M}}_A\}P_Bn_3\,d^2\mathbf{x}_L\\&=-j\omega\int_{\mathscr{V}}P_A\{(\hat{\mathscr{K}}_B-\hat{\mathscr{K}}_A)\\&\quad-\hat{\mathscr{M}}_A(\varrho_B-\varrho_A)\hat{\mathscr{M}}_B\}P_B\,d^3\mathbf{x}\\&\quad+P_A(\mathbf{x}_B)q_B\chi(\mathbf{x}_B)-q_A\chi(\mathbf{x}_A)P_B(\mathbf{x}_A),\end{aligned} \tag{B11}$$

where $\chi(\mathbf{x})$ is the characteristic function for volume $\mathscr{V}$. Note that this reciprocity theorem is fully expressed in terms of the total acoustic pressure. A further discussion is beyond the scope of this paper.

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