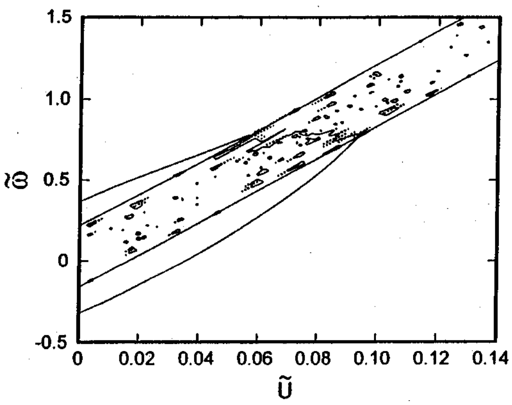

Fig. 2 The recalculated $\tilde{\omega}$ versus $\tilde{U}$ plot for $n=0$

could then be obtained, it does not truly solve the problem; solutions of zero displacement amplitude are as strange as solutions of infinite pressure for this linear homogeneous problem. Thus, instead of islets of nonsolution (infinite pressure) we would have islets of zero-amplitude solutions.

The same applies to Chen and Bert's results, in view of the similarities in the expressions for the perturbation pressures.

Conclusion

It is not known how the two sets of previous authors have overcome these difficulties and have presented full curves.[3] It is of course possible to ignore the islets of nonsolution and join the valid regions with a trusty French curve, though *any* solution within the no-solution band is really questionable. Alternatively, the previous authors may have used the device suggested by the reviewer. In either case a mathematical/physical difficulty exists.

Fundamentally, the question is this: does the mathematical difficulty have the physical meaning that flutter of arbitrary amplitude in such cases is impossible, even though its existence seems physically reasonable? Of course, the whole analysis of the physical system is highly idealized by ignoring viscous effects. Their incorporation, however, is anything but trivial. Perhaps, as is often the case, added realism will also overcome the mathematical difficulties. This line of research is being pursued.

It was nevertheless thought that the research community should be made aware that the results of the 1970s analyses may be flawed in some regions of the parameter space.

[2] As suggested by the same reviewer, at these points the fluid would then behave as a vibration absorber.

[3] All of those who actually did the calculations being unreachable.

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Reciprocity Theorems for Diffusion, Flow, and Waves

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Diffusion, flow, and wave phenomena can each be captured by a unified differential equation in matrix-vector form. This equation forms the basis for the derivation of unified reciprocity theorems for diffusion, flow and wave phenomena.
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Introduction

Diffusion, flow, and wave phenomena can each be captured by the following differential equation in matrix-vector form:

$$\underset{\sim}{\mathbf{A}}\frac{D\underset{\sim}{\mathbf{u}}}{Dt}+\mathbf{B}\mathbf{u}+\mathbf{D}_x\mathbf{u}=\mathbf{s}, \tag{1}$$

where $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$ is a vector containing space and time-dependent field quantities, $\mathbf{s}=\mathbf{s}(\mathbf{x},t)$ is a source vector, $\mathbf{A}=\mathbf{A}(\mathbf{x})$ and $\mathbf{B}=\mathbf{B}(\mathbf{x})$ are matrices containing space-dependent material parameters, and $\mathbf{D}_\mathbf{x}$ is a matrix containing the spatial differential operators $\partial/\partial x_1$, $\partial/\partial x_2$, and $\partial/\partial x_3$. Finally, D/Dt denotes the material time derivative, defined as $D/Dt=\partial/\partial t+\mathbf{v}\cdot\boldsymbol{\nabla}=\partial/\partial t+v_k\partial/\partial x_k$, where $\partial/\partial t$ denotes the time derivative in the reference frame and $\mathbf{v}=\mathbf{v}(\mathbf{x})$ is the space-dependent flow velocity of the material; v_k denotes the kth component of $\mathbf{v}$. Throughout this paper the summation convention applies to repeated subscripts; lowercase Latin subscripts run from 1 to 3. The vectors and matrices in Eq. (1) are further defined in the appendices for diffusion (Appendix A), acoustic wave propagation in moving fluids (Appendix B), momentum transport (Appendix C), and coupled elastodynamic and electromagnetic wave propagation in porous solids (Appendix D). In this paper we use Eq. (1) as the basis for deriving unified reciprocity theorems for these phenomena. In general, a reciprocity theorem interrelates the quantities that characterize two admissible physical states that could occur in one and the same domain, [1]. One can distinguish between convolution type and correlation type reciprocity theorems, [2]. Generally speaking, these two types of reciprocity theorems find their applications in forward and inverse problems, respectively. Both types of reciprocity theorems will be derived for the field vector $\mathbf{u}$.

The Differential Equation in the Frequency Domain

Reciprocity theorems can be derived in the time domain, the Laplace domain and the frequency domain, [3]. Here we only consider the frequency domain. We define the Fourier transform of a time-dependent function $f(t)$ as $\hat{f}(\omega)=\int_{-\infty}^{\infty}f(t)\exp(-j\omega t)\mathrm{d}t$, where j is the imaginary unit and ω denotes the angular frequency. We apply the Fourier transform to all terms in Eq. (1), under the assumption that this equation is linear in $\mathbf{u}$. Hence, we only consider those cases in which the field quantities in $\mathbf{u}$ do not appear in any of the matrices or operators in Eq. (1). In particular, this is why the term $D\mathbf{u}/Dt$ in the momentum transport Eq. (C7) is replaced by $\partial\mathbf{u}/\partial t$ in (C10). Transforming Eq. (1) to the frequency domain yields

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$$\mathbf{A}(j\omega+\mathbf{v}\cdot\boldsymbol{\nabla})\hat{\mathbf{u}}+\mathbf{B}\hat{\mathbf{u}}+\mathbf{D}_x\hat{\mathbf{u}}=\hat{\mathbf{s}}, \tag{2}$$

where $\hat{\mathbf{u}}=\hat{\mathbf{u}}(\mathbf{x},\omega)$ is the space and frequency-dependent field vector and $\hat{\mathbf{s}}=\hat{\mathbf{s}}(\mathbf{x},\omega)$ is the space and frequency-dependent source vector. The term $\mathbf{v}\cdot\boldsymbol{\nabla}$ should be dropped for linearized momentum transport (Appendix C) as well as for wave phenomena in nonmoving media (Appendix D). Finally we remark that in a number of cases matrix $\mathbf{B}$ contains temporal convolution kernels in the time domain (Appendix B) or, equivalently, complex frequency-dependent material parameters in the frequency domain (Appendices B and D).

Modification of Gauss' Divergence Theorem

The reciprocity theorem will be derived for a volume $\mathcal{V}$ enclosed by surface $\partial\mathcal{V}$ with outward pointing normal vector $\mathbf{n}$. Note that $\partial\mathcal{V}$ does not necessarily coincide with a physical boundary. Gauss' divergence theorem plays a central role in the derivation. For a scalar field $a(\mathbf{x})$, this theorem reads

$$\int_{\mathcal{V}}\frac{\partial a(\mathbf{x})}{\partial x_i}\,\mathrm{d}^3\mathbf{x}=\oint_{\partial\mathcal{V}}a(\mathbf{x})n_i\mathrm{d}^2\mathbf{x}, \tag{3}$$

where n_i denotes the ith component of $\mathbf{n}$. In this section we will modify this theorem for the differential operator matrix $\mathbf{D}_x$ appearing in Eqs. (1) and (2). Note that $\mathbf{D}_x=\mathbf{D}_x^T$ for all forms of $\mathbf{D}_x$ appearing in the appendices (here superscript ^{T} denotes matrix transposition only; it does not denote operator transposition). Let D_{IJ} denote the operator in row I and column J of matrix $\mathbf{D}_x$. The symmetry of $\mathbf{D}_x$ implies $D_{IJ}=D_{JI}$. We define a matrix $\mathbf{N}_\mathbf{x}$ which contains the components of the normal vector $\mathbf{n}$, organized in a similar way as matrix $\mathbf{D}_x$, see the appendices for details. Hence, if N_{IJ} denotes the element in row I and column J of matrix $\mathbf{N}_x$, we have $N_{IJ}=N_{JI}$. For example, for matrices $\mathbf{D}_x$ and $\mathbf{N}_x$ in Eqs. ($A3$) and ($A5$) we have $D_{12}=D_{21}=\partial/\partial x_1$ and $N_{12}=N_{21}=n_1$. If we now replace the scalar field $a(\mathbf{x})$ by $a_I(\mathbf{x})b_J(\mathbf{x})$, we may generalize Eq. (3) to

$$\int_{\mathcal{V}}D_{IJ}[a_I(\mathbf{x})b_J(\mathbf{x})]\mathrm{d}^3\mathbf{x}=\oint_{\partial\mathcal{V}}a_I(\mathbf{x})b_J(\mathbf{x})N_{IJ}\mathrm{d}^2\mathbf{x}, \tag{4}$$

where the summation convention applies for repeated capital Latin subscripts (which may run from 1 to 4, 12 or 22, depending on the choice of operator $\mathbf{D}_x$). Applying the product rule for differentiation and using the symmetry property $D_{IJ}=D_{JI}$, we obtain for the integrand in the left-hand side of Eq. (4)

$$D_{IJ}(a_Ib_J)=a_ID_{IJ}b_J+(D_{JI}a_I)b_J=\mathbf{a}^T\mathbf{D}_x\mathbf{b}+(\mathbf{D}_x\mathbf{a})^T\mathbf{b}, \tag{5}$$

where $\mathbf{a}$ and $\mathbf{b}$ are vector functions, containing the scalar functions $a_I(\mathbf{x})$ and $b_J(\mathbf{x})$, respectively. Rewriting the integrand in the right-hand side of Eq. (4) in a similar way, we thus obtain

$$\int_{\mathcal{V}}[\mathbf{a}^T\mathbf{D}_x\mathbf{b}+(\mathbf{D}_x\mathbf{a})^T\mathbf{b}]\mathrm{d}^3\mathbf{x}=\oint_{\partial\mathcal{V}}\mathbf{a}^T\mathbf{N}_x\mathbf{b}\mathrm{d}^2\mathbf{x}. \tag{6}$$

Finally we consider a variant of this equation. We replace $\mathbf{a}$ by $\mathbf{Ka}$, where $\mathbf{K}$ is a diagonal matrix with the following property:

$$\mathbf{D}_x\mathbf{K}=-\mathbf{K}\mathbf{D}_x, \tag{7}$$

see the appendices for details. With this replacement, Eq. (6) becomes

$$\int_{\mathcal{V}}[\mathbf{a}^T\mathbf{K}\mathbf{D}_x\mathbf{b}-(\mathbf{D}_x\mathbf{a})^T\mathbf{K}\mathbf{b}]\mathrm{d}^3\mathbf{x}=\oint_{\partial\mathcal{V}}\mathbf{a}^T\mathbf{K}\mathbf{N}_x\mathbf{b}\mathrm{d}^2\mathbf{x}. \tag{8}$$

Reciprocity Theorem of the Convolution Type

We consider two physical states in volume $\mathcal{V}$. The field quantities, the material parameters, the flow velocity as well as the source functions may be different in both states and they will be distinguished with subscripts A and B (of course the summation convention does not apply for these subscripts). We substitute $\hat{\mathbf{u}}_A$ and $\hat{\mathbf{u}}_B$ for $\mathbf{a}$ and $\mathbf{b}$ in Eq. (8), apply Eq. (2) for states A and B and use the symmetry properties

$$\mathbf{A}_A^T\mathbf{K}=\mathbf{K}\mathbf{A}_A\quad\text{and}\quad\mathbf{B}_A^T\mathbf{K}=\mathbf{K}\mathbf{B}_A \tag{9}$$

(see the appendices). This yields

$$\begin{aligned}\oint_{\partial\mathcal{V}}\hat{\mathbf{u}}_A^T\mathbf{K}\mathbf{N}_x\mathbf{u}_B\mathrm{d}^2\mathbf{x}=&\int_{\mathcal{V}}[\mathbf{u}_A^T\mathbf{K}\hat{\mathbf{s}}_B-\mathbf{s}_A^T\mathbf{K}\mathbf{u}_B\mathrm{d}^3\mathbf{x}\\&+\int_{\mathcal{V}}\mathbf{u}_A^T\mathbf{K}[j\omega(\mathbf{A}_A-\mathbf{A}_B)+(\mathbf{B}_A-\mathbf{B}_B)]\mathbf{u}_B\mathrm{d}^3\mathbf{x}\\&+\int_{\mathcal{V}}[((\mathbf{v}_A\cdot\boldsymbol{\nabla})\mathbf{u}_A)^T\mathbf{K}\mathbf{A}_A\\&-\mathbf{u}_A^T\mathbf{K}\mathbf{A}_B(\mathbf{v}_B\cdot\boldsymbol{\nabla})]\mathbf{u}_B\mathrm{d}^3\mathbf{x}.\end{aligned} \tag{10}$$

The first term in the last integral can be written as

$$\begin{aligned}((\mathbf{v}_A\cdot\boldsymbol{\nabla})\mathbf{u}_A{}^T\mathbf{K}\mathbf{A}_A\hat{\mathbf{u}}_B=&\boldsymbol{\nabla}\cdot(\mathbf{v}_A\hat{\mathbf{u}}_A^T\mathbf{K}\mathbf{A}_A\mathbf{u}_B\\&-\hat{\mathbf{u}}_A^T\mathbf{K}\frac{\partial(v_{i,A}\mathbf{A}_A)}{\partial x_i}\mathbf{u}_B-\mathbf{u}_A^T\mathbf{K}\mathbf{A}_A(\mathbf{v}_A\cdot\boldsymbol{\nabla})\mathbf{u}_B.\end{aligned} \tag{11}$$

For diffusion (Appendix A) the term $\partial(v_{i,A}\mathbf{A}_A)/\partial x_i$ vanishes on account of the equation of continuity. For acoustic wave propagation this term (with $v_{i,A}$ replaced by $v^0_{i,A}$, Appendix B) is negligible in comparison with the spatial derivatives of the wave fields $\hat{\mathbf{u}}_A$ and $\hat{\mathbf{u}}_B$. For the other situations considered in the appendices, $v_{i,A}$ is taken equal to zero. Hence, the term containing $\partial(v_{i,A}\mathbf{A}_A)/\partial x_i$ will be dropped. Substituting the remainder of the right-hand side of Eq. (11) into Eq. (10) and applying the theorem of Gauss for the term containing the divergence operator, yields

$$\begin{aligned}\oint_{\partial\mathcal{V}}\hat{\mathbf{u}}_A^T\mathbf{K}\mathbf{N}_x\hat{\mathbf{u}}_B\mathrm{d}^2\mathbf{x}=&\int_{\mathcal{V}}[\hat{\mathbf{u}}_A^T\mathbf{K}\hat{\mathbf{s}}_B-\hat{\mathbf{s}}_A^T\mathbf{K}\hat{\mathbf{u}}_B]\mathrm{d}^3\mathbf{x}\\&+\int_{\mathcal{V}}\hat{\mathbf{u}}_A^T\mathbf{K}[j\omega(\mathbf{A}_A-\mathbf{A}_B)+(\mathbf{B}_A-\mathbf{B}_B)]\hat{\mathbf{u}}_B\mathrm{d}^3\mathbf{x}\\&-\int_{\mathcal{V}}\hat{\mathbf{u}}_A^T\mathbf{K}[\mathbf{A}_A(\mathbf{v}_A\cdot\boldsymbol{\nabla})+\mathbf{A}_B(\mathbf{v}_B\cdot\boldsymbol{\nabla})]\hat{\mathbf{u}}_B\mathrm{d}^3\mathbf{x}\\&+\oint_{\partial\mathcal{V}}(\hat{\mathbf{u}}_A^T\mathbf{K}\mathbf{A}_A\hat{\mathbf{u}}_B)\mathbf{v}_A\boldsymbol{\cdot}\mathbf{n}\mathrm{d}^2\mathbf{x}.\end{aligned} \tag{12}$$

This is the unified reciprocity theorem of the convolution type (we speak of convolution type, because the multiplications in the frequency domain correspond to convolutions in the time domain). It interrelates the field quantities (contained in $\hat{\mathbf{u}}_A$ and $\hat{\mathbf{u}}_B$), the material parameters (contained in $\mathbf{A}_A$, $\mathbf{B}_A$, $\mathbf{A}_B$, and $\mathbf{B}_B$), the flow velocities ($\mathbf{v}_A$ and $\mathbf{v}_B$) as well as the source functions (contained in $\hat{\mathbf{s}}_A$ and $\hat{\mathbf{s}}_B$) of states A and B. The left-hand side is a boundary integral which contains a specific combination of the field quantities of states A and B at the boundary of the volume $\mathcal{V}$. The first integral on the right-hand side interrelates the field quantities and the source functions in $\mathcal{V}$. The second integral contains the differences of the medium parameters in both states; obviously this integral vanishes when the medium parameters in both states are identical. The third integral on the right-hand side contains the flow velocities in $\mathcal{V}$; this integral vanishes when the medium parameters in both states are identical and the flow velocities in both states are opposite to each other. The last integral on the right-hand side is a boundary integral containing the normal component of the flow velocity in state A; it vanishes when this flow velocity is tangential to the boundary $\partial\mathcal{V}$. Depending on the type of application, states A and B can be both physical states, or both mathematical states (e.g., Green's states), or one can be a physical state and the other a mathematical state (the latter situation leads to representation integrals). For further discussions on convolution-type reciprocity theorems in different fields of application we refer to Lyamshev [4], De Hoop and Stam [1], Fokkema and Van den Berg [3], Allard et al. [5], Pride and Haartsen [6], and Belinskiy [7].

Reciprocity Theorem of the Correlation Type

We substitute $\hat{\mathbf{u}}_A^*$ and $\hat{\mathbf{u}}_B$ for $\mathbf{a}$ and $\mathbf{b}$ in Eq. (6), where* denotes complex conjugation. Following the same procedure as in the previous section, using the symmetry property

$$\mathbf{A}_A^H = \mathbf{A}_A\,, \tag{13}$$

where ^{H} denotes complex conjugation and transposition, we obtain

$$\begin{aligned}\oint_{\partial\mathcal{V}} \hat{\mathbf{u}}_A^H \mathbf{N}_x \hat{\mathbf{u}}_B \mathrm{d}^2\mathbf{x} = {}& \int_{\mathcal{V}} [\hat{\mathbf{u}}_A^H \hat{\mathbf{s}}_B + \hat{\mathbf{s}}_A^H \hat{\mathbf{u}}_B] \mathrm{d}^3\mathbf{x} \\ &+ \int_{\mathcal{V}} \hat{\mathbf{u}}_A^H [j\omega(\mathbf{A}_A - \mathbf{A}_B) - (\mathbf{B}_A^H + \mathbf{B}_B)] \hat{\mathbf{u}}_B \mathrm{d}^3\mathbf{x} \\ &+ \int_{\mathcal{V}} \hat{\mathbf{u}}_A^H [\mathbf{A}_A(\mathbf{v}_A \cdot \boldsymbol{\nabla}) - \mathbf{A}_B(\mathbf{v}_B \cdot \boldsymbol{\nabla})] \hat{\mathbf{u}}_B \mathrm{d}^3\mathbf{x} \\ &- \oint_{\partial\mathcal{V}} (\hat{\mathbf{u}}_A^H \mathbf{A}_A \hat{\mathbf{u}}_B) \mathbf{v}_A \cdot \mathbf{n} \mathrm{d}^2\mathbf{x}.\end{aligned} \tag{14}$$

This is the unified reciprocity theorem of the correlation type (we speak of correlation type, because the multiplications in the frequency domain correspond to correlations in the time domain). The term $\hat{\mathbf{u}}_A^H$ contains "back-propagating" field quantities in state A, [2]. When we compare this reciprocity theorem with Eq. (12), we observe that, apart from the complex conjugation, the diagonal matrix $\mathbf{K}$ is absent in all integrals and that some plus and minus signs have been changed. In particular, the term $(\mathbf{B}_A - \mathbf{B}_B)$ has been replaced by $(\mathbf{B}_A^H + \mathbf{B}_B)$, which means that the second integral on the right-hand side no longer vanishes when the medium parameters in both states are identical. Moreover, the term $[\mathbf{A}_A(\mathbf{v}_A \cdot \boldsymbol{\nabla}) + \mathbf{A}_B(\mathbf{v}_B \cdot \boldsymbol{\nabla})]$ has been replaced by $[\mathbf{A}_A(\mathbf{v}_A \cdot \boldsymbol{\nabla}) - \mathbf{A}_B(\mathbf{v}_B \cdot \boldsymbol{\nabla})]$, which means that the third integral on the right-hand side vanishes when the medium parameters contained in matrix $\mathbf{A}$ as well as the flow velocities are identical in both states. For a discussion on the application of correlation-type reciprocity theorems to inverse problems we refer to Fisher and Langenberg [8] and De Hoop and Stam [1].

Conclusions

We have formulated a general differential equation in matrix-vector form (equation (1)), which applies to diffusion (Appendix A), acoustic wave propagation in moving fluids (Appendix B), momentum transport (Appendix C) and coupled elastodynamic and electromagnetic wave propagation in fluid-saturated porous solids (Appendix D). For linear phenomena (which excludes non-linear momentum transport) we have transformed the general equation from the time domain to the frequency domain (Eq. (2)). Based on this general equation as well as the symmetry properties (7), (9), and (13) we have derived unified reciprocity theorems of the convolution type (Eq. (12)) and of the correlation type (Eq. (14)), respectively.

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Appendix A

Mass Diffusion. The equation of continuity for species k in a mixture of fluids reads

$$\varrho \frac{DY^{(k)}}{Dt} + \frac{\partial J_j^{(k)}}{\partial x_j} = \dot{\omega}^{(k)}, \tag{A1}$$

where $Y^{(k)}$ is the mass fraction of species k, $J_j^{(k)}$ its mass flux relative to the mixture, ϱ is the mass density of the mixture and $\dot{\omega}^{(k)}$ the mass production rate density of species k (due to chemical reactions). Fick's first law of diffusion reads

$$J_j^{(k)} + \varrho \mathcal{D}^{(k)} \frac{\partial Y^{(k)}}{\partial x_j} = 0, \tag{A2}$$

where $\mathcal{D}^{(k)}$ is the diffusion coefficient for species k. Equations ($A1$) and ($A2$) can be combined to yield Eq. (1), with

$$\mathbf{u} = \begin{pmatrix} Y^{(k)} \\ J_1^{(k)} \\ J_2^{(k)} \\ J_3^{(k)} \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} \dot{\omega}^{(k)} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$
$$\mathbf{D}_x = \begin{pmatrix} 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & 0 & 0 & 0 \\ \frac{\partial}{\partial x_3} & 0 & 0 & 0 \end{pmatrix}, \tag{A3}$$

$$\mathbf{A} = \begin{pmatrix} \varrho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$
$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\varrho\mathcal{D}^{(k)}} & 0 & 0 \\ 0 & 0 & \frac{1}{\varrho\mathcal{D}^{(k)}} & 0 \\ 0 & 0 & 0 & \frac{1}{\varrho\mathcal{D}^{(k)}} \end{pmatrix}. \tag{A4}$$

Matrices $\mathbf{N}_x$ and $\mathbf{K}$, appearing in the modified divergence theorems (6) and (8), read

$$\mathbf{N}_x = \begin{pmatrix} 0 & n_1 & n_2 & n_3 \\ n_1 & 0 & 0 & 0 \\ n_2 & 0 & 0 & 0 \\ n_3 & 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$
$$\mathbf{K} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A5}$$

Note that matrices $\mathbf{D}_x$, $\mathbf{A}$ and $\mathbf{B}$ obey Eqs. (7), (9), and (13). Other diffusion phenomena can be formulated in a similar way.

Appendix B

Acoustic Wave Propagation in Moving Fluids. The linearized equation of motion in a moving fluid reads

$$\varrho \frac{Dv_i}{Dt} + b^v * v_i + \frac{\partial p}{\partial x_i} = f_i, \tag{B1}$$

with

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_k^0 \frac{\partial}{\partial x_k}, \tag{B2}$$

where p is the acoustic pressure, v_i the particle velocity associated to the acoustic wave motion (which is to be distinguished from the flow velocity v_k^0 in the operator D/Dt), ϱ the mass density of the medium in equilibrium, f_i the volume density of external force, and b^v a causal loss function ($*$ denotes a temporal convolution). The linearized stress-strain relation reads

$$\frac{1}{K}\frac{Dp}{Dt} + b^p * p + \frac{\partial v_i}{\partial x_i} = q, \tag{B3}$$

where K is the bulk compression modulus, q the volume injection rate density, and b^p a causal loss function.

Equations ($B1$) and ($B3$) can be combined to yield the general matrix-vector Eq. (1), with D/Dt defined in Eq. ($B2$) and

$$\mathbf{u} = \begin{pmatrix} p \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} q \\ f_1 \\ f_2 \\ f_3 \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} \frac{1}{K} & 0 & 0 & 0 \\ 0 & \varrho & 0 & 0 \\ 0 & 0 & \varrho & 0 \\ 0 & 0 & 0 & \varrho \end{pmatrix} \quad \text{and} \tag{B4}$$

$$\mathbf{B} = \begin{pmatrix} b^p* & 0 & 0 & 0 \\ 0 & b^v* & 0 & 0 \\ 0 & 0 & b^v* & 0 \\ 0 & 0 & 0 & b^v* \end{pmatrix}.$$

Matrices $\mathbf{D}_x$, $\mathbf{N}_x$, and $\mathbf{K}$ are the same as in Appendix A. The symmetry properties described by Eqs. (7), (9), and (13) are easily confirmed. Finally, note that in the frequency domain formulation, the temporal convolution kernels $b^p(\mathbf{x},t)*$ and $b^v(\mathbf{x},t)*$ in matrix $\mathbf{B}$ are replaced by complex frequency-dependent functions $\hat{b}^p(\mathbf{x},\omega)$ and $\hat{b}^v(\mathbf{x},\omega)$, respectively.

Appendix C

Momentum Transport. The nonlinear equation of motion for a viscous fluid reads

$$\varrho \frac{Dv_i}{Dt} - \frac{\partial \tau_{ij}}{\partial x_j} = f_i, \tag{C1}$$

where v_i is the particle velocity, τ_{ij} the stress tensor, ϱ the mass density, and f_i the volume density of external force. Stoke's stress-strain rate relation reads

$$-\tau_{ij} + \eta_{ijkl}\frac{\partial v_k}{\partial x_l} = p\,\delta_{ij}, \tag{C2}$$

where η_{ijkl} is the anisotropic viscosity tensor and p the hydrostatic pressure. The viscosity tensor obeys the following symmetry relation $\eta_{ijkl} = \eta_{jikl} = \eta_{ijlk} = \eta_{klij}$. For isotropic fluids the viscosity tensor reads $\eta_{ijkl} = \eta(-2/3\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where η is the isotropic viscosity parameter. Equations ($C1$) and ($C2$) can be combined to yield the general matrix-vector Eq. (1). To this end we first rewrite these equations as

$$\varrho \frac{D\mathbf{v}}{Dt} - \frac{\partial \boldsymbol{\tau}_j}{\partial x_j} = \mathbf{f}, \tag{C3}$$

$$-\boldsymbol{\tau}_j + \mathbf{h}_{jl}\frac{\partial \mathbf{v}}{\partial x_l} = p\,\boldsymbol{\delta}_j, \tag{C4}$$

with

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \boldsymbol{\tau}_j = \begin{pmatrix} \tau_{1j} \\ \tau_{2j} \\ \tau_{3j} \end{pmatrix}, \quad \boldsymbol{\delta}_j = \begin{pmatrix} \delta_{1j} \\ \delta_{2j} \\ \delta_{3j} \end{pmatrix} \quad \text{and} \tag{C5}$$

$$\mathbf{h}_{jl} = \begin{pmatrix} \eta_{1j1l} & \eta_{1j2l} & \eta_{1j3l} \\ \eta_{2j1l} & \eta_{2j2l} & \eta_{2j3l} \\ \eta_{3j1l} & \eta_{3j2l} & \eta_{3j3l} \end{pmatrix}.$$

Note that

$$\mathbf{h}_{jl} = \mathbf{h}_{lj}^T \tag{C6}$$

on account of the symmetry properties of η_{ijkl}. Hence, we obtain

$$\bar{\mathbf{A}}\frac{D\mathbf{u}}{Dt} + \bar{\mathbf{B}}\mathbf{u} + \mathbf{C}\mathbf{D}_x\mathbf{u} = \bar{\mathbf{s}}, \tag{C7}$$

with

$$\mathbf{u} = \begin{pmatrix} \mathbf{v} \\ -\boldsymbol{\tau}_1 \\ -\boldsymbol{\tau}_2 \\ -\boldsymbol{\tau}_3 \end{pmatrix}, \quad \bar{\mathbf{s}} = \begin{pmatrix} \mathbf{f} \\ p\,\boldsymbol{\delta}_1 \\ p\,\boldsymbol{\delta}_2 \\ p\,\boldsymbol{\delta}_3 \end{pmatrix},$$

$$\bar{\mathbf{A}} = \begin{pmatrix} \varrho\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}, \quad \bar{\mathbf{B}} = \begin{pmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}, \tag{C8}$$

$$\mathbf{C} = \begin{pmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{h}_{11} & \mathbf{h}_{12} & \mathbf{h}_{13} \\ \mathbf{O} & \mathbf{h}_{21} & \mathbf{h}_{22} & \mathbf{h}_{23} \\ \mathbf{O} & \mathbf{h}_{31} & \mathbf{h}_{32} & \mathbf{h}_{33} \end{pmatrix},$$

$$\mathbf{D}_x = \begin{pmatrix} \mathbf{O} & \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 \\ \mathbf{D}_1 & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{D}_2 & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{D}_3 & \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix} \quad \text{and} \tag{C9}$$

$$\mathbf{D}_j = \begin{pmatrix} \frac{\partial}{\partial x_j} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_j} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_j} \end{pmatrix},$$

for $j = 1, 2, 3$, $\mathbf{I}$ being the 3×3 identity matrix and $\mathbf{O}$ the 3×3 null matrix. Multiplication of all terms in Eq. ($C7$) by the inverse of $\mathbf{C}$ and linearization of the term $D\mathbf{u}/Dt$ yields

$$\mathbf{A}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{B}\mathbf{u} + \mathbf{D}_x\mathbf{u} = \mathbf{s}, \tag{C10}$$

with

$$\mathbf{A} = \mathbf{C}^{-1}\bar{\mathbf{A}} = \bar{\mathbf{A}}, \quad \mathbf{B} = \mathbf{C}^{-1}\bar{\mathbf{B}} \quad \text{and} \quad \mathbf{s} = \mathbf{C}^{-1}\bar{\mathbf{s}}. \tag{C11}$$

Matrices $\mathbf{N}_x$ and $\mathbf{K}$, appearing in the modified divergence theorems (6) and (8), read

$$\mathbf{N_x}=\begin{pmatrix} \mathbf{O} & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 \\ \mathbf{N}_1 & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{N}_2 & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{N}_3 & \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix} \quad \text{and} \quad \mathbf{N}_j=\begin{pmatrix} n_j & 0 & 0 \\ 0 & n_j & 0 \\ 0 & 0 & n_j \end{pmatrix}, \tag{C12}$$

for $j=1, 2, 3$ and

$$\mathbf{K}=\mathrm{diag}(\mathbf{1},-\mathbf{1},-\mathbf{1},-\mathbf{1}), \quad \text{with} \quad \mathbf{1}=(1,1,1). \tag{C13}$$

Based on the structure of the matrices $\bar{\mathbf{A}},\bar{\mathbf{B}},\mathbf{C}$, $\mathbf{D_x}$, and $\mathbf{K}$ as well as the symmetry relation (C6), we find that the symmetry properties described by Eqs. (7), (9), and (13) are obeyed.

Appendix D

Coupled Elastodynamic and Electromagnetic Wave Propagation in Porous Solids. We briefly review the theory for elastodynamic waves coupled to electromagnetic fields in a dissipative inhomogeneous anisotropic fluid-saturated porous solid, [6,9]. The linearized equations of motion read in the frequency domain (using the vector notation introduced in Appendix C)

$$j\omega\varrho^b\hat{\mathbf{v}}^s+j\omega\varrho^f\hat{\mathbf{w}}-\frac{\partial\hat{\boldsymbol{\tau}}_j^b}{\partial x_j}=\hat{\mathbf{f}}^b, \tag{D1}$$

$$j\omega\varrho^f\hat{\mathbf{v}}^s+\eta\hat{\mathbf{k}}^{-1}(\hat{\mathbf{w}}-\hat{\mathbf{L}}\hat{\mathbf{E}})+\boldsymbol{\nabla}\hat{p}=\hat{\mathbf{f}}^f, \tag{D2}$$

with $\hat{\mathbf{w}}=\phi(\hat{\mathbf{v}}^f-\hat{\mathbf{v}}^s)$. Here $\hat{\mathbf{v}}^s$ and $\hat{\mathbf{v}}^f$ are the averaged solid and fluid particle velocities associated to the wave motion, $\hat{\mathbf{w}}$ is the filtration velocity, ϕ the porosity, $\hat{\boldsymbol{\tau}}_j^b$ the averaged bulk stress, $\hat{p}$ the averaged fluid pressure, and $\hat{\mathbf{E}}$ the averaged electric field strength. The source functions $\hat{\mathbf{f}}^b$ and $\hat{\mathbf{f}}^f$ are the volume densities of external force on the bulk and on the fluid, respectively. The constitutive parameters ϱ^b and ϱ^f are the anisotropic bulk and fluid mass densities, respectively, [10]. In the following we assume that these tensors are symmetric, according to $\varrho^b=(\varrho^b)^T$ and $\varrho^f=(\varrho^f)^T$, which is for example the case when the anisotropy is the result of parallel fine layering at a scale much smaller than the wavelength. The complex frequency-dependent tensor $\hat{\mathbf{k}}$ is the dynamic permeability tensor of the porous material, with $\hat{\mathbf{k}}=\hat{\mathbf{k}}^T$, and η is the fluid viscosity parameter. Finally, the complex frequency-dependent tensor $\hat{\mathbf{L}}$ accounts for the coupling between the elastodynamic and electromagnetic waves. In the following we will assume that this tensor is symmetric as well, according to $\hat{\mathbf{L}}=\hat{\mathbf{L}}^T$ (Pride and Haartsen [6] discuss the conditions for this symmetry).

The linearized stress-strain relations read

$$-j\omega\hat{\boldsymbol{\tau}}_j^b+\mathbf{c}_{jl}\frac{\partial\hat{\mathbf{v}}^s}{\partial x_l}+\mathbf{d}_j\boldsymbol{\nabla}\cdot\hat{\mathbf{w}}=\mathbf{0}, \tag{D3}$$

$$j\omega\hat{p}+\mathbf{d}_l^T\frac{\partial\hat{\mathbf{v}}^s}{\partial x_l}+M\boldsymbol{\nabla}\cdot\hat{\mathbf{w}}=0, \tag{D4}$$

with $\mathbf{0}$ a 3×1 null vector and $\mathbf{d}_j$ and $\mathbf{c}_{jl}$ defined similar as $\boldsymbol{\delta}_j$ and $\mathbf{h}_{jl}$ in Eq. (C5), i.e., $(\mathbf{d}_j)_i=d_{ij}$, with $d_{ij}=d_{ji}$, and $(\mathbf{c}_{jl})_{ik}=c_{ijkl}$, with $c_{ijkl}=c_{jikl}=c_{ijlk}=c_{klij}$. Note that $\mathbf{c}_{jl}=\mathbf{c}_{lj}^T$. M, d_{ij} and c_{ijkl} are the stiffness parameters of the porous solid.

Maxwell's electromagnetic field equations read

$$j\omega\boldsymbol{\epsilon}\hat{\mathbf{E}}+\hat{\mathbf{J}}-\boldsymbol{\nabla}\times\hat{\mathbf{H}}=-\hat{\mathbf{J}}^e, \tag{D5}$$

$$j\omega\boldsymbol{\mu}\hat{\mathbf{H}}+\boldsymbol{\nabla}\times\hat{\mathbf{E}}=-\hat{\mathbf{J}}^m, \tag{D6}$$

where $\hat{\mathbf{H}}$ is the averaged magnetic field strength, $\hat{\mathbf{J}}$ the averaged induced electric current density, $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are the anisotropic permittivity and permeability, with $\boldsymbol{\epsilon}=\boldsymbol{\epsilon}^T$ and $\boldsymbol{\mu}=\boldsymbol{\mu}^T$, and $\mathbf{J}^e$ and $\mathbf{J}^m$ are source functions in terms of the external electric and magnetic current densities. The induced electric current density is coupled to the elastodynamic wave motion, according to

$$\hat{\mathbf{J}}=\hat{\boldsymbol{\sigma}}\hat{\mathbf{E}}-\hat{\mathbf{L}}[\boldsymbol{\nabla}\hat{p}+j\omega\varrho^f\hat{\mathbf{v}}^s-\hat{\mathbf{f}}^f], \tag{D7}$$

where $\hat{\boldsymbol{\sigma}}$ is the complex frequency dependent conductivity, with $\hat{\boldsymbol{\sigma}}=\hat{\boldsymbol{\sigma}}^T$. Substituting the constitutive relation (D7) into the Maxwell Eq. (D5), and adding $\hat{\mathbf{L}}$ times Eq. (D2) to Eq. (D5) in order to compensate for the term $-\hat{\mathbf{L}}[\boldsymbol{\nabla}\hat{p}+j\omega\varrho^f\hat{\mathbf{v}}^s-\hat{\mathbf{f}}^f]$, yields

$$j\omega\boldsymbol{\epsilon}\hat{\mathbf{E}}+(\hat{\boldsymbol{\sigma}}-\eta\hat{\mathbf{L}}\hat{\mathbf{k}}^{-1}\hat{\mathbf{L}})\hat{\mathbf{E}}+\eta\hat{\mathbf{L}}\hat{\mathbf{k}}^{-1}\hat{\mathbf{w}}-\boldsymbol{\nabla}\times\hat{\mathbf{H}}=-\hat{\mathbf{J}}^e. \tag{D8}$$

Equations (D8) and (D6), together with Eqs. (D1), (D2), (D3), and (D4) can be combined to yield

$$j\omega\bar{\mathbf{A}}\hat{\mathbf{u}}+\bar{\mathbf{B}}\hat{\mathbf{u}}+\mathbf{C}\mathbf{D}_x\hat{\mathbf{u}}=\hat{\hat{\mathbf{s}}}, \tag{D9}$$

where

$$\hat{\mathbf{u}}=\begin{pmatrix} \hat{\mathbf{u}}_1 \\ \hat{\mathbf{u}}_2 \\ \hat{\mathbf{u}}_3 \end{pmatrix}, \quad \hat{\hat{\mathbf{s}}}=\begin{pmatrix} \hat{\hat{\mathbf{s}}}_1 \\ \hat{\hat{\mathbf{s}}}_2 \\ \hat{\hat{\mathbf{s}}}_3 \end{pmatrix}, \quad \bar{\mathbf{A}}=\begin{pmatrix} \bar{\mathbf{A}}_{11} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \bar{\mathbf{A}}_{22} & \bar{\mathbf{A}}_{23} \\ \mathbf{O} & \bar{\mathbf{A}}_{23}^T & \bar{\mathbf{A}}_{33} \end{pmatrix}, \tag{D10}$$

$$\bar{\mathbf{B}}=\begin{pmatrix} \bar{\mathbf{B}}_{11} & \mathbf{O} & \bar{\mathbf{B}}_{13} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\bar{\mathbf{B}}_{13}^T & \mathbf{O} & \bar{\mathbf{B}}_{33} \end{pmatrix},$$

$$\mathbf{C}=\begin{pmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{C}_{22} & \mathbf{C}_{23} \\ \mathbf{O} & \mathbf{C}_{23}^T & \mathbf{C}_{33} \end{pmatrix} \quad \text{and} \quad \mathbf{D_x}=\begin{pmatrix} \mathbf{D}_{11} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{D}_{22} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{D}_{33} \end{pmatrix}, \tag{D11}$$

where $\mathbf{I}$ and $\mathbf{O}$ are identity and null matrices of appropriate size and

$$\hat{\mathbf{u}}_1=\begin{pmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{H}} \end{pmatrix}, \quad \hat{\mathbf{u}}_2=\begin{pmatrix} \hat{\mathbf{v}}^s \\ -\hat{\boldsymbol{\tau}}_1^b \\ -\hat{\boldsymbol{\tau}}_2^b \\ -\hat{\boldsymbol{\tau}}_3^b \end{pmatrix}, \quad \hat{\mathbf{u}}_3=\begin{pmatrix} \hat{\mathbf{w}} \\ \hat{p} \end{pmatrix},$$

$$\hat{\hat{\mathbf{s}}}_1=\begin{pmatrix} -\hat{\mathbf{J}}^e \\ -\hat{\mathbf{J}}^m \end{pmatrix}, \quad \hat{\hat{\mathbf{s}}}_2=\begin{pmatrix} \hat{\mathbf{f}}^b \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad \hat{\hat{\mathbf{s}}}_3=\begin{pmatrix} \hat{\mathbf{f}}^f \\ 0 \end{pmatrix}, \tag{D12}$$

$$\bar{\mathbf{A}}_{11}=\begin{pmatrix} \boldsymbol{\epsilon} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{\mu} \end{pmatrix}, \quad \bar{\mathbf{A}}_{22}=\begin{pmatrix} \varrho^b & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}, \tag{D13}$$

$$\bar{\mathbf{A}}_{23}=\begin{pmatrix} \varrho^f & \mathbf{0} \\ \mathbf{O} & \mathbf{0} \\ \mathbf{O} & \mathbf{0} \\ \mathbf{O} & \mathbf{0} \end{pmatrix}, \quad \bar{\mathbf{A}}_{33}=\begin{pmatrix} \mathbf{O} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix},$$

$$\overline{\mathbf{B}}_{11}=\begin{pmatrix}(\hat{\boldsymbol{\sigma}}-\eta\hat{\mathbf{L}}\hat{\mathbf{k}}^{-1}\hat{\mathbf{L}}) & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix},\quad \overline{\mathbf{B}}_{13}=\begin{pmatrix}\eta\hat{\mathbf{L}}\hat{\mathbf{k}}^{-1} & \mathbf{0}\\ \mathbf{O} & \mathbf{0}\end{pmatrix},$$
$$\overline{\mathbf{B}}_{33}=\begin{pmatrix}\eta\hat{\mathbf{k}}^{-1} & \mathbf{0}\\ \mathbf{0}^T & 0\end{pmatrix}, \tag{D14}$$

$$\mathbf{C}_{22}=\begin{pmatrix}\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & \mathbf{c}_{11} & \mathbf{c}_{12} & \mathbf{c}_{13}\\ \mathbf{O} & \mathbf{c}_{21} & \mathbf{c}_{22} & \mathbf{c}_{23}\\ \mathbf{O} & \mathbf{c}_{31} & \mathbf{c}_{32} & \mathbf{c}_{33}\end{pmatrix},\quad \mathbf{C}_{23}=\begin{pmatrix}\mathbf{O} & \mathbf{0}\\ \mathbf{O} & \mathbf{d}_1\\ \mathbf{O} & \mathbf{d}_2\\ \mathbf{O} & \mathbf{d}_3\end{pmatrix},$$
$$\mathbf{C}_{33}=\begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0}^T & M\end{pmatrix}, \tag{D15}$$

$$\mathbf{D}_{11}=\begin{pmatrix}\mathbf{O} & \mathbf{D}_0^T\\ \mathbf{D}_0 & \mathbf{O}\end{pmatrix},\quad \mathbf{D}_0=\begin{pmatrix}0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2}\\ \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1}\\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0\end{pmatrix},$$
$$\mathbf{D}_{33}=\begin{pmatrix}\mathbf{O} & \boldsymbol{\nabla}\\ \boldsymbol{\nabla}^T & 0\end{pmatrix} \tag{D16}$$

and $\mathbf{D}_{22}$ equal to $\mathbf{D}_\mathbf{x}$ in Eq. (C9). Multiplying all terms in Eq. (D9) by the inverse of $\mathbf{C}$ finally yields

$$j\omega\mathbf{A}\hat{\mathbf{u}}+\mathbf{B}\hat{\mathbf{u}}+\mathbf{D}_\mathbf{x}\hat{\mathbf{u}}=\hat{\mathbf{s}}, \tag{D17}$$

with $\mathbf{A}=\mathbf{C}^{-1}\overline{\mathbf{A}}$, $\mathbf{B}=\mathbf{C}^{-1}\overline{\mathbf{B}}=\overline{\mathbf{B}}$ and $\hat{\mathbf{s}}=\mathbf{C}^{-1}\hat{\overline{\mathbf{s}}}=\hat{\overline{\mathbf{s}}}$. Matrices $\mathbf{N}_\mathbf{x}$ and $\mathbf{K}$, appearing in the modified divergence theorems (6) and (8), read

$$\mathbf{N}_\mathbf{x}=\begin{pmatrix}\mathbf{N}_{11} & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & \mathbf{N}_{22} & \mathbf{O}\\ \mathbf{O} & \mathbf{O} & \mathbf{N}_{33}\end{pmatrix},\quad \mathbf{N}_{11}=\begin{pmatrix}\mathbf{O} & \mathbf{N}_0^T\\ \mathbf{N}_0 & \mathbf{O}\end{pmatrix},$$
$$\mathbf{N}_0=\begin{pmatrix}0 & -n_3 & n_2\\ n_3 & 0 & -n_1\\ -n_2 & n_1 & 0\end{pmatrix},\quad \mathbf{N}_{33}=\begin{pmatrix}\mathbf{O} & \mathbf{n}\\ \mathbf{n}^T & 0\end{pmatrix}, \tag{D18}$$

$$\mathbf{K}=\mathrm{diag}(-\mathbf{1},\mathbf{1},\mathbf{1},-\mathbf{1},-\mathbf{1},-\mathbf{1},\mathbf{1},-1), \tag{D19}$$

and $\mathbf{N}_{22}$ equal to $\mathbf{N}_x$ in Eq. (C12). Based on the structure of the matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$, $\mathbf{C}$, $\mathbf{D}_x$, and $\mathbf{K}$ as well as the symmetry relations discussed above, we find that the symmetry properties described by Eqs. (7), (9), and (13) are obeyed.

Finally, note that when the coupling tensor $\hat{\mathbf{L}}$ is zero, the matrix $\overline{\mathbf{B}}_{13}$ vanishes and hence equation (D9) decouples into the electromagnetic wave equation for the wave vector $\hat{\mathbf{u}}_1$ and Biot's poroelastic wave equation for the wave vector $(\hat{\mathbf{u}}_2^T,\hat{\mathbf{u}}_3^T)^T$, [11]. For a nonporous solid the matrices $\overline{\mathbf{A}}_{23}$ and $\mathbf{C}_{23}$ vanish as well, so Biot's wave equation reduces to the elastodynamic wave equation for the wave vector $\hat{\mathbf{u}}_2$. Obviously the symmetry properties described by Eqs. (7), (9), and (13) are obeyed for the matrices appearing in the electromagnetic wave equation, Biot's poroelastic wave equation and the elastodynamic wave equation, respectively.

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Erratum: "Reciprocity Theorems for Diffusion, Flow, and Waves" [Journal of Applied Mechanics, 2004, 71(1), pp. 145–150]

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Matrix $\mathbf{C}$ defined in Appendices C and D is singular and hence expressions containing the inverse of $\mathbf{C}$ cannot be used as such. The singularity is a consequence of the chosen organization of the matrix-vector differential equation in these appendices. The field vector $\mathbf{u}$ contains nine stress components of which only six are independent. By removing the three redundant stress components from $\mathbf{u}$ and reorganizing the matrix-vector equation accordingly, we obtain a matrix $\mathbf{C}$ that is invertible. The redefined matrices $\mathbf{A}=\mathbf{C}^{-1}\overline{\mathbf{A}}$ and $\mathbf{B}=\mathbf{C}^{-1}\overline{\mathbf{B}}$ in Appendices C and D obey symmetry relations (9) and (13) in the body of the paper. Hence, the unified reciprocity theorems (12) and (14) are valid for the modified matrix-vector differential equation in these appendices. Explicit expressions for the modified matrices and vectors can be found at http://geodus1.ta.tudelft.nl/PrivatePages/C.P.A.Wapenaar/4_Journals/J.Appl.Mech/AppM_04.pdf.

We take this opportunity to indicate some printing errors in the paper. The tildes below $\mathbf{A}$ and $\mathbf{u}$ in Eq. (1) should be removed. Circumflexes should be added above all vectors $\mathbf{u}$ and $\mathbf{s}$ in Eqs. (10) and (11). A right-bracket] should be inserted after the first $\hat{\mathbf{u}}_B$ at the right-hand side of Eq. (10). Right-parentheses) should be inserted after $\hat{\mathbf{u}}_A$ at the left-hand side of Eq. (11) and after the first $\hat{\mathbf{u}}_B$ at the right-hand side of Eq. (11).

We thank Stefan Stijlen for bringing the singularity of matrix $\mathbf{C}$ to our attention.