Reciprocity theorems for diffusion, flow and waves*

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Diffusion, flow and wave phenomena can each be captured by a unified differential equation in matrix-vector form. This equation forms the basis for the derivation of unified reciprocity theorems for diffusion, flow and wave phenomena.

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I. INTRODUCTION

Diffusion, flow and wave phenomena can each be captured by the following differential equation in matrix-vector form

$$\mathbf{A}\frac{D\mathbf{u}}{Dt} + \mathbf{B}\mathbf{u} + \mathbf{D_x}\mathbf{u} = \mathbf{s}, \tag{1}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is a vector containing space- and time-dependent field quantities, $\mathbf{s} = \mathbf{s}(\mathbf{x}, t)$ is a source vector, $\mathbf{A} = \mathbf{A}(\mathbf{x})$ and $\mathbf{B} = \mathbf{B}(\mathbf{x})$ are matrices containing space-dependent material parameters and $\mathbf{D_x}$ is a matrix containing the spatial differential operators $\partial/\partial x_1$, $\partial/\partial x_2$ and $\partial/\partial x_3$. Finally, D/Dt denotes the material time derivative, defined as $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} = \frac{\partial}{\partial t} + v_k\frac{\partial}{\partial x_k}$, where $\partial/\partial t$ denotes the time derivative in the reference frame and $\mathbf{v} = \mathbf{v}(\mathbf{x})$ is the space-dependent flow velocity of the material; v_k denotes the kth component of $\mathbf{v}$. Throughout this paper the summation convention applies to repeated subscripts; lower-case Latin subscripts run from 1 to 3. The vectors and matrices in equation (1) are further defined in the appendices for diffusion (Appendix A), acoustic wave propagation in moving fluids (Appendix B), momentum transport (Appendix C), elastodynamic wave propagation in solids (Appendix D) electromagnetic diffusion and wave propagation (Appendix E), and coupled elastodynamic and electromagnetic wave propagation in porous solids (Appendix F). In this paper we use equation (1) as the basis for deriving unified reciprocity theorems for these phenomena. In general, a reciprocity theorem interrelates the quantities that characterize two admissible physical states that could occur in one and the same domain [1]. One can distinguish between convolution type and correlation type reciprocity theorems [2]. Generally speaking, these two types of reciprocity theorems find their applications in forward and inverse problems, respectively. Both types of reciprocity theorems will be derived for the field vector $\mathbf{u}$.

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II. THE DIFFERENTIAL EQUATION IN THE FREQUENCY DOMAIN

Reciprocity theorems can be derived in the time domain, the Laplace domain and the frequency domain [3]. Here we only consider the frequency domain. We define the Fourier transform of a time-dependent function $f(t)$ as $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)\mathrm{d}t$, where j is the imaginary unit and ω denotes the angular frequency. We apply the Fourier transform to all terms in equation (1), under the assumption that this equation is linear in $\mathbf{u}$. Hence, we only consider those cases in which the field quantities in $\mathbf{u}$ do not appear in any of the matrices or operators in equation (1). In particular, this is why the term $D\mathbf{u}/Dt$ in the momentum transport equation (C15) is replaced by $\partial\mathbf{u}/\partial t$ in (C20). Transforming equation (1) to the frequency domain yields

$$\mathbf{A}\Big(j\omega + \mathbf{v}\cdot\boldsymbol{\nabla}\Big)\hat{\mathbf{u}} + \mathbf{B}\hat{\mathbf{u}} + \mathbf{D_x}\hat{\mathbf{u}} = \hat{\mathbf{s}}, \tag{2}$$

where $\hat{\mathbf{u}} = \hat{\mathbf{u}}(\mathbf{x}, \omega)$ is the space- and frequency dependent field vector and $\hat{\mathbf{s}} = \hat{\mathbf{s}}(\mathbf{x}, \omega)$ is the space- and frequency dependent source vector. The term $\mathbf{v}\cdot\boldsymbol{\nabla}$ should be dropped for linearized momentum transport (Appendix C) as well as for wave phenomena in non-moving media (Appendices D − F). Finally we remark that in a number of cases matrix $\mathbf{B}$ contains temporal convolution kernels in the time domain (Appendix B) or, equivalently, complex frequency-dependent material parameters in the frequency domain (Appendices B and F).

III. MODIFICATION OF GAUSS' DIVERGENCE THEOREM

The reciprocity theorem will be derived for a volume $\mathcal{V}$ enclosed by surface $\partial\mathcal{V}$ with outward pointing normal vector $\mathbf{n}$. Note that $\partial\mathcal{V}$ does not necessarily coincide with a physical boundary. Gauss' divergence theorem plays a central role in the derivation. For a scalar field $a(\mathbf{x})$, this theorem reads

$$\int_{\mathcal{V}} \frac{\partial a(\mathbf{x})}{\partial x_i}\,\mathrm{d}^3\mathbf{x} = \oint_{\partial\mathcal{V}} a(\mathbf{x})n_i\,\mathrm{d}^2\mathbf{x}, \tag{3}$$

where n_i denotes the ith component of $\mathbf{n}$. In this section we will modify this theorem for the differential operator matrix $\mathbf{D_x}$ appearing in equations (1) and (2). Note that $\mathbf{D_x} = \mathbf{D_x}^T$ for all forms of $\mathbf{D_x}$ appearing in the appendices (here superscript ^{T} denotes matrix transposition only; it does not denote operator transposition). Let D_{IJ} denote the operator in row I and column J of matrix $\mathbf{D_x}$. The symmetry of $\mathbf{D_x}$ implies $D_{IJ} = D_{JI}$. We define a matrix $\mathbf{N_x}$ which contains the components of the normal vector $\mathbf{n}$, organized in a similar way as matrix $\mathbf{D_x}$, see the appendices for details. Hence, if N_{IJ} denotes the element in row I and column J of matrix $\mathbf{N_x}$, we have $N_{IJ} = N_{JI}$. For example, for matrices $\mathbf{D_x}$ and $\mathbf{N_x}$ in equations (A4) and (A6) we have $D_{12} = D_{21} = \partial/\partial x_1$ and $N_{12} = N_{21} = n_1$. If we now replace the scalar field $a(\mathbf{x})$ by $a_I(\mathbf{x})b_J(\mathbf{x})$, we may generalize equation (3) to

$$\int_{\mathcal{V}} D_{IJ}[a_I(\mathbf{x})b_J(\mathbf{x})]\,\mathrm{d}^3\mathbf{x} = \oint_{\partial\mathcal{V}} a_I(\mathbf{x})b_J(\mathbf{x})N_{IJ}\,\mathrm{d}^2\mathbf{x}, \quad (4)$$

where the summation convention applies for repeated capital Latin subscripts (which may run from 1 to 4, 6, 9 or 19, depending on the choice of operator $\mathbf{D_x}$). Applying the product rule for differentiation and using the symmetry property $D_{IJ} = D_{JI}$, we obtain for the integrand in the left-hand side of equation (4)

$$\begin{aligned} D_{IJ}(a_Ib_J) &= a_ID_{IJ}b_J + (D_{JI}a_I)b_J \\ &= \mathbf{a}^T\mathbf{D_x}\mathbf{b} + (\mathbf{D_x}\mathbf{a})^T\mathbf{b}, \end{aligned} \quad (5)$$

where $\mathbf{a}$ and $\mathbf{b}$ are vector functions, containing the scalar functions $a_I(\mathbf{x})$ and $b_J(\mathbf{x})$, respectively. Rewriting the integrand in the right-hand side of equation (4) in a similar way, we thus obtain

$$\int_{\mathcal{V}} [\mathbf{a}^T\mathbf{D_x}\mathbf{b} + (\mathbf{D_x}\mathbf{a})^T\mathbf{b}]\,\mathrm{d}^3\mathbf{x} = \oint_{\partial\mathcal{V}} \mathbf{a}^T\mathbf{N_x}\mathbf{b}\,\mathrm{d}^2\mathbf{x}. \quad (6)$$

Finally we consider a variant of this equation. We replace $\mathbf{a}$ by $\mathbf{Ka}$, where $\mathbf{K}$ is a diagonal matrix with the following property

$$\mathbf{D_x}\mathbf{K} = -\mathbf{K}\mathbf{D_x}, \quad (7)$$

see the appendices for details. With this replacement, equation (6) becomes

$$\int_{\mathcal{V}} [\mathbf{a}^T\mathbf{K}\mathbf{D_x}\mathbf{b} - (\mathbf{D_x}\mathbf{a})^T\mathbf{K}\mathbf{b}]\,\mathrm{d}^3\mathbf{x} = \oint_{\partial\mathcal{V}} \mathbf{a}^T\mathbf{K}\mathbf{N_x}\mathbf{b}\,\mathrm{d}^2\mathbf{x}. \quad (8)$$

IV. RECIPROCITY THEOREM OF THE CONVOLUTION TYPE

We consider two physical states in volume $\mathcal{V}$. The field quantities, the material parameters, the flow velocity as well as the source functions may be different in both states and they will be distinguished with subscripts A and B (of course the summation convention does not apply for these subscripts). We substitute $\hat{\mathbf{u}}_A$ and $\hat{\mathbf{u}}_B$ for $\mathbf{a}$ and $\mathbf{b}$ in equation (8), apply equation (2) for states A and B and use the symmetry properties

$$\mathbf{A}_A^T\mathbf{K} = \mathbf{K}\mathbf{A}_A \quad \text{and} \quad \mathbf{B}_A^T\mathbf{K} = \mathbf{K}\mathbf{B}_A \quad (9)$$

(see the appendices). This yields

$$\begin{aligned} \oint_{\partial\mathcal{V}} \hat{\mathbf{u}}_A^T\mathbf{K}\mathbf{N_x}\hat{\mathbf{u}}_B\,\mathrm{d}^2\mathbf{x} &= \int_{\mathcal{V}} \Big[\hat{\mathbf{u}}_A^T\mathbf{K}\hat{\mathbf{s}}_B - \hat{\mathbf{s}}_A^T\mathbf{K}\hat{\mathbf{u}}_B\Big]\,\mathrm{d}^3\mathbf{x} \\ &+ \int_{\mathcal{V}} \hat{\mathbf{u}}_A^T\mathbf{K}\big[j\omega(\mathbf{A}_A - \mathbf{A}_B) + (\mathbf{B}_A - \mathbf{B}_B)\big]\hat{\mathbf{u}}_B\,\mathrm{d}^3\mathbf{x} + \\ &\int_{\mathcal{V}} \Big[\big((\mathbf{v}_A\cdot\boldsymbol{\nabla})\hat{\mathbf{u}}_A\big)^T\mathbf{K}\mathbf{A}_A - \hat{\mathbf{u}}_A^T\mathbf{K}\mathbf{A}_B(\mathbf{v}_B\cdot\boldsymbol{\nabla})\Big]\hat{\mathbf{u}}_B\,\mathrm{d}^3\mathbf{x}. \end{aligned} \quad (10)$$

The first term in the last integral can be written as

$$\begin{aligned} \big((\mathbf{v}_A\cdot\boldsymbol{\nabla})\hat{\mathbf{u}}_A\big)^T\mathbf{K}\mathbf{A}_A\hat{\mathbf{u}}_B &= \boldsymbol{\nabla}\cdot\big(\mathbf{v}_A\hat{\mathbf{u}}_A^T\mathbf{K}\mathbf{A}_A\hat{\mathbf{u}}_B\big) \\ &-\hat{\mathbf{u}}_A^T\mathbf{K}\frac{\partial(v_{i,A}\mathbf{A}_A)}{\partial x_i}\hat{\mathbf{u}}_B - \hat{\mathbf{u}}_A^T\mathbf{K}\mathbf{A}_A(\mathbf{v}_A\cdot\boldsymbol{\nabla})\hat{\mathbf{u}}_B. \end{aligned} \quad (11)$$

For diffusion (Appendix A) the term $\partial(v_{i,A}\mathbf{A}_A)/\partial x_i$ vanishes on account of the equation of continuity. For acoustic wave propagation this term (with $v_{i,A}$ replaced by $v^0_{i,A}$, Appendix B) is negligible in comparison with the spatial derivatives of the wave fields $\hat{\mathbf{u}}_A$ and $\hat{\mathbf{u}}_B$. For the other situations considered in the appendices, $v_{i,A}$ is taken equal to zero. Hence, the term containing $\partial(v_{i,A}\mathbf{A}_A)/\partial x_i$ will be dropped. Substituting the remainder of the right-hand side of equation (11) into equation (10) and applying the theorem of Gauss for the term containing the divergence operator, yields

$$\begin{aligned} \oint_{\partial\mathcal{V}} \hat{\mathbf{u}}_A^T\mathbf{K}\mathbf{N_x}\hat{\mathbf{u}}_B\,\mathrm{d}^2\mathbf{x} &= \int_{\mathcal{V}} \Big[\hat{\mathbf{u}}_A^T\mathbf{K}\hat{\mathbf{s}}_B - \hat{\mathbf{s}}_A^T\mathbf{K}\hat{\mathbf{u}}_B\Big]\,\mathrm{d}^3\mathbf{x} \\ &+ \int_{\mathcal{V}} \hat{\mathbf{u}}_A^T\mathbf{K}\big[j\omega(\mathbf{A}_A - \mathbf{A}_B) + (\mathbf{B}_A - \mathbf{B}_B)\big]\hat{\mathbf{u}}_B\,\mathrm{d}^3\mathbf{x} \\ &- \int_{\mathcal{V}} \hat{\mathbf{u}}_A^T\mathbf{K}\big[\mathbf{A}_A(\mathbf{v}_A\cdot\boldsymbol{\nabla}) + \mathbf{A}_B(\mathbf{v}_B\cdot\boldsymbol{\nabla})\big]\hat{\mathbf{u}}_B\,\mathrm{d}^3\mathbf{x} \\ &+ \oint_{\partial\mathcal{V}} \big(\hat{\mathbf{u}}_A^T\mathbf{K}\mathbf{A}_A\hat{\mathbf{u}}_B\big)\mathbf{v}_A\cdot\mathbf{n}\,\mathrm{d}^2\mathbf{x}. \end{aligned} \quad (12)$$

This is the unified reciprocity theorem of the convolution type (we speak of convolution type, because the multiplications in the frequency domain correspond to convolutions in the time domain). It interrelates the field quantities (contained in $\hat{\mathbf{u}}_A$ and $\hat{\mathbf{u}}_B$), the material parameters (contained in $\mathbf{A}_A$, $\mathbf{B}_A$, $\mathbf{A}_B$ and $\mathbf{B}_B$), the flow velocities

($\mathbf{v}_A$ and $\mathbf{v}_B$) as well as the source functions (contained in $\hat{\mathbf{s}}_A$ and $\hat{\mathbf{s}}_B$) of states A and B. The left-hand side is a boundary integral which contains a specific combination of the field quantities of states A and B at the boundary of the volume $\mathcal{V}$. The first integral on the right-hand side interrelates the field quantities and the source functions in $\mathcal{V}$. The second integral contains the differences of the medium parameters in both states; obviously this integral vanishes when the medium parameters in both states are identical. The third integral on the right-hand side contains the flow velocities in $\mathcal{V}$; this integral vanishes when the medium parameters in both states are identical and the flow velocities in both states are opposite to each other. The last integral on the right-hand side is a boundary integral containing the normal component of the flow velocity in state A; it vanishes when this flow velocity is tangential to the boundary $\partial\mathcal{V}$. Depending on the type of application, states A and B can be both physical states, or both mathematical states (e.g. Green's states), or one can be a physical state and the other a mathematical state (the latter situation leads to representation integrals). For further discussions on convolution type reciprocity theorems in different fields of application we refer to Lyamshev [4], De Hoop and Stam [1], Fokkema and Van den Berg [3], Allard et al. [5], Pride and Haartsen [6] and Belinskiy [7].

V. RECIPROCITY THEOREM OF THE CORRELATION TYPE

We substitute $\hat{\mathbf{u}}_A^*$ and $\hat{\mathbf{u}}_B$ for $\mathbf{a}$ and $\mathbf{b}$ in equation (6), where * denotes complex conjugation. Following the same procedure as in the previous section, using the symmetry property

$$\mathbf{A}_A^H = \mathbf{A}_A, \tag{13}$$

where ^{H} denotes complex conjugation and transposition, we obtain

$$\begin{aligned}\oint_{\partial\mathcal{V}} \hat{\mathbf{u}}_A^H \mathbf{N_x} \hat{\mathbf{u}}_B \,\mathrm{d}^2\mathbf{x} &= \int_{\mathcal{V}} \Big[\hat{\mathbf{u}}_A^H \hat{\mathbf{s}}_B + \hat{\mathbf{s}}_A^H \hat{\mathbf{u}}_B\Big] \,\mathrm{d}^3\mathbf{x} \\ &+ \int_{\mathcal{V}} \hat{\mathbf{u}}_A^H \big[j\omega(\mathbf{A}_A - \mathbf{A}_B) - (\mathbf{B}_A^H + \mathbf{B}_B)\big] \hat{\mathbf{u}}_B \,\mathrm{d}^3\mathbf{x} \\ &+ \int_{\mathcal{V}} \hat{\mathbf{u}}_A^H \big[\mathbf{A}_A(\mathbf{v}_A \cdot \boldsymbol{\nabla}) - \mathbf{A}_B(\mathbf{v}_B \cdot \boldsymbol{\nabla})\big] \hat{\mathbf{u}}_B \,\mathrm{d}^3\mathbf{x} \\ &- \oint_{\partial\mathcal{V}} \big(\hat{\mathbf{u}}_A^H \mathbf{A}_A \hat{\mathbf{u}}_B\big) \mathbf{v}_A \cdot \mathbf{n} \,\mathrm{d}^2\mathbf{x}.\end{aligned} \tag{14}$$

This is the unified reciprocity theorem of the correlation type (we speak of correlation type, because the multiplications in the frequency domain correspond to correlations in the time domain). The term $\hat{\mathbf{u}}_A^H$ contains 'back-propagating' field quantities in state A, [2]. When we compare this reciprocity theorem with equation (12), we observe that, apart from the complex conjugation, the diagonal matrix $\mathbf{K}$ is absent in all integrals and that some plus and minus signs have been changed. In particular, the term $(\mathbf{B}_A - \mathbf{B}_B)$ has been replaced by $(\mathbf{B}_A^H + \mathbf{B}_B)$, which means that the second integral on the right-hand side no longer vanishes when the medium parameters in both states are identical. Moreover, the term $\big[\mathbf{A}_A(\mathbf{v}_A \cdot \boldsymbol{\nabla}) + \mathbf{A}_B(\mathbf{v}_B \cdot \boldsymbol{\nabla})\big]$ has been replaced by $\big[\mathbf{A}_A(\mathbf{v}_A \cdot \boldsymbol{\nabla}) - \mathbf{A}_B(\mathbf{v}_B \cdot \boldsymbol{\nabla})\big]$, which means that the third integral on the right-hand side vanishes when the medium parameters contained in matrix $\mathbf{A}$ as well as the flow velocities are identical in both states. For a discussion on the application of correlation type reciprocity theorems to inverse problems we refer to Fisher and Langenberg [8] and De Hoop and Stam [1].

VI. CONCLUSIONS

We have formulated a general differential equation in matrix-vector form [equation (1)], which applies to diffusion (Appendix A), acoustic wave propagation in moving fluids (Appendix B), momentum transport (Appendix C), elastodynamic wave propagation in solids (Appendix D) electromagnetic diffusion and wave propagation (Appendix E) and coupled elastodynamic and electromagnetic wave propagation in fluid-saturated porous solids (Appendix F). For linear phenomena (which excludes non-linear momentum transport) we have transformed the general equation from the time domain to the frequency domain [equation (2)]. Based on this general equation as well as the symmetry properties (7), (9) and (13) we have derived unified reciprocity theorems of the convolution type [equation (12)] and of the correlation type [equation (14)], respectively.

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APPENDIX A: MASS DIFFUSION

The equation of continuity for species k in a mixture of fluids reads

$$\rho \frac{DY^{(k)}}{Dt} + \frac{\partial J_j^{(k)}}{\partial x_j} = \dot{\omega}^{(k)}, \tag{A1}$$

where $Y^{(k)}$ is the mass fraction of species k, $J_j^{(k)}$ its mass flux relative to the mixture, ρ is the mass density of the mixture and $\dot{\omega}^{(k)}$ the mass production rate density of species k (due to chemical reactions). Fick's first law of diffusion reads

$$J_j^{(k)} + \rho \mathcal{D}^{(k)} \frac{\partial Y^{(k)}}{\partial x_j} = 0, \tag{A2}$$

where $\mathcal{D}^{(k)}$ is the diffusion coefficient for species k. Eliminating $J_j^{(k)}$ from equations (A1) and (A2) yields the mass diffusion equation, according to

$$\rho \frac{DY^{(k)}}{Dt} - \frac{\partial}{\partial x_j} \Big(\rho \mathcal{D}^{(k)} \frac{\partial Y^{(k)}}{\partial x_j} \Big) = \dot{\omega}^{(k)}. \tag{A3}$$

On the other hand, equations (A1) and (A2) can be combined to yield equation (1), with

$$\mathbf{u} = \begin{pmatrix} Y^{(k)} \\ J_1^{(k)} \\ J_2^{(k)} \\ J_3^{(k)} \end{pmatrix}, \mathbf{s} = \begin{pmatrix} \dot{\omega}^{(k)} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{D_x} = \begin{pmatrix} 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & 0 & 0 & 0 \\ \frac{\partial}{\partial x_3} & 0 & 0 & 0 \end{pmatrix}, \tag{A4}$$

$$\mathbf{A} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\rho \mathcal{D}^{(k)}} & 0 & 0 \\ 0 & 0 & \frac{1}{\rho \mathcal{D}^{(k)}} & 0 \\ 0 & 0 & 0 & \frac{1}{\rho \mathcal{D}^{(k)}} \end{pmatrix}. \tag{A5}$$

Matrices $\mathbf{N_x}$ and $\mathbf{K}$, appearing in the modified divergence theorems (6) and (8), read

$$\mathbf{N_x} = \begin{pmatrix} 0 & n_1 & n_2 & n_3 \\ n_1 & 0 & 0 & 0 \\ n_2 & 0 & 0 & 0 \\ n_3 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{K} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A6}$$

Note that matrices $\mathbf{D_x}$, $\mathbf{A}$ and $\mathbf{B}$ obey equations (7), (9) and (13). Other diffusion phenomena can be formulated in a similar way.

APPENDIX B: ACOUSTIC WAVE PROPAGATION IN A MOVING FLUID

The linearized equation of motion in a moving fluid reads

$$\rho \frac{Dv_i}{Dt} + b^v * v_i + \frac{\partial p}{\partial x_i} = f_i, \tag{B1}$$

with

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_k^0 \frac{\partial}{\partial x_k}, \tag{B2}$$

where p is the acoustic pressure, v_i the particle velocity associated to the acoustic wave motion (which is to be distinguished from the flow velocity v_k^0 in the operator D/Dt), ρ the mass density of the medium in equilibrium, f_i the volume density of external force and b^v a causal loss function ($*$ denotes a temporal convolution). The spatial variations of the flow velocity are assumed small in comparison with those of the particle velocity of the acoustic wave field, i.e. $\partial v_i^0 / \partial x_j << \partial v_i / \partial x_j$ (this assumption can be relaxed, but then the equations become more involved [9]). The linearized stress-strain relation reads

$$\kappa \frac{Dp}{Dt} + b^p * p + \frac{\partial v_i}{\partial x_i} = q, \tag{B3}$$

where κ is the compressibility, q the volume injection rate density and b^p a causal loss function.

Eliminating v_i from equations (B1) and (B3) for the lossless situation ($b^v = b^p = 0$), yields the acoustic wave equation, according to

$$\frac{D}{Dt} \Big(\kappa \frac{Dp}{Dt} \Big) - \frac{\partial}{\partial x_i} \Big(\frac{1}{\rho} \frac{\partial p}{\partial x_i} \Big) = - \frac{\partial}{\partial x_i} \Big(\frac{f_i}{\rho} \Big) + \frac{Dq}{Dt}. \tag{B4}$$

On the other hand, equations (B1) and (B3) can be combined to yield the general matrix-vector equation (1), with D/Dt defined in equation (B2) and

$$\mathbf{u} = \begin{pmatrix} p \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}, \mathbf{s} = \begin{pmatrix} q \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & \rho \end{pmatrix}$$

$$\text{and} \quad \mathbf{B} = \begin{pmatrix} b^p * & 0 & 0 & 0 \\ 0 & b^v * & 0 & 0 \\ 0 & 0 & b^v * & 0 \\ 0 & 0 & 0 & b^v * \end{pmatrix}. \tag{B5}$$

Matrices $\mathbf{D_x}$, $\mathbf{N_x}$ and $\mathbf{K}$ are the same as in Appendix A. The symmetry properties described by equations (7), (9) and (13) are easily confirmed. Finally, note that in the frequency domain formulation, the temporal convolution kernels $b^p(\mathbf{x}, t)*$ and $b^v(\mathbf{x}, t)*$ in matrix $\mathbf{B}$ are replaced by complex frequency-dependent functions $\hat{b}^p(\mathbf{x}, \omega)$ and $\hat{b}^v(\mathbf{x}, \omega)$, respectively.

APPENDIX C: MOMENTUM TRANSPORT

The non-linear equation of motion for a viscous fluid reads [10, 11]

$$\rho \frac{Dv_i}{Dt} - \frac{\partial \tau_{ij}}{\partial x_j} = f_i, \tag{C1}$$

where v_i is the particle velocity, τ_{ij} the stress tensor, ρ the mass density and f_i the volume density of external force. The stress tensor is symmetric, i.e., $\tau_{ij} = \tau_{ji}$.

Stoke's stress-strainrate relation reads

$$-\tau_{ij} + \eta_{ijkl} \frac{\partial v_k}{\partial x_l} = p \delta_{ij}, \tag{C2}$$

where η_{ijkl} is the anisotropic viscosity tensor and p the hydrostatic pressure. The viscosity tensor obeys the following symmetry relation

$$\eta_{ijkl} = \eta_{jikl} = \eta_{ijlk} = \eta_{klij}. \tag{C3}$$

For isotropic fluids the viscosity tensor reads

$$\eta_{ijkl} = \big(\zeta - \tfrac{2}{3}\eta\big)\delta_{ij}\delta_{kl} + \eta\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big), \tag{C4}$$

where η is the dynamic viscosity coefficient and ζ a volume viscosity coefficient (or second viscosity). From energy considerations it follows that η is positive and from entropy considerations that ζ is positive [12]. Eliminating τ_{ij} from equations (C1) and (C2) yields the Navier-Stokes equation, according to

$$\rho\frac{Dv_i}{Dt} - \frac{\partial}{\partial x_i}\Big(\big(\zeta - \tfrac{2}{3}\eta\big)\frac{\partial v_k}{\partial x_k}\Big)$$

$$-\frac{\partial}{\partial x_j}\left[\eta\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)\right] = f_i - \frac{\partial p}{\partial x_i}, \tag{C5}$$

or, for constant η and ζ,

$$\rho\frac{D\mathbf{v}}{Dt} - \big(\zeta + \tfrac{1}{3}\eta\big)\nabla(\nabla\cdot\mathbf{v}) - \eta\nabla^2\mathbf{v} + \nabla p = \mathbf{f}, \tag{C6}$$

where

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad\text{and}\quad \mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{C7}$$

On the other hand, equations (C1) and (C2) can be combined to yield the general matrix-vector equation (1). To this end we first rewrite these equations as

$$\rho\frac{D\mathbf{v}}{Dt} - \mathbf{D}_\alpha\boldsymbol{\tau}_\alpha \;=\; \mathbf{f}, \tag{C8}$$

$$-\boldsymbol{\tau}_\alpha + \mathbf{h}_{\alpha\beta}\mathbf{D}_\beta\mathbf{v} \;=\; p\boldsymbol{\delta}_\alpha, \tag{C9}$$

(lower case Greek subscripts take on the values 1 and 2), where

$$\boldsymbol{\tau}_1 = \begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \end{pmatrix}, \quad \boldsymbol{\tau}_2 = \begin{pmatrix} \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{pmatrix}, \tag{C10}$$

$$\boldsymbol{\delta}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{\delta}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{C11}$$

$$\mathbf{h}_{11} = \begin{pmatrix} \eta_{1111} & \eta_{1122} & \eta_{1133} \\ \eta_{1122} & \eta_{2222} & \eta_{2233} \\ \eta_{1133} & \eta_{2233} & \eta_{3333} \end{pmatrix}, \mathbf{h}_{12} = \begin{pmatrix} \eta_{1123} & \eta_{1131} & \eta_{1112} \\ \eta_{2223} & \eta_{2231} & \eta_{2212} \\ \eta_{3323} & \eta_{3331} & \eta_{3312} \end{pmatrix} \tag{C12}$$

$$\mathbf{h}_{21} = \mathbf{h}_{12}^T, \quad \mathbf{h}_{22} = \begin{pmatrix} \eta_{2323} & \eta_{2331} & \eta_{2312} \\ \eta_{2331} & \eta_{3131} & \eta_{3112} \\ \eta_{2312} & \eta_{3112} & \eta_{1212} \end{pmatrix}, \tag{C13}$$

and

$$\mathbf{D}_1 = \begin{pmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \end{pmatrix}, \mathbf{D}_2 = \begin{pmatrix} 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{pmatrix}. \tag{C14}$$

Hence, we obtain

$$\bar{\mathbf{A}}\frac{D\mathbf{u}}{Dt} + \bar{\mathbf{B}}\mathbf{u} + \mathbf{C}\mathbf{D}_\mathbf{x}\mathbf{u} = \bar{\mathbf{s}}, \tag{C15}$$

with

$$\mathbf{u} = \begin{pmatrix} \mathbf{v} \\ -\boldsymbol{\tau}_1 \\ -\boldsymbol{\tau}_2 \end{pmatrix}, \bar{\mathbf{s}} = \begin{pmatrix} \mathbf{f} \\ p\boldsymbol{\delta}_1 \\ \mathbf{0} \end{pmatrix}, \tag{C16}$$

$$\bar{\mathbf{A}} = \begin{pmatrix} \rho\mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}, \bar{\mathbf{B}} = \begin{pmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}, \tag{C17}$$

$$\mathbf{C} = \begin{pmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{h}_{11} & \mathbf{h}_{12} \\ \mathbf{O} & \mathbf{h}_{21} & \mathbf{h}_{22} \end{pmatrix}, \mathbf{D}_\mathbf{x} = \begin{pmatrix} \mathbf{O} & \mathbf{D}_1 & \mathbf{D}_2 \\ \mathbf{D}_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{D}_2 & \mathbf{O} & \mathbf{O} \end{pmatrix}, \tag{C18}$$

$\mathbf{I}$ being the 3×3 identity matrix and $\mathbf{O}$ the 3×3 null matrix. For the isotropic situation we have

$$\mathbf{h}_{11} = \begin{pmatrix} \zeta + \frac{4}{3}\eta & \zeta - \frac{2}{3}\eta & \zeta - \frac{2}{3}\eta \\ \zeta - \frac{2}{3}\eta & \zeta + \frac{4}{3}\eta & \zeta - \frac{2}{3}\eta \\ \zeta - \frac{2}{3}\eta & \zeta - \frac{2}{3}\eta & \zeta + \frac{4}{3}\eta \end{pmatrix}, \mathbf{h}_{22} = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta \end{pmatrix}, \tag{C19}$$

and $\mathbf{h}_{12} = \mathbf{h}_{21} = \mathbf{O}$. The determinant of $\mathbf{C}$ equals $12\zeta\eta^5$. Since $\eta > 0$ and $\zeta > 0$ the inverse of $\mathbf{C}$ exists for the isotropic situation. Whether the inverse of $\mathbf{C}$ exists in general for anisotropic fluids remains to be investigated. Multiplication of all terms in equation (C15) by the inverse of $\mathbf{C}$ and linearization of the term $D\mathbf{u}/Dt$ yields

$$\mathbf{A}\frac{\partial\mathbf{u}}{\partial t} + \mathbf{B}\mathbf{u} + \mathbf{D}_\mathbf{x}\mathbf{u} = \mathbf{s}, \tag{C20}$$

with

$$\mathbf{A} = \mathbf{C}^{-1}\bar{\mathbf{A}} = \bar{\mathbf{A}}, \quad \mathbf{B} = \mathbf{C}^{-1}\bar{\mathbf{B}} \quad\text{and}\quad \mathbf{s} = \mathbf{C}^{-1}\bar{\mathbf{s}}. \tag{C21}$$

Matrices $\mathbf{N}_\mathbf{x}$ and $\mathbf{K}$, appearing in the modified divergence theorems (6) and (8), read

$$\mathbf{N}_\mathbf{x} = \begin{pmatrix} \mathbf{O} & \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{N}_2 & \mathbf{O} & \mathbf{O} \end{pmatrix}, \tag{C22}$$

$$\mathbf{N}_1 = \begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{pmatrix}, \mathbf{N}_2 = \begin{pmatrix} 0 & n_3 & n_2 \\ n_3 & 0 & n_1 \\ n_2 & n_1 & 0 \end{pmatrix} \tag{C23}$$

and

$$\mathbf{K} = \mathrm{diag}(\mathbf{1}, -\mathbf{1}, -\mathbf{1}), \text{ with } \mathbf{1} = (1,1,1). \tag{C24}$$

Based on the structure of the matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\mathbf{C}$, $\mathbf{D}_\mathbf{x}$ and $\mathbf{K}$ we find that the symmetry properties described by equations (7), (9) and (13) are obeyed.

APPENDIX D: ELASTODYNAMIC WAVE PROPAGATION IN A SOLID

The linearized equation of motion in solids reads [13–15]

$$\rho \frac{\partial v_i}{\partial t} - \frac{\partial \tau_{ij}}{\partial x_j} = f_i, \tag{D1}$$

where v_i is the particle velocity associated to the elastodynamic wave motion, τ_{ij} the stress tensor, ρ the mass density of the medium in equilibrium and f_i the volume density of external force. The stress tensor is symmetric, i.e., $\tau_{ij} = \tau_{ji}$.

Hooke's linearized stress-strain relation reads

$$-\frac{\partial \tau_{ij}}{\partial t} + c_{ijkl}\frac{\partial v_k}{\partial x_l} = c_{ijkl} h_{kl}, \tag{D2}$$

where h_{kl} is the external deformation rate, with $h_{kl} = h_{lk}$, and c_{ijkl} is the anisotropic stiffness tensor. The stiffness tensor obeys the following symmetry relation [16]

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \tag{D3}$$

For isotropic solids this tensor reads

$$c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{D4}$$

where λ and μ are the Lamé parameters. Eliminating τ_{ij} from equations (D1) and (D2), using equation (D4) and taking $h_{kl} = 0$, yields the elastodynamic wave equation, according to

$$\rho \frac{\partial^2 v_i}{\partial t^2} - \frac{\partial}{\partial x_i}\Big(\lambda \frac{\partial v_k}{\partial x_k}\Big) - \frac{\partial}{\partial x_j}\Big[\mu\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)\Big] = \frac{\partial f_i}{\partial t}, \tag{D5}$$

or, for constant λ and μ,

$$\rho \frac{\partial^2 \mathbf{v}}{\partial t^2} - (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{v}) + \mu \nabla \times \nabla \times \mathbf{v} = \frac{\partial \mathbf{f}}{\partial t}, \tag{D6}$$

with

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{D7}$$

On the other hand, equations (D1) and (D2) can be combined to yield the general matrix-vector equation (1). To this end we first rewrite these equations (for the anisotropic situation) as

$$\rho \frac{\partial \mathbf{v}}{\partial t} - \mathbf{D}_\alpha \boldsymbol{\tau}_\alpha = \mathbf{f} \tag{D8}$$

and

$$-\frac{\partial \boldsymbol{\tau}_\alpha}{\partial t} + \mathbf{c}_{\alpha\beta}\mathbf{D}_\beta \mathbf{v} = \mathbf{c}_{\alpha\beta}\mathbf{h}_\beta \tag{D9}$$

(lower case Greek subscripts take on the values 1 and 2), where

$$\boldsymbol{\tau}_1 = \begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \end{pmatrix}, \quad \boldsymbol{\tau}_2 = \begin{pmatrix} \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{pmatrix}, \tag{D10}$$

$$\mathbf{h}_1 = \begin{pmatrix} h_{11} \\ h_{22} \\ h_{33} \end{pmatrix}, \quad \mathbf{h}_2 = \begin{pmatrix} 2h_{23} \\ 2h_{31} \\ 2h_{12} \end{pmatrix}, \tag{D11}$$

$$\mathbf{c}_{11} = \begin{pmatrix} c_{1111} & c_{1122} & c_{1133} \\ c_{1122} & c_{2222} & c_{2233} \\ c_{1133} & c_{2233} & c_{3333} \end{pmatrix}, \mathbf{c}_{12} = \begin{pmatrix} c_{1123} & c_{1131} & c_{1112} \\ c_{2223} & c_{2231} & c_{2212} \\ c_{3323} & c_{3331} & c_{3312} \end{pmatrix} \tag{D12}$$

$$\mathbf{c}_{21} = \mathbf{c}_{12}^T, \quad \mathbf{c}_{22} = \begin{pmatrix} c_{2323} & c_{2331} & c_{2312} \\ c_{2331} & c_{3131} & c_{3112} \\ c_{2312} & c_{3112} & c_{1212} \end{pmatrix}, \tag{D13}$$

and

$$\mathbf{D}_1 = \begin{pmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \end{pmatrix}, \mathbf{D}_2 = \begin{pmatrix} 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{pmatrix}. \tag{D14}$$

Hence, we obtain

$$\bar{\mathbf{A}}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{C}\mathbf{D}_\mathbf{x}\mathbf{u} = \bar{\mathbf{s}}, \tag{D15}$$

with

$$\mathbf{u} = \begin{pmatrix} \mathbf{v} \\ -\boldsymbol{\tau}_1 \\ -\boldsymbol{\tau}_2 \end{pmatrix}, \bar{\mathbf{s}} = \begin{pmatrix} \mathbf{f} \\ \mathbf{c}_{1\beta}\mathbf{h}_\beta \\ \mathbf{c}_{2\beta}\mathbf{h}_\beta \end{pmatrix}, \bar{\mathbf{A}} = \begin{pmatrix} \rho\mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}, \tag{D16}$$

$$\mathbf{C} = \begin{pmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{c}_{11} & \mathbf{c}_{12} \\ \mathbf{O} & \mathbf{c}_{21} & \mathbf{c}_{22} \end{pmatrix}, \mathbf{D}_\mathbf{x} = \begin{pmatrix} \mathbf{O} & \mathbf{D}_1 & \mathbf{D}_2 \\ \mathbf{D}_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{D}_2 & \mathbf{O} & \mathbf{O} \end{pmatrix}, \tag{D17}$$

$\mathbf{I}$ being the 3×3 identity matrix and $\mathbf{O}$ the 3×3 null matrix. Note that $\mathbf{C}$ is a symmetric real-valued matrix. From energy considerations it follows that it is positive definite [15], hence its inverse exists. Multiplying all terms in equation (D15) by the inverse of $\mathbf{C}$ yields

$$\mathbf{A}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{D}_\mathbf{x}\mathbf{u} = \mathbf{s}, \tag{D18}$$

with

$$\mathbf{A} = \mathbf{C}^{-1}\bar{\mathbf{A}} \quad \text{and} \quad \mathbf{s} = \mathbf{C}^{-1}\bar{\mathbf{s}}, \tag{D19}$$

or

$$\mathbf{A} = \begin{pmatrix} \rho\mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{s}_{11} & 2\mathbf{s}_{12} \\ \mathbf{O} & 2\mathbf{s}_{21} & 4\mathbf{s}_{22} \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} \mathbf{f} \\ \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix}, \tag{D20}$$

with

$$\mathbf{s}_{11} = \begin{pmatrix} s_{1111} & s_{1122} & s_{1133} \\ s_{1122} & s_{2222} & s_{2233} \\ s_{1133} & s_{2233} & s_{3333} \end{pmatrix}, \mathbf{s}_{12} = \begin{pmatrix} s_{1123} & s_{1131} & s_{1112} \\ s_{2223} & s_{2231} & s_{2212} \\ s_{3323} & s_{3331} & s_{3312} \end{pmatrix} \tag{D21}$$

$$\mathbf{s}_{21} = \mathbf{s}_{12}^T, \quad \mathbf{s}_{22} = \begin{pmatrix} s_{2323} & s_{2331} & s_{2312} \\ s_{2331} & s_{3131} & s_{3112} \\ s_{2312} & s_{3112} & s_{1212} \end{pmatrix}, \tag{D22}$$

where the s_{ijkl} are the elements of the compliance tensor, with $s_{ijkl} = s_{jikl} = s_{ijlk} = s_{klij}$. Note that

$$c_{ijkl}s_{klmn} = s_{ijkl}c_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}). \tag{D23}$$

For an isotropic solid we have

$$\mathbf{c}_{11} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda \\ \lambda & \lambda+2\mu & \lambda \\ \lambda & \lambda & \lambda+2\mu \end{pmatrix}, \mathbf{c}_{22} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \tag{D24}$$

$$\mathbf{c}_{21} = \mathbf{c}_{12}^T = \mathbf{O}, \tag{D25}$$

and

$$\mathbf{s}_{11} = \begin{pmatrix} \Lambda+2M & \Lambda & \Lambda \\ \Lambda & \Lambda+2M & \Lambda \\ \Lambda & \Lambda & \Lambda+2M \end{pmatrix}, \mathbf{s}_{22} = \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{pmatrix}, \tag{D26}$$

$$\mathbf{s}_{21} = \mathbf{s}_{12}^T = \mathbf{O}, \tag{D27}$$

with

$$\Lambda = -\frac{\lambda}{(3\lambda+2\mu)2\mu}, \quad M = \frac{1}{4\mu}. \tag{D28}$$

Matrices $\mathbf{N_x}$ and $\mathbf{K}$, appearing in the modified divergence theorems (6) and (8), read

$$\mathbf{N_x} = \begin{pmatrix} \mathbf{O} & \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{N}_2 & \mathbf{O} & \mathbf{O} \end{pmatrix}, \tag{D29}$$

$$\mathbf{N}_1 = \begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{pmatrix}, \mathbf{N}_2 = \begin{pmatrix} 0 & n_3 & n_2 \\ n_3 & 0 & n_1 \\ n_2 & n_1 & 0 \end{pmatrix} \tag{D30}$$

and

$$\mathbf{K} = \mathrm{diag}(\mathbf{1}, -\mathbf{1}, -\mathbf{1}), \text{ with } \mathbf{1} = (1,1,1). \tag{D31}$$

Based on the structure of the matrices $\bar{\mathbf{A}}$, $\mathbf{C}$, $\mathbf{D_x}$ and $\mathbf{K}$ we find that the symmetry properties described by equations (7), (9) and (13) are obeyed.

APPENDIX E: ELECTROMAGNETIC DIFFUSION AND WAVE PROPAGATION

The Maxwell equations for electromagnetic phenomena read [15, 17]

$$\epsilon\frac{\partial \mathbf{E}}{\partial t} + \sigma\mathbf{E} - \nabla\times\mathbf{H} = -\mathbf{J}^e, \tag{E1}$$

$$\mu\frac{\partial \mathbf{H}}{\partial t} + \nabla\times\mathbf{E} = -\mathbf{J}^m, \tag{E2}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field strengths, ϵ, μ and σ are the permittivity, permeability and conductivity, respectively and, finally, $\mathbf{J}^e$ and $\mathbf{J}^m$ are source functions in terms of the external electric and magnetic current densities. Eliminating the magnetic field strength $\mathbf{H}$ from equations (E1) and (E2) yields the electromagnetic wave equation, according to

$$\sigma\frac{\partial \mathbf{E}}{\partial t} + \epsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} + \nabla\times\Big(\frac{1}{\mu}\nabla\times\mathbf{E}\Big) = -\frac{\partial \mathbf{J}^e}{\partial t} - \nabla\times\Big(\frac{1}{\mu}\mathbf{J}^m\Big). \tag{E3}$$

Note that first and second order time derivatives occur in this equation, which means that this equation accounts for diffusion as well as wave propagation. On the other hand, equations (E1) and (E2) can be combined to

$$\mathbf{A}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{B}\mathbf{u} + \mathbf{D_x}\mathbf{u} = \mathbf{s}, \tag{E4}$$

with

$$\mathbf{u} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \mathbf{s} = \begin{pmatrix} -\mathbf{J}^e \\ -\mathbf{J}^m \end{pmatrix}, \mathbf{A} = \begin{pmatrix} \epsilon\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mu\mathbf{I} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \sigma\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \tag{E5}$$

$$\mathbf{D_x} = \begin{pmatrix} \mathbf{O} & \mathbf{D}_0^T \\ \mathbf{D}_0 & \mathbf{O} \end{pmatrix}, \mathbf{D}_0 = \begin{pmatrix} 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{pmatrix}, \tag{E6}$$

with $\mathbf{I}$ being the 3×3 identity matrix and $\mathbf{O}$ the 3×3 null matrix. Matrices $\mathbf{N_x}$ and $\mathbf{K}$, appearing in the modified divergence theorems (6) and (8), read

$$\mathbf{N_x} = \begin{pmatrix} \mathbf{O} & \mathbf{N}_0^T \\ \mathbf{N}_0 & \mathbf{O} \end{pmatrix}, \mathbf{N}_0 = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}, \tag{E7}$$

$$\mathbf{K} = \mathrm{diag}(-\mathbf{1}, \mathbf{1}), \ \mathbf{1} = (1,1,1). \tag{E8}$$

The symmetry properties described by equations (7), (9) and (13) are easily confirmed.

APPENDIX F: COUPLED ELASTODYNAMIC AND ELECTROMAGNETIC WAVE PROPAGATION IN POROUS SOLIDS

We briefly review the theory for elastodynamic waves coupled to electromagnetic fields in a dissipative inhomogeneous anisotropic fluid-saturated porous solid [6, 18].

The linearized equations of motion read in the frequency domain (using the vector notation introduced in Appendix D)

$$j\omega\boldsymbol{\rho}^b\hat{\mathbf{v}}^s + j\omega\boldsymbol{\rho}^f\hat{\mathbf{w}} - \mathbf{D}_\alpha\hat{\boldsymbol{\tau}}^b_\alpha \;=\; \hat{\mathbf{f}}^b, \qquad \text{(F1)}$$

$$j\omega\boldsymbol{\rho}^f\hat{\mathbf{v}}^s + \eta\hat{\mathbf{k}}^{-1}\Big(\hat{\mathbf{w}} - \hat{\mathbf{L}}\hat{\mathbf{E}}\Big) + \boldsymbol{\nabla}\hat{p} \;=\; \hat{\mathbf{f}}^f, \qquad \text{(F2)}$$

with $\hat{\mathbf{w}} = \phi(\hat{\mathbf{v}}^f - \hat{\mathbf{v}}^s)$. Here $\hat{\mathbf{v}}^s$ and $\hat{\mathbf{v}}^f$ are the averaged solid and fluid particle velocities associated to the wave motion, $\hat{\mathbf{w}}$ is the filtration velocity, ϕ the porosity, $\hat{\boldsymbol{\tau}}^b_\alpha$ the averaged bulk stress (organized as in equation (D10)), $\hat{p}$ the averaged fluid pressure and $\hat{\mathbf{E}}$ the averaged electric field strength. Matrix $\mathbf{D}_\alpha$ contains the spatial differential operators and is defined in equation (D14). The source functions $\hat{\mathbf{f}}^b$ and $\hat{\mathbf{f}}^f$ are the volume densities of external force on the bulk and on the fluid, respectively. The constitutive parameters $\boldsymbol{\rho}^b$ and $\boldsymbol{\rho}^f$ are the anisotropic bulk and fluid mass densities, respectively [19]. In the following we assume that these tensors are symmetric, according to $\boldsymbol{\rho}^b = (\boldsymbol{\rho}^b)^T$ and $\boldsymbol{\rho}^f = (\boldsymbol{\rho}^f)^T$, which is for example the case when the anisotropy is the result of parallel fine layering at a scale much smaller than the wavelength. The complex frequency-dependent tensor $\hat{\mathbf{k}}$ is the dynamic permeability tensor of the porous material, with $\hat{\mathbf{k}} = \hat{\mathbf{k}}^T$, and η is the fluid viscosity parameter. Finally, the complex frequency-dependent tensor $\hat{\mathbf{L}}$ accounts for the coupling between the elastodynamic and electromagnetic waves. In the following we will assume that this tensor is symmetric as well, according to $\hat{\mathbf{L}} = \hat{\mathbf{L}}^T$ (Pride and Haartsen [6] discuss the conditions for this symmetry).

The linearized stress-strain relations read

$$-j\omega\hat{\boldsymbol{\tau}}^b_\alpha + \mathbf{c}_{\alpha\beta}\mathbf{D}_\beta\hat{\mathbf{v}}^s + \mathbf{d}_\alpha\boldsymbol{\nabla}\cdot\hat{\mathbf{w}} \;=\; \mathbf{0}, \qquad \text{(F3)}$$

$$j\omega\hat{p} + \mathbf{d}^T_\alpha\mathbf{D}_\alpha\hat{\mathbf{v}}^s + M\boldsymbol{\nabla}\cdot\hat{\mathbf{w}} \;=\; 0, \qquad \text{(F4)}$$

with

$$\mathbf{d}_1 = \begin{pmatrix} d_{11}\\ d_{22}\\ d_{33}\end{pmatrix}, \quad \mathbf{d}_2 = \begin{pmatrix} d_{23}\\ d_{31}\\ d_{12}\end{pmatrix}, \qquad \text{(F5)}$$

$\mathbf{c}_{\alpha\beta}$ defined in equations (D12) and (D13) and $\mathbf{0}$ a 3×1 null vector. M, d_{ij} and c_{ijkl} are the real-valued stiffness parameters of the fluid-saturated porous solid.

Maxwell's electromagnetic field equations read

$$j\omega\boldsymbol{\epsilon}\hat{\mathbf{E}} + \hat{\mathbf{J}} - \boldsymbol{\nabla}\times\hat{\mathbf{H}} \;=\; -\hat{\mathbf{J}}^e, \qquad \text{(F6)}$$

$$j\omega\boldsymbol{\mu}\hat{\mathbf{H}} + \boldsymbol{\nabla}\times\hat{\mathbf{E}} \;=\; -\hat{\mathbf{J}}^m, \qquad \text{(F7)}$$

where $\hat{\mathbf{H}}$ is the averaged magnetic field strength, $\hat{\mathbf{J}}$ the averaged induced electric current density, $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are the anisotropic permittivity and permeability, with $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^T$ and $\boldsymbol{\mu} = \boldsymbol{\mu}^T$, and $\mathbf{J}^e$ and $\mathbf{J}^m$ are source functions in terms of the external electric and magnetic current densities. The induced electric current density is coupled to the elastodynamic wave motion, according to

$$\hat{\mathbf{J}} \;=\; \hat{\boldsymbol{\sigma}}\hat{\mathbf{E}} - \hat{\mathbf{L}}\Big[\boldsymbol{\nabla}\hat{p} + j\omega\boldsymbol{\rho}^f\hat{\mathbf{v}}^s - \hat{\mathbf{f}}^f\Big], \qquad \text{(F8)}$$

where $\hat{\boldsymbol{\sigma}}$ is the complex frequency dependent conductivity, with $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}^T$. Substituting the constitutive relation (F8) into the Maxwell equation (F6), and adding $\hat{\mathbf{L}}$ times equation (F2) to equation (F6) in order to compensate for the term $-\hat{\mathbf{L}}[\boldsymbol{\nabla}\hat{p} + j\omega\boldsymbol{\rho}^f\hat{\mathbf{v}}^s - \hat{\mathbf{f}}^f]$, yields

$$j\omega\boldsymbol{\epsilon}\hat{\mathbf{E}} + \Big(\hat{\boldsymbol{\sigma}} - \eta\hat{\mathbf{L}}\hat{\mathbf{k}}^{-1}\hat{\mathbf{L}}\Big)\hat{\mathbf{E}} + \eta\hat{\mathbf{L}}\hat{\mathbf{k}}^{-1}\hat{\mathbf{w}} - \boldsymbol{\nabla}\times\hat{\mathbf{H}} = -\hat{\mathbf{J}}^e. \quad \text{(F9)}$$

Equations (F9) and (F7), together with equations (F1), (F2), (F3) and (F4) can be combined to yield

$$j\omega\bar{\mathbf{A}}\hat{\mathbf{u}} + \bar{\mathbf{B}}\hat{\mathbf{u}} + \mathbf{C}\mathbf{D_x}\hat{\mathbf{u}} = \hat{\bar{\mathbf{s}}}, \qquad \text{(F10)}$$

where

$$\hat{\mathbf{u}} = \begin{pmatrix}\hat{\mathbf{u}}_1\\ \hat{\mathbf{u}}_2\\ \hat{\mathbf{u}}_3\end{pmatrix}, \quad \hat{\bar{\mathbf{s}}} = \begin{pmatrix}\hat{\bar{\mathbf{s}}}_\mathbf{1}\\ \hat{\bar{\mathbf{s}}}_\mathbf{2}\\ \hat{\bar{\mathbf{s}}}_\mathbf{3}\end{pmatrix}, \bar{\mathbf{A}} = \begin{pmatrix}\bar{\mathbf{A}}_{11} & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & \bar{\mathbf{A}}_{22} & \bar{\mathbf{A}}_{23}\\ \mathbf{O} & \bar{\mathbf{A}}^T_{23} & \bar{\mathbf{A}}_{33}\end{pmatrix},$$

$$\bar{\mathbf{B}} = \begin{pmatrix}\bar{\mathbf{B}}_{11} & \mathbf{O} & \bar{\mathbf{B}}_{13}\\ \mathbf{O} & \mathbf{O} & \mathbf{O}\\ -\bar{\mathbf{B}}^T_{13} & \mathbf{O} & \bar{\mathbf{B}}_{33}\end{pmatrix}, \qquad \text{(F11)}$$

$$\mathbf{C} = \begin{pmatrix}\mathbf{I} & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & \mathbf{C}_{22} & \mathbf{C}_{23}\\ \mathbf{O} & \mathbf{C}^T_{23} & \mathbf{C}_{33}\end{pmatrix} \quad \text{and} \quad \mathbf{D_x} = \begin{pmatrix}\mathbf{D}_{11} & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & \mathbf{D}_{22} & \mathbf{O}\\ \mathbf{O} & \mathbf{O} & \mathbf{D}_{33}\end{pmatrix}, \qquad \text{(F12)}$$

where $\mathbf{I}$ and $\mathbf{O}$ are identity and null matrices of appropriate size and

$$\hat{\mathbf{u}}_1 = \begin{pmatrix}\hat{\mathbf{E}}\\ \hat{\mathbf{H}}\end{pmatrix}, \hat{\mathbf{u}}_2 = \begin{pmatrix}\hat{\mathbf{v}}^s\\ -\hat{\boldsymbol{\tau}}^b_1\\ -\hat{\boldsymbol{\tau}}^b_2\end{pmatrix}, \hat{\mathbf{u}}_3 = \begin{pmatrix}\hat{\mathbf{w}}\\ \hat{p}\end{pmatrix},$$

$$\hat{\bar{\mathbf{s}}}_\mathbf{1} = \begin{pmatrix}-\hat{\mathbf{J}}^e\\ -\hat{\mathbf{J}}^m\end{pmatrix}, \hat{\bar{\mathbf{s}}}_\mathbf{2} = \begin{pmatrix}\hat{\mathbf{f}}^b\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix}, \hat{\bar{\mathbf{s}}}_\mathbf{3} = \begin{pmatrix}\hat{\mathbf{f}}^f\\ 0\end{pmatrix}, \qquad \text{(F13)}$$

$$\bar{\mathbf{A}}_{11} = \begin{pmatrix}\boldsymbol{\epsilon} & \mathbf{O}\\ \mathbf{O} & \boldsymbol{\mu}\end{pmatrix}, \bar{\mathbf{A}}_{22} = \begin{pmatrix}\boldsymbol{\rho}^b & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & \mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{O} & \mathbf{I}\end{pmatrix},$$

$$\bar{\mathbf{A}}_{23} = \begin{pmatrix}\boldsymbol{\rho}^f & \mathbf{0}\\ \mathbf{O} & \mathbf{0}\\ \mathbf{O} & \mathbf{0}\end{pmatrix}, \bar{\mathbf{A}}_{33} = \begin{pmatrix}\mathbf{O} & \mathbf{0}\\ \mathbf{0}^T & 1\end{pmatrix}, \qquad \text{(F14)}$$

$$\bar{\mathbf{B}}_{11} = \begin{pmatrix}(\hat{\boldsymbol{\sigma}} - \eta\hat{\mathbf{L}}\hat{\mathbf{k}}^{-1}\hat{\mathbf{L}}) & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix}, \bar{\mathbf{B}}_{13} = \begin{pmatrix}\eta\hat{\mathbf{L}}\hat{\mathbf{k}}^{-1} & \mathbf{0}\\ \mathbf{O} & \mathbf{0}\end{pmatrix},$$

$$\bar{\mathbf{B}}_{33} = \begin{pmatrix}\eta\hat{\mathbf{k}}^{-1} & \mathbf{0}\\ \mathbf{0}^T & 0\end{pmatrix}, \qquad \text{(F15)}$$

$$\mathbf{C}_{22} = \begin{pmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{c}_{11} & \mathbf{c}_{12} \\ \mathbf{O} & \mathbf{c}_{21} & \mathbf{c}_{22} \end{pmatrix}, \mathbf{C}_{23} = \begin{pmatrix} \mathbf{O} & \mathbf{0} \\ \mathbf{O} & \mathbf{d}_1 \\ \mathbf{O} & \mathbf{d}_2 \end{pmatrix},$$

$$\mathbf{C}_{33} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^T & M \end{pmatrix}, \qquad \text{(F16)}$$

$$\mathbf{D}_{11} = \begin{pmatrix} \mathbf{O} & \mathbf{D}_0^T \\ \mathbf{D}_0 & \mathbf{O} \end{pmatrix}, \mathbf{D}_0 = \begin{pmatrix} 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{pmatrix},$$

$$\mathbf{D}_{33} = \begin{pmatrix} \mathbf{O} & \boldsymbol{\nabla} \\ \boldsymbol{\nabla}^T & 0 \end{pmatrix} \qquad \text{(F17)}$$

and $\mathbf{D}_{22}$ equal to $\mathbf{D_x}$ in equation (D17). Note that $\mathbf{C}$ in equation (F12) is a symmetric real-valued matrix. From energy considerations it follows that it is positive definite [6, 15], hence its inverse exists. Multiplying all terms in equation (F10) by the inverse of $\mathbf{C}$ finally yields

$$j\omega\mathbf{A}\hat{\mathbf{u}} + \mathbf{B}\hat{\mathbf{u}} + \mathbf{D_x}\hat{\mathbf{u}} = \hat{\mathbf{s}}, \qquad \text{(F18)}$$

with $\mathbf{A} = \mathbf{C}^{-1}\bar{\mathbf{A}}$, $\mathbf{B} = \mathbf{C}^{-1}\bar{\mathbf{B}} = \bar{\mathbf{B}}$ and $\hat{\mathbf{s}} = \mathbf{C}^{-1}\hat{\bar{\mathbf{s}}} = \hat{\bar{\mathbf{s}}}$. Matrices $\mathbf{N_x}$ and $\mathbf{K}$, appearing in the modified divergence theorems (6) and (8), read

$$\mathbf{N_x} = \begin{pmatrix} \mathbf{N}_{11} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{N}_{22} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{N}_{33} \end{pmatrix}, \mathbf{N}_{11} = \begin{pmatrix} \mathbf{O} & \mathbf{N}_0^T \\ \mathbf{N}_0 & \mathbf{O} \end{pmatrix},$$

$$\mathbf{N}_0 = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}, \mathbf{N}_{33} = \begin{pmatrix} \mathbf{O} & \mathbf{n} \\ \mathbf{n}^T & 0 \end{pmatrix}, \text{(F19)}$$

$$\mathbf{K} = \mathrm{diag}(-\mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, -1), \qquad \text{(F20)}$$

and $\mathbf{N}_{22}$ equal to $\mathbf{N_x}$ in equation (D29). Based on the structure of the matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\mathbf{C}$, $\mathbf{D_x}$ and $\mathbf{K}$ as well as the symmetry relations discussed above, we find that the symmetry properties described by equations (7), (9) and (13) are obeyed.

Finally, note that when the coupling tensor $\hat{\mathbf{L}}$ is zero, the matrix $\bar{\mathbf{B}}_{13}$ vanishes and hence equation (F10) decouples into the electromagnetic wave equation for the wave vector $\hat{\mathbf{u}}_1$ and Biot's poroelastic wave equation for the wave vector $(\hat{\mathbf{u}}_2^T, \hat{\mathbf{u}}_3^T)^T$, [20]. For a non-porous solid the matrices $\bar{\mathbf{A}}_{23}$ and $\mathbf{C}_{23}$ vanish as well, so Biot's wave equation reduces to the elastodynamic wave equation for the wave vector $\hat{\mathbf{u}}_2$. Obviously the symmetry properties described by equations (7), (9) and (13) are obeyed for the matrices appearing in the electromagnetic wave equation, Biot's poroelastic wave equation and the elastodynamic wave equation, respectively.

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