Reciprocity and power balance for piecewise continuous media with imperfect interfaces

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[1] The interaction of elastodynamic waves with imperfect interfaces is usually described by the so-called linear slip model. In this model it is assumed that the particle displacement of an elastic wave at an interface jumps by a finite amount, linearly proportional to the stress at the interface. In this paper we postulate general boundary conditions at arbitrarily shaped imperfect interfaces for acoustic waves in fluids, elastodynamic waves in solids, electromagnetic waves in matter, poroelastic waves in porous solids, and seismoelectric waves in porous solids in such a way that they cover the linear slip model and other existing models for imperfect interfaces. These boundary conditions are expressed by a single matrix-vector equation in the space-frequency domain. Using this equation, we extend two unified reciprocity theorems (one of the convolution type and one of the correlation type) for the various wave phenomena, with an extra integral over the imperfect interfaces. By considering two special cases of these reciprocity theorems, we observe that (1) source-receiver reciprocity remains valid when the source and receiver are separated by imperfect interfaces and that (2) the extra integral in the correlation-type reciprocity theorem quantifies the power dissipated by the imperfect interfaces. *INDEX TERMS:* 0935 Exploration Geophysics: Seismic methods (3025); 7260 Seismology: Theory and modeling; 3210 Mathematical Geophysics: Modeling; 0689 Electromagnetics: Wave propagation (4275); 5114 Physical Properties of Rocks: Permeability and porosity; *KEYWORDS:* reciprocity, linear slip, boundary permeability

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1. Introduction

[2] The interaction of elastodynamic waves with imperfect interfaces has been investigated by many authors for various situations (throughout this paper, by "imperfect interface" we mean an interface between two media that are not in perfect contact). *Schoenberg* [1980] introduced the linear slip model for an imperfect interface between two elastic media. In this model it is assumed that the particle displacement of an elastic wave at an interface jumps by a finite amount, linearly proportional to the stress at the interface. The stress itself is assumed continuous across the interface. The ratio of the stress and the displacement jump is the specific boundary stiffness. *Schoenberg*'s [1980] linear slip model is, for example, suited for modelling the interaction of elastodynamic waves with fractures in which viscosity may be ignored. *Pyrak-Nolte et al.* [1990] give many references to laboratory and field experiments which show evidence of wave attenuation due to fractures. To account for this attenuation, they proposed to extend the linear slip model with a specific boundary viscosity, which is the ratio of stress and a velocity jump across the interface. *Nakagawa et al.* [2000] discuss laboratory experiments which show that a fracture under shear stress may give rise to wave conversion at normal incidence. Since this effect cannot be explained by either of the two models discussed above, they extended the linear slip model with coupling stiffnesses which relate the normal stress to the tangential displacement jump and vice versa. *Gurevich and Schoenberg* [1999] discuss interface conditions for Biot's equations at the boundary between two poroelastic media. In their model the fluid pressure jumps by a finite amount, linearly proportional to the filtration velocity, which is assumed continuous across the interface. The ratio of the filtration velocity and the pressure jump is the hydraulic boundary permeability. Imperfect interfaces have also been investigated for electromagnetic fields. *Kaufman and Keller* [1983] discuss interface conditions, in the diffusive approximation to Maxwell's equations, for a strongly conductive interface between two electric media. In the case of a conductive interface charges cannot be built up and the current that is generated at the interface gives rise to a jump in the tangential magnetic field components, linearly proportional to the tangential electric field components (the normal component of the electric field is zero at a perfectly conducting interface). The ratio of the electric field and the magnetic field jump is the specific boundary resistivity.

[3] In this paper we analyze reciprocity theorems and power balances for wave fields in piecewise continuous inhomogeneous media, containing arbitrarily shaped imperfect interfaces. First we postulate general boundary con-

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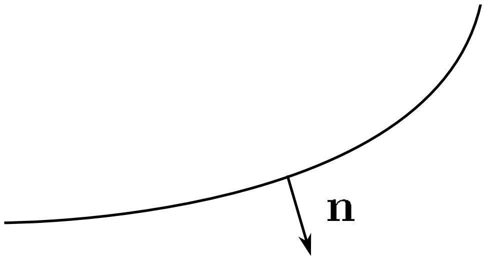

Figure 1. Interface between two media.

ditions at imperfect interfaces for (1) acoustic waves in fluids, (2) elastodynamic waves in solids, (3) electromagnetic waves in matter, (4) poroelastic waves in porous solids and, finally, (5) poroelastic waves coupled to electromagnetic waves (also known as seismoelectric waves) in porous solids. These boundary conditions are general in the sense that none of the wave field quantities are assumed continuous across an interface and the boundary parameters are arbitrary complex-valued frequency-dependent functions. Since in all cases we consider linear wave theory, we assume that the boundary conditions are linear as well. The postulated boundary conditions for each of the five wave phenomena can be cast in a single matrix-vector equation, according to

$$[\mathbf{M}\hat{\mathbf{u}}] = -j\omega\hat{\mathbf{Y}}\langle\mathbf{M}\hat{\mathbf{u}}\rangle, \tag{1}$$

where $\hat{\mathbf{u}}$ is the wave field vector (quantities with a circumflex denote temporal Fourier transforms, see equation (2)), $\mathbf{M}$ is a matrix that contracts this wave vector to the components that are involved in the boundary conditions, $\hat{\mathbf{Y}}$ is a matrix containing the boundary parameters, j is the imaginary unit and ω the angular frequency. Brackets and angle brackets represent the jump and the average across the interface, respectively (see equations (5) and (6)). Next we briefly review two unified reciprocity theorems in a similar matrix-vector notation, that have previously been formulated for the various wave phenomena in piecewise continuous inhomogeneous media, containing perfect interfaces (that is, interfaces between media that are in perfect contact). Using the matrix-vector form of the general boundary conditions (equation (1)) we extend these reciprocity theorems for the situation of piecewise continuous media with imperfect interfaces. The extended reciprocity theorems contain an extra integral over all imperfect interfaces. Finally we discuss some consequences of this extra integral in the two unified reciprocity theorems.

2. General Boundary Conditions for Wave Fields at Imperfect Interfaces

2.1. Some Definitions

[4] To represent position in space we use the vector $\mathbf{x} = (x_1, x_2, x_3)^T$, defined in a right-handed Cartesian coordinate system (superscript ^{T} denotes transposition). We use a subscript notation for the components of vectorial and tensorial quantities. Lowercase Latin subscripts take on the values 1, 2 and 3. Einstein's summation convention applies for repeated subscripts, hence, $p_i q_i$ stands for $\Sigma_{i=1}^{3} p_i q_i$. The time coordinate will be denoted by t. We define the temporal Fourier transform of a space- and time-dependent real-valued function $p(\mathbf{x}, t)$ as

$$\hat{p}(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} \exp(-j\omega t) p(\mathbf{x}, t) \mathrm{d}t. \tag{2}$$

2.2. Boundary Conditions for Acoustic Waves in Fluids

[5] An acoustic wave field in a fluid is described in terms of the acoustic pressure $p(\mathbf{x}, t)$ and the particle velocity $v_i(\mathbf{x}, t)$, or, in the frequency domain, by $\hat{p}(\mathbf{x}, \omega)$ and $\hat{v}_i(\mathbf{x}, \omega)$. Consider an interface with normal vector $\mathbf{n} = (n_1, n_2, n_3)^T$ between two fluids with different space-dependent medium parameters, see Figure 1. When the two fluids are in perfect contact at the interface, the boundary conditions require that the acoustic pressure ($\hat{p}$) and the normal component of the particle velocity ($\hat{v}_i n_i$) are continuous. The tangential components of the particle velocity need not be continuous since at the interface the fluids may slide along each other. It can be shown that these conditions are consistent with the acoustic equation of motion and the stress-strain relation [*de Hoop*, 1995]. When the fluids at both sides of an interface are not in perfect contact, the wave quantities $\hat{p}$ and $\hat{v}_i n_i$ may exhibit a finite jump across the interface. For example, when the media are separated by a thin permeable membrane there may be a pressure drop approximately proportional to the normal velocity (here "thin" means that the thickness of the membrane is very small in comparison with the wave length). On the other hand, there will be a velocity discontinuity when the fluids mix at the interface or when the interface is a thin compressible layer. In the latter case the discontinuity of the normal velocity will be approximately proportional to the pressure rate. In general, the boundary conditions for imperfect interfaces cannot be derived from the equation of motion and the stress-strain relation, hence they should be specified independently. We postulate general boundary conditions for an acoustic wave field at an imperfect interface as follows:

$$[\hat{p}] = -j\omega\hat{\varrho}^{\mathrm{b}}\langle\hat{v}_i n_i\rangle \tag{3}$$

$$[\hat{v}_i n_i] = -j\omega\hat{\kappa}^{\mathrm{b}}\langle\hat{p}\rangle. \tag{4}$$

The parameters $\hat{\varrho}^{\mathrm{b}}$ and $\hat{\kappa}^{\mathrm{b}}$ are space- and frequency-dependent complex-valued functions, hence $\hat{\varrho}^{\mathrm{b}} = \hat{\varrho}^{\mathrm{b}}(\mathbf{x}, \omega)$ and $\hat{\kappa}^{\mathrm{b}} = \hat{\kappa}^{\mathrm{b}}(\mathbf{x}, \omega)$. As mentioned in the introduction, brackets and angle brackets represent the jump and the average across the interface, respectively; hence

$$[\hat{p}(\mathbf{x}, \omega)] = \lim_{h \downarrow 0} (\hat{p}(\mathbf{x} + h\mathbf{n}, \omega) - \hat{p}(\mathbf{x} - h\mathbf{n}, \omega)) \tag{5}$$

$$\langle\hat{p}(\mathbf{x}, \omega)\rangle = \lim_{h \downarrow 0} (\hat{p}(\mathbf{x} + h\mathbf{n}, \omega) + \hat{p}(\mathbf{x} - h\mathbf{n}, \omega))/2, \tag{6}$$

(where $\mathbf{x}$ is chosen at the interface) and similar relations for $[\hat{v}_i n_i]$ and $\langle\hat{v}_i n_i\rangle$. Of course the direction of $\mathbf{n}$ on a given interface is not unique: $\mathbf{n}$ as well as $-\mathbf{n}$ are equally valid choices. Note, however, that with the definitions above, the boundary condition equations (3) and (4) are independent of the choice of the sign of $\mathbf{n}$ (changing the sign of $\mathbf{n}$ implies that both sides of equation (3) change sign, whereas both sides of equation (4) remain unchanged).

[6] Note that equations (3) and (4) have been written in a form that resembles the equation of motion and the stress-strain relation for acoustic waves in a volume. The main difference is that the left-hand sides contain jumps instead of spatial derivatives and the right-hand sides contain boundary parameters ($\hat{\varrho}^{\rm b}$ and $\hat{\kappa}^{\rm b}$) instead of volumetric parameters ($\hat{\varrho}$ and $\hat{\kappa}$). The dimension of the boundary parameters is equal to the dimension of the volumetric parameters multiplied by meter.

[7] For vanishing $\hat{\varrho}^{\rm b}$ and $\hat{\kappa}^{\rm b}$, equations (3) and (4) reduce to the standard boundary conditions for perfectly coupled fluids. When $\hat{\varrho}^{\rm b}$ and $\hat{\kappa}^{\rm b}$ are nonzero, $1/j\omega\hat{\varrho}^{\rm b}$ is the hydraulic boundary permeability, $\hat{\kappa}^{\rm b}$ the boundary compressibility and $1/\hat{\kappa}^{\rm b} = \hat{K}^{\rm b}$ the boundary stiffness. Note that the dimension of the boundary stiffness $\hat{K}^{\rm b}$ is that of stiffness per meter (i.e., Pa/m). Therefore $\hat{K}^{\rm b}$ is also called the specific boundary stiffness.

[8] Boundary condition equations (3) and (4) can be expressed by matrix-vector equation (1), with

$$\hat{\mathbf{u}} = \begin{pmatrix} \hat{p} \\ \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_1 & n_2 & n_3 \end{pmatrix}, \quad \hat{\mathbf{Y}} = \begin{pmatrix} 0 & \hat{\varrho}^{\rm b} \\ \hat{\kappa}^{\rm b} & 0 \end{pmatrix}, \tag{7}$$

where $\hat{\mathbf{u}} = \hat{\mathbf{u}}(\mathbf{x}, \omega)$ is the acoustic wave field vector, $\mathbf{M} = \mathbf{M}(\mathbf{x})$ the acoustic contraction matrix and $\hat{\mathbf{Y}} = \hat{\mathbf{Y}}(\mathbf{x}, \omega)$ the acoustic boundary matrix. Note that $\hat{\mathbf{Y}}$ obeys the symmetry relations

$$\hat{\mathbf{Y}}^T\mathbf{N} = -\mathbf{N}\hat{\mathbf{Y}} \qquad \hat{\mathbf{Y}}^H\mathbf{J} = \mathbf{J}\hat{\mathbf{Y}}^*, \tag{8}$$

with matrices $\mathbf{N}$ and $\mathbf{J}$ defined in equation (A1) in Appendix A (superscript * denotes complex conjugation and ^{H} denotes complex conjugation and transposition). These symmetry relations will be used in the derivation of the reciprocity theorems.

2.3. Boundary Conditions for Elastodynamic Waves in Solids

[9] An elastodynamic wave field in a solid is described in the frequency domain in terms of the stress $\hat{\tau}_{ij}(\mathbf{x}, \omega)$ and the particle velocity $\hat{v}_i(\mathbf{x}, \omega)$. At an interface between two solids that are in perfect contact, the boundary conditions require that the traction ($\hat{\tau}_{ij}n_j$) and the particle velocity ($\hat{v}_i$) are continuous. These conditions are consistent with the elastodynamic equation of motion and the stress-strain relation [*de Hoop*, 1995]. When the solids at both sides of an interface are not in perfect contact, the wave quantities $\hat{\tau}_{ij}n_j$ and $\hat{v}_i$ may exhibit a finite jump across the interface. In general, the boundary conditions for imperfect interfaces cannot be derived from the equation of motion and the stress-strain relation, hence they should be specified independently. We postulate general boundary conditions for an elastodynamic wave field at an imperfect interface as follows:

$$\left[\hat{\tau}_{ij}n_j\right] = j\omega\hat{\varrho}^{\rm b}_{ik}\langle\hat{v}_k\rangle \tag{9}$$

$$[\hat{v}_i] = j\omega\hat{S}^{\rm b}_{ik}\langle\hat{\tau}_{kj}n_j\rangle. \tag{10}$$

Here $\hat{\varrho}^{\rm b}_{ik} = \hat{\varrho}^{\rm b}_{ik}(\mathbf{x}, \mathbf{n}, \omega)$ and $\hat{S}^{\rm b}_{ik} = \hat{S}^{\rm b}_{ik}(\mathbf{x}, \mathbf{n}, \omega)$ are the anisotropic space- and frequency-dependent complex-valued boundary density and compliance; $\mathbf{n}$ denotes that these parameters are defined for an interface with normal $\mathbf{n}$. We assume that these boundary parameters are symmetric, hence $\hat{\varrho}^{\rm b}_{ik} = \hat{\varrho}^{\rm b}_{ki}$ and $\hat{S}^{\rm b}_{ik} = \hat{S}^{\rm b}_{ki}$.

[10] Boundary condition equations (9) and (10) can be expressed by matrix-vector equation (1), with

$$\hat{\mathbf{u}} = \begin{pmatrix} \hat{\mathbf{v}} \\ -\hat{\boldsymbol{\tau}}_1 \\ -\hat{\boldsymbol{\tau}}_2 \\ -\hat{\boldsymbol{\tau}}_3 \end{pmatrix}, \mathbf{M} = \begin{pmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & n_1\mathbf{I} & n_2\mathbf{I} & n_3\mathbf{I} \end{pmatrix}, \hat{\mathbf{Y}} = \begin{pmatrix} \mathbf{O} & \hat{\mathbf{S}}^{\rm b} \\ \hat{\boldsymbol{\varrho}}^{\rm b} & \mathbf{O} \end{pmatrix}, \tag{11}$$

where $(\hat{\mathbf{v}})_i = \hat{v}_i$, $(\hat{\boldsymbol{\tau}}_j)_i = \hat{\tau}_{ij}$, $(\hat{\boldsymbol{\varrho}}^{\rm b})_{ik} = \hat{\varrho}^{\rm b}_{ik}$, $(\hat{\mathbf{S}}^{\rm b})_{ik} = \hat{S}^{\rm b}_{ik}$, $(\mathbf{I})_{ik} = \delta_{ik}$ and $(\mathbf{O})_{ik} = 0$. Note that $\hat{\mathbf{u}}$ is a 12×1 vector, $\mathbf{M}$ a 6×12 matrix and $\hat{\mathbf{Y}}$ a 6×6 matrix. Furthermore, note that $\hat{\mathbf{Y}}$ obeys symmetry relations of the form of equation (8), with matrices $\mathbf{N}$ and $\mathbf{J}$ defined in equation (A3) in Appendix A.

[11] We consider some special situations. When $\hat{\boldsymbol{\varrho}}^{\rm b} = \mathbf{O}$ (which is usually a good approximation) and $\mathbf{n} = (0, 0, 1)^T$ (i.e., the interface is horizontal), then equations (1) and (11) yield

$$[\hat{\boldsymbol{\tau}}_3] = \mathbf{0} \quad \Rightarrow \quad \langle\hat{\boldsymbol{\tau}}_3\rangle = \hat{\boldsymbol{\tau}}_3 \tag{12}$$

and

$$[\hat{\mathbf{v}}] = j\omega\hat{\mathbf{S}}^{\rm b}\hat{\boldsymbol{\tau}}_3, \tag{13}$$

or, equivalently,

$$\hat{\boldsymbol{\tau}}_3 = \left(\hat{\mathbf{S}}^{\rm b}\right)^{-1}[\hat{\mathbf{v}}/j\omega], \tag{14}$$

where $\hat{\mathbf{v}}/j\omega$ is the displacement vector and $(\mathbf{0})_i = 0$. Equation (14) corresponds with the model of *Nakagawa et al.* [2000], with $(\hat{\mathbf{S}}^{\rm b})^{-1}$ being the specific boundary stiffness tensor. The off-diagonal elements of this tensor are the coupling stiffnesses. When the coupling stiffnesses are zero and $(\hat{\mathbf{S}}^{\rm b})^{-1}$ can be written as

$$\left(\hat{\mathbf{S}}^{\rm b}\right)^{-1} = \begin{pmatrix} K^{\rm b}_1 + j\omega\eta^{\rm b} & 0 & 0 \\ 0 & K^{\rm b}_2 + j\omega\eta^{\rm b} & 0 \\ 0 & 0 & K^{\rm b}_3 \end{pmatrix}, \tag{15}$$

then equation (14) yields

$$\hat{\tau}_{13} = K^{\rm b}_1[\hat{v}_1/j\omega] + \eta^{\rm b}[\hat{v}_1], \tag{16}$$

$$\hat{\tau}_{23} = K^{\rm b}_2[\hat{v}_2/j\omega] + \eta^{\rm b}[\hat{v}_2], \tag{17}$$

$$\hat{\tau}_{33} = K^{\rm b}_3[\hat{v}_3/j\omega]. \tag{18}$$

These equations represent the frequency domain equivalent of the extended linear slip model of *Pyrak-Nolte et al.* [1990]. The first term on the right-hand side of each of these equations represents the specific boundary stiffness K_i^b multiplied with the displacement jump $[\hat{v}_i/j\omega]$; the second term in equations (16) and (17) is the product of the specific boundary viscosity η^b and the velocity jump $[\hat{v}_i]$. When $\eta^b = 0$ and $K_1^b = K_2^b$, equations (16)–(18) reduce to the linear slip model of *Schoenberg* [1980].

2.4. Boundary Conditions for Electromagnetic Waves in Matter

[12] An electromagnetic wave field in matter is described in the frequency domain in terms of the electric field strength $\hat{E}_l(\mathbf{x}, \omega)$ and the magnetic field strength $\hat{H}_j(\mathbf{x}, \omega)$. At an interface between two materials in perfect contact, the boundary conditions require that the tangential components of the electric and magnetic field vectors, i.e., $\varepsilon_{klm}\hat{E}_l n_m$ and $\varepsilon_{ijk}\hat{H}_j n_k$, are continuous. Here ε_{ijk} is the alternating tensor (or Levi-Civita tensor), with $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = -\varepsilon_{213} = -\varepsilon_{321} = -\varepsilon_{132} = 1$ and the other elements are equal to zero (hence, $\varepsilon_{ijk}\hat{H}_j n_k$ represents the i component of $\hat{\mathbf{H}} \times \mathbf{n}$, etc.). These boundary conditions are consistent with Maxwell's field equations [*de Hoop*, 1995]. When the materials at both sides of an interface are not in perfect contact, the wave quantities $\varepsilon_{klm}\hat{E}_l n_m$ and $\varepsilon_{ijk}\hat{H}_j n_k$ may exhibit a finite jump across the interface. In general, the boundary conditions for imperfect interfaces cannot be derived from Maxwell's field equations, hence they should be specified independently. We postulate general boundary conditions for an electromagnetic wave field at an imperfect interface as follows:

$$\left[\varepsilon_{ijk} n_j \varepsilon_{klm} \hat{E}_l n_m\right] = -j\omega \hat{\mu}^b_{in} \langle \varepsilon_{npq} \hat{H}_p n_q \rangle \tag{19}$$

$$\left[\varepsilon_{ijk} \hat{H}_j n_k\right] = -j\omega \hat{\epsilon}^b_{il} \langle \varepsilon_{lmn} n_m \varepsilon_{npq} \hat{E}_p n_q \rangle. \tag{20}$$

Here $\hat{\mu}^b_{in} = \hat{\mu}^b_{in}(\mathbf{x}, \mathbf{n}, \omega)$ and $\hat{\epsilon}^b_{il} = \hat{\epsilon}^b_{il}(\mathbf{x}, \mathbf{n}, \omega)$ are the anisotropic space- and frequency-dependent boundary permeability and permittivity; $\mathbf{n}$ denotes that these parameters are defined for an interface with normal $\mathbf{n}$. We assume that these boundary parameters are symmetric, hence $\hat{\mu}^b_{in} = \hat{\mu}^b_{ni}$ and $\hat{\epsilon}^b_{il} = \hat{\epsilon}^b_{li}$. Note that $\varepsilon_{ijk} n_j \varepsilon_{klm} \hat{E}_l n_m$ represents the i component of $\mathbf{n} \times \hat{\mathbf{E}} \times \mathbf{n}$, etc. Hence the tangential field components in the left- and right-hand sides of these equations are mutually perpendicular (alternatively we could have formulated these boundary conditions in terms of $\varepsilon_{ijk}\hat{E}_j n_k$ and $\varepsilon_{ijk} n_j \varepsilon_{klm} \hat{H}_l n_m$).

[13] Boundary condition equations (19) and (20) can be expressed by matrix-vector equation (1), with

$$\hat{\mathbf{u}} = \begin{pmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{H}} \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} \mathbf{N}_0\mathbf{N}_0^T & \mathbf{O} \\ \mathbf{O} & \mathbf{N}_0^T \end{pmatrix}, \quad \hat{\mathbf{Y}} = \begin{pmatrix} \mathbf{O} & \hat{\boldsymbol{\mu}}^b \\ \hat{\boldsymbol{\epsilon}}^b & \mathbf{O} \end{pmatrix}, \tag{21}$$

where $(\hat{\mathbf{E}})_l = \hat{E}_l$, $(\hat{\mathbf{H}})_j = \hat{H}_j$, $(\hat{\boldsymbol{\mu}}^b)_{in} = \hat{\mu}^b_{in}$, $(\hat{\boldsymbol{\epsilon}}^b)_{il} = \hat{\epsilon}^b_{il}$ and $(\mathbf{N}_0)_{ik} = \varepsilon_{ijk} n_j$. Note that $\hat{\mathbf{u}}$ is a 6×1 vector, $\mathbf{M}$ a 6×6 matrix and $\hat{\mathbf{Y}}$ also a 6×6 matrix. Furthermore, note that $\hat{\mathbf{Y}}$ obeys symmetry relations of the form of equation (8), with matrices $\mathbf{N}$ and $\mathbf{J}$ defined in equation (A5) in Appendix A.

[14] We consider some special situations. When $\hat{\boldsymbol{\mu}}^b = \mathbf{O}$ (which is usually a good approximation) and $\mathbf{n} = (0, 0, 1)^T$ (i.e., the interface is horizontal), then equations (1) and (21) yield

$$\begin{pmatrix} [\hat{H}_2] \\ -[\hat{H}_1] \\ 0 \end{pmatrix} = -j\omega \hat{\boldsymbol{\epsilon}}^b \begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \\ 0 \end{pmatrix}, \tag{22}$$

where $\hat{\boldsymbol{\epsilon}}^b$ can be written as

$$\hat{\boldsymbol{\epsilon}}^b = \begin{pmatrix} \hat{\epsilon}^b_{11} & \hat{\epsilon}^b_{12} & 0 \\ \hat{\epsilon}^b_{21} & \hat{\epsilon}^b_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{23}$$

and hence

$$\hat{E}_1 = -\frac{1}{j\omega} \frac{\hat{\epsilon}^b_{12}[\hat{H}_1] + \hat{\epsilon}^b_{22}[\hat{H}_2]}{\hat{\epsilon}^b_{11}\hat{\epsilon}^b_{22} - \hat{\epsilon}^b_{12}\hat{\epsilon}^b_{21}} \tag{24}$$

$$\hat{E}_2 = \frac{1}{j\omega} \frac{\hat{\epsilon}^b_{11}[\hat{H}_1] + \hat{\epsilon}^b_{21}[\hat{H}_2]}{\hat{\epsilon}^b_{11}\hat{\epsilon}^b_{22} - \hat{\epsilon}^b_{12}\hat{\epsilon}^b_{21}}. \tag{25}$$

Note that $\hat{\epsilon}^b_{il} = \epsilon^b_{il} + \sigma^b_{il}/j\omega$, where σ^b_{il} is the boundary conductivity. In the isotropic situation and in the diffusive field approximation equations (23)–(25) reduce to

$$\hat{\boldsymbol{\epsilon}}^b = \frac{1}{j\omega} \begin{pmatrix} \sigma^b & 0 & 0 \\ 0 & \sigma^b & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{26}$$

and

$$\hat{E}_1 = -\frac{1}{\sigma^b}[\hat{H}_2] \tag{27}$$

$$\hat{E}_2 = \frac{1}{\sigma^b}[\hat{H}_1]. \tag{28}$$

This corresponds to the model of *Kaufman and Keller* [1983] for a conductive interface, with $(\sigma^b)^{-1}$ being the specific boundary resistivity. We refer to equations (22)–(25) as the extended *Kaufman and Keller* [1983] model.

2.5. Boundary Conditions for Poroelastic Waves in Porous Solids

[15] A poroelastic wave field in a porous solid is described in the frequency domain in terms of the averaged bulk stress $\hat{\tau}^b_{ij}(\mathbf{x}, \omega)$, the averaged fluid pressure $\hat{p}(\mathbf{x}, \omega)$, the averaged solid particle velocity $\hat{v}^s_i(\mathbf{x}, \omega)$ and the averaged filtration velocity $\hat{w}_i(\mathbf{x}, \omega)$ [*Biot*, 1956; *Smeulders et al.*, 1992; *Gurevich and Schoenberg*, 1999]. All averages are volumetric averages. At an interface between two porous solids in perfect contact with open pores, the boundary conditions require that the bulk traction $(\hat{\tau}^b_{ij} n_j)$, the fluid pressure $(\hat{p})$, the solid particle velocity $(\hat{v}^s_i)$ and the normal component of the filtration velocity $(\hat{w}_i n_i)$ are continuous [*Deresiewicz and*

Skalak, 1963]. These conditions are consistent with the poroelastic equations of motion and the stress-strain relations [*Gurevich and Schoenberg*, 1999]. When the porous solids at both sides of an interface are not in perfect contact, the wave quantities $\hat{\tau}^b_{ij}n_j$, $\hat{p}$, $\hat{v}^s_i$ and $\hat{w}_i n_i$ may exhibit a finite jump across the interface. In general, the boundary conditions for imperfect interfaces cannot be derived from the poroelastic equations of motion and the stress-strain relations, hence they should be specified independently. We postulate that the general boundary conditions for a poroelastic wave field at an imperfect interface can be expressed by equation (1), with

$$\hat{\mathbf{u}} = \begin{pmatrix} \hat{\mathbf{v}}^s \\ -\hat{\boldsymbol{\tau}}^b_1 \\ -\hat{\boldsymbol{\tau}}^b_2 \\ -\hat{\boldsymbol{\tau}}^b_3 \\ \hat{\mathbf{w}} \\ \hat{p} \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} \\ \mathbf{O} & n_1\mathbf{I} & n_2\mathbf{I} & n_3\mathbf{I} & \mathbf{O} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{n}^T & 0 \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & 1 \end{pmatrix}, \tag{29}$$

where $(\hat{\mathbf{v}}^s)_i = \hat{v}^s_i$, $(\hat{\boldsymbol{\tau}}^b_j)_i = \hat{\tau}^b_{ij}$ and $(\hat{\mathbf{w}})_i = \hat{w}_i$. Note that $\hat{\mathbf{u}}$ is a 16×1 vector, $\mathbf{M}$ a 8×16 matrix and $\hat{\mathbf{Y}}$ in equation (1) a 8×8 matrix with zero block matrices on the diagonal. We assume again that the nonzero block matrices in $\hat{\mathbf{Y}}$ are symmetric, such that $\hat{\mathbf{Y}}$ obeys symmetry relations of the form of equation (8), with matrices $\mathbf{N}$ and $\mathbf{J}$ defined in equation (A7) in Appendix A.

[16] For the special case that $(\hat{\mathbf{Y}})_{8,7} = \hat{\varrho}^{\mathrm{b}}$ and all other elements of $\hat{\mathbf{Y}}$ are zero, equations (1) and (29) yield

$$[\hat{p}] = -j\omega\hat{\varrho}^{\mathrm{b}}\hat{w}_i n_i, \tag{30}$$

where $1/j\omega\hat{\varrho}^{\mathrm{b}}$ is the hydraulic boundary permeability. Equation (30) corresponds to the model of *Gurevich and Schoenberg* [1999] for an imperfect interface between porous solids.

2.6. Boundary Conditions for Seismoelectric Waves in Porous Solids

[17] A seismoelectric wave field in a porous solid is described in the frequency domain in terms of the averaged electric field strength $\hat{E}_l(\mathbf{x}, \omega)$, the averaged magnetic field strength $\hat{H}_j(\mathbf{x}, \omega)$, the averaged bulk stress $\hat{\tau}^b_{ij}(\mathbf{x}, \omega)$, the averaged fluid pressure $\hat{p}(\mathbf{x}, \omega)$, the averaged solid particle velocity $\hat{v}^s_i(\mathbf{x}, \omega)$ and the averaged filtration velocity $\hat{w}_i(\mathbf{x}, \omega)$ [*Pride*, 1994; *Pride and Haartsen*, 1996]. All averages are volumetric averages. At an interface between two porous solids in perfect contact with open pores, the boundary conditions require that the tangential components of the electric and magnetic field vectors, i.e., $\varepsilon_{klm}\hat{E}_l n_m$ and $\varepsilon_{ijk}\hat{H}_j n_k$, the bulk traction ($\hat{\tau}^b_{ij}n_j$), the fluid pressure ($\hat{p}$), the solid particle velocity ($\hat{v}^s_i$) and the normal component of the filtration velocity ($\hat{w}_i n_i$) are continuous. When the porous solids at both sides of an interface are not in perfect contact, the wave quantities $\varepsilon_{klm}\hat{E}_l n_m$, $\varepsilon_{ijk}\hat{H}_j n_k$, $\hat{\tau}^b_{ij}n_j$, $\hat{p}$, $\hat{v}^s_i$ and $\hat{w}_i n_i$ may exhibit a finite jump across the interface. We postulate that the general boundary conditions for a seismoelectric wave field at an imperfect interface can be expressed by equation (1), with

$$\hat{\mathbf{u}} = \begin{pmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{H}} \\ \hat{\mathbf{v}}^s \\ -\hat{\boldsymbol{\tau}}^b_1 \\ -\hat{\boldsymbol{\tau}}^b_2 \\ -\hat{\boldsymbol{\tau}}^b_3 \\ \hat{\mathbf{w}} \\ \hat{p} \end{pmatrix} \tag{31}$$

$$\mathbf{M} = \begin{pmatrix} \mathbf{N}_0\mathbf{N}_0^T & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} \\ \mathbf{O} & \mathbf{N}_0^T & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & n_1\mathbf{I} & n_2\mathbf{I} & n_3\mathbf{I} & \mathbf{O} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{n}^T & 0 \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & 1 \end{pmatrix}.$$

Note that $\hat{\mathbf{u}}$ is a 22×1 vector, $\mathbf{M}$ a 14×22 matrix and $\hat{\mathbf{Y}}$ in equation (1) a 14×14 matrix with zero block matrices on the diagonal. We assume again that the nonzero block matrices in $\hat{\mathbf{Y}}$ are symmetric, such that $\hat{\mathbf{Y}}$ obeys symmetry relations of the form of equation (8), with matrices $\mathbf{N}$ and $\mathbf{J}$ defined in equation (A9) in Appendix A.

3. Unified Reciprocity Theorems

[18] For any of the wave vectors $\hat{\mathbf{u}}$ defined above, we previously formulated two unified reciprocity theorems [*Wapenaar and Fokkema*, 2004], which are briefly reviewed here. We consider two physical states in a volume $\mathcal{V}$, enclosed by surface $\partial\mathcal{V}$ with outward pointing normal vector $\mathbf{n}$. The field quantities, the material parameters, as well as the source functions may be different in both states; they will be distinguished with subscripts A and B (of course the summation convention does not apply for these subscripts). For the moment we assume that there are no interfaces in $\mathcal{V}$. In the frequency domain, the reciprocity theorem of the convolution type reads

$$\oint_{\partial\mathcal{V}} \hat{\mathbf{u}}_A^T\mathbf{K}\mathbf{N}_\mathbf{x}\hat{\mathbf{u}}_B\,\mathrm{d}^2\mathbf{x} = \int_{\mathcal{V}} \left(\hat{\mathbf{u}}_A^T\mathbf{K}\hat{\mathbf{s}}_B - \hat{\mathbf{s}}_A^T\mathbf{K}\hat{\mathbf{u}}_B\right)\mathrm{d}^3\mathbf{x} + \int_{\mathcal{V}} \hat{\mathbf{u}}_A^T\mathbf{K}(j\omega(\mathbf{A}_A - \mathbf{A}_B) + (\mathbf{B}_A - \mathbf{B}_B))\hat{\mathbf{u}}_B\,\mathrm{d}^3\mathbf{x}. \tag{32}$$

We speak of a convolution-type theorem, because the multiplications in the frequency domain correspond to convolutions in the time domain. This theorem interrelates the wave field quantities (contained in $\hat{\mathbf{u}}_A$ and $\hat{\mathbf{u}}_B$), the material parameters (contained in $\mathbf{A}_A$, $\mathbf{B}_A$, $\mathbf{A}_B$ and $\mathbf{B}_B$) as well as the source functions (contained in $\hat{\mathbf{S}}_A$ and $\hat{\mathbf{S}}_B$) of states A and B. The material parameter matrices and the source vectors are given by *Wapenaar and Fokkema* [2004] for the various wave phenomena discussed above. They are not repeated here because they play no role in the derivations in this paper. The matrices $\mathbf{N}_\mathbf{x}$ and $\mathbf{K}$ are given in Appendix A for the various wave phenomena. Note that $\mathbf{N}_\mathbf{x}$ contains the components of the normal vector $\mathbf{n}$ in a particular way; $\mathbf{K}$ is a diagonal matrix with +1s and −1s on its diagonal. The left-hand side of

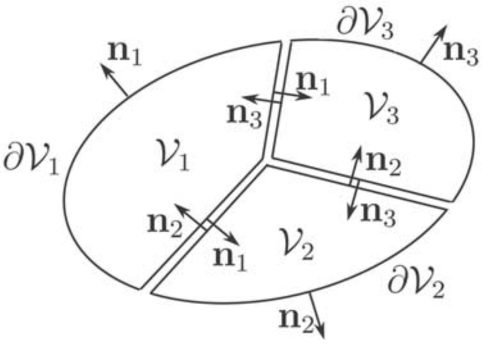

Figure 2. Piecewise continuous volume $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cdots \cup \mathcal{V}_L$.

equation (32) is a boundary integral which contains a specific combination of the field quantities of states A and B at the boundary of the volume $\mathcal{V}$. The first integral on the right-hand side interrelates the field quantities and the source functions in $\mathcal{V}$; the second integral contains the differences of the medium parameters in both states (in the work of *Wapenaar and Fokkema* [2004] the right-hand side contains two extra integrals that are applicable for acoustic waves in a moving medium). Depending on the type of application, states A and B can be both physical states, or both mathematical states (e.g., Green's states), or one can be a physical state and the other a mathematical state. The latter situation leads to representation integrals [see *Gangi*, 1970]. When the medium parameters are identical in both states and the medium outside $\partial\mathcal{V}$ is source-free, equation (32) reduces to

$$\int_{\mathcal{V}} \hat{\mathbf{u}}_A^T \mathbf{K} \hat{\mathbf{s}}_B \, \mathrm{d}^3\mathbf{x} = \int_{\mathcal{V}} \hat{\mathbf{s}}_A^T \mathbf{K} \hat{\mathbf{u}}_B \, \mathrm{d}^3\mathbf{x}. \tag{33}$$

By choosing spatial delta functions for the source distributions, this equation formulates source-receiver reciprocity for each of the wave phenomena treated in this paper. Note that this important property is just one result of the unified reciprocity theorem of equation (32). For further discussions on convolution type reciprocity theorems in different fields of application we refer to *de Hoop and Stam* [1988], *Fokkema and van den Berg* [1993], *Gangi* [2000], *Allard et al.* [1993], *Pride and Haartsen* [1996], and *Belinskiy* [2001].

[19] The reciprocity theorem of the correlation type reads

$$\oint_{\partial\mathcal{V}} \hat{\mathbf{u}}_A^H \mathbf{N}_\mathbf{x} \hat{\mathbf{u}}_B \, \mathrm{d}^2\mathbf{x} = \int_{\mathcal{V}} \left(\hat{\mathbf{u}}_A^H \hat{\mathbf{s}}_B + \hat{\mathbf{s}}_A^H \hat{\mathbf{u}}_B \right) \mathrm{d}^3\mathbf{x} + \int_{\mathcal{V}} \hat{\mathbf{u}}_A^H \big(j\omega(\mathbf{A}_A - \mathbf{A}_B) - \left(\mathbf{B}_A^H + \mathbf{B}_B\right)\big) \hat{\mathbf{u}}_B \, \mathrm{d}^3\mathbf{x}. \tag{34}$$

We speak of a correlation-type theorem, because the multiplications in the frequency domain correspond to correlations in the time domain. The term $\hat{\mathbf{u}}_A^H$ contains "back-propagating" wave field quantities in state A [see *Bojarski*, 1983]. When we compare this reciprocity theorem with equation (32), we observe that, apart from the complex conjugation, the diagonal matrix $\mathbf{K}$ is absent in all integrals and that some plus and minus signs have been changed. When the medium parameters, sources and wave fields are identical in both states, this reciprocity theorem reduces to (omitting subscripts A and B)

$$\int_{\mathcal{V}} \left(\hat{\mathbf{u}}^H \hat{\mathbf{s}} + \hat{\mathbf{s}}^H \hat{\mathbf{u}} \right) \mathrm{d}^3\mathbf{x} = \oint_{\partial\mathcal{V}} \hat{\mathbf{u}}^H \mathbf{N}_\mathbf{x} \hat{\mathbf{u}} \, \mathrm{d}^2\mathbf{x} + \int_{\mathcal{V}} \hat{\mathbf{u}}^H \left(\mathbf{B}^H + \mathbf{B}\right) \hat{\mathbf{u}} \, \mathrm{d}^3\mathbf{x}. \tag{35}$$

Note that this form of the reciprocity theorem represents a power balance for each of the wave phenomena treated in this paper. The term on the left-hand side represents the power, generated by the sources in $\mathcal{V}$. The first term on the right-hand side represents the power flux propagating outward through ∂ $\mathcal{V}$ and the second term the power dissipated by the medium in $\mathcal{V}$. Note that this is just one result of the unified reciprocity theorem of equation (34). For a discussion on the application of correlation type reciprocity theorems to inverse problems we refer to *Fisher and Langenberg* [1984] and *de Hoop and Stam* [1988].

4. Reciprocity Theorem for Media With Imperfect Interfaces

[20] We now extend the reciprocity theorems for the situation in which $\mathcal{V}$ contains imperfect internal interfaces. To this end we subdivide $\mathcal{V}$ into L continuous regions, according to $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cdots \cup \mathcal{V}_L$, see Figure 2. Region $\mathcal{V}_l$ is enclosed by surface $\partial\mathcal{V}_l$ with outward pointing normal vector $\mathbf{n}_l$. The boundaries between these regions represent the imperfect internal interfaces. Note that each internal interface is part of two surfaces $\partial\mathcal{V}_l$, with oppositely pointing normal vectors $\mathbf{n}_l$, see Figure 2.

[21] Since the medium parameters in region $\mathcal{V}_l$ are continuous, the reciprocity theorem equations (32) and (34) apply to each of these regions. Summing both sides of these equations over l yields again equations (32) and (34) for the total volume $\mathcal{V}$, with in the left-hand sides extra integrals over the internal interfaces, according to

$$\int_{\Sigma_{\rm int}} \left(\left(\hat{\mathbf{u}}_A^T \mathbf{K} \mathbf{N}_\mathbf{x} \hat{\mathbf{u}}_B\right)_1 + \left(\hat{\mathbf{u}}_A^T \mathbf{K} \mathbf{N}_\mathbf{x} \hat{\mathbf{u}}_B\right)_2 \right) \mathrm{d}^2\mathbf{x} \tag{36}$$

and

$$\int_{\Sigma_{\rm int}} \left(\left(\hat{\mathbf{u}}_A^H \mathbf{N}_\mathbf{x} \hat{\mathbf{u}}_B\right)_1 + \left(\hat{\mathbf{u}}_A^H \mathbf{N}_\mathbf{x} \hat{\mathbf{u}}_B\right)_2 \right) \mathrm{d}^2\mathbf{x}, \tag{37}$$

respectively, where $\Sigma_{\rm int}$ constitutes the total of all internal interfaces; the subscripts $_1$ and $_2$ denote the two sides of the internal interfaces. In the following we evaluate these integrals, using the general boundary condition of equation (1). To this end we first express $\mathbf{K}\mathbf{N}_\mathbf{x}$ and $\mathbf{N}_\mathbf{x}$ as

$$\mathbf{K}\mathbf{N}_\mathbf{x} = \mathbf{M}^T \mathbf{N} \mathbf{M} \tag{38}$$

$$\mathbf{N}_\mathbf{x} = \mathbf{M}^H \mathbf{J} \mathbf{M}. \tag{39}$$

Matrices $\mathbf{N}$ and $\mathbf{J}$ are given in Appendix A for the various wave phenomena. Since the normal vectors contained in

$\mathbf{N_x}$ and $\mathbf{M}$ have different signs at opposite sides of an interface (see Figure 2), we have

$$\mathbf{M}_2^T\mathbf{N}\mathbf{M}_2 = -\mathbf{M}_1^T\mathbf{N}\mathbf{M}_1 \tag{40}$$

$$\mathbf{M}_2^H\mathbf{J}\mathbf{M}_2 = -\mathbf{M}_1^H\mathbf{J}\mathbf{M}_1. \tag{41}$$

We use equations (38)–(41) to rewrite the interface integral equations (36) and (37) as

$$\int_{\Sigma_{\rm int}}\left(\hat{\mathbf{u}}_{1,A}^T\mathbf{M}^T\mathbf{N}\mathbf{M}\hat{\mathbf{u}}_{1,B} - \hat{\mathbf{u}}_{2,A}^T\mathbf{M}^T\mathbf{N}\mathbf{M}\hat{\mathbf{u}}_{2,B}\right){\rm d}^2\mathbf{x} \tag{42}$$

and

$$\int_{\Sigma_{\rm int}}\left(\hat{\mathbf{u}}_{1,A}^H\mathbf{M}^H\mathbf{J}\mathbf{M}\hat{\mathbf{u}}_{1,B} - \hat{\mathbf{u}}_{2,A}^H\mathbf{M}^H\mathbf{J}\mathbf{M}\hat{\mathbf{u}}_{2,B}\right){\rm d}^2\mathbf{x}, \tag{43}$$

respectively, where $\mathbf{M}$ stands for $\mathbf{M}_1$. In case of perfect interfaces, we have $\mathbf{M}\hat{\mathbf{u}}_2 = \mathbf{M}\hat{\mathbf{u}}_1$ for state A as well as state B, hence, the internal interface integrals vanish. This means that the reciprocity theorem equations (32) and (34) are valid for a piecewise continuous medium (as in Figure 2) with perfect interfaces.

[22] Of course the more interesting case is the one with imperfect interfaces. For this situation we rewrite the general boundary condition equation (1) as

$$\mathbf{M}(\hat{\mathbf{u}}_2 - \hat{\mathbf{u}}_1) = -j\omega\hat{\mathbf{Y}}\mathbf{M}(\hat{\mathbf{u}}_2 + \hat{\mathbf{u}}_1)/2. \tag{44}$$

Bringing $\hat{\mathbf{u}}_1$ to one side of the equation and $\hat{\mathbf{u}}_2$ to the other yields

$$\left(\mathbf{I} + j\omega\hat{\mathbf{Y}}/2\right)\mathbf{M}\hat{\mathbf{u}}_2 = \left(\mathbf{I} - j\omega\hat{\mathbf{Y}}/2\right)\mathbf{M}\hat{\mathbf{u}}_1 \tag{45}$$

or

$$\mathbf{M}\hat{\mathbf{u}}_2 = \hat{\mathbf{Z}}\,\mathbf{M}\hat{\mathbf{u}}_1, \tag{46}$$

with

$$\hat{\mathbf{Z}} = \left(\mathbf{I} + j\omega\hat{\mathbf{Y}}/2\right)^{-1}\left(\mathbf{I} - j\omega\hat{\mathbf{Y}}/2\right). \tag{47}$$

Using the symmetry relations of equation (8) we obtain

$$\hat{\mathbf{Z}}^T\mathbf{N} = \mathbf{N}\hat{\mathbf{Z}}^{-1} \qquad \hat{\mathbf{Z}}^H\mathbf{J} = \mathbf{J}\left(\hat{\mathbf{Z}}'\right)^{-1}, \tag{48}$$

where $\hat{\mathbf{Z}}'$ is defined by equation (47), with $\hat{\mathbf{Y}}$ replaced by $\hat{\mathbf{Y}}^*$, and hence

$$\hat{\mathbf{Z}}' = \left(\mathbf{I} + j\omega\hat{\mathbf{Y}}^*/2\right)^{-1}\left(\mathbf{I} - j\omega\hat{\mathbf{Y}}^*/2\right). \tag{49}$$

We substitute equation (46) for states A and B into the interface integral equations (42) and (43), use the symmetry relations of equation (48), and add the resulting integrals to the left-hand sides of the reciprocity theorems (32) and (34). This yields for the convolution-type reciprocity theorem

$$\oint_{\partial\mathcal{V}}\hat{\mathbf{u}}_A^T\mathbf{K}\mathbf{N_x}\hat{\mathbf{u}}_B\,{\rm d}^2\mathbf{x} + \int_{\Sigma_{\rm int}}\hat{\mathbf{u}}_A^T\mathbf{M}^T\mathbf{N}\left(\mathbf{I} - \hat{\mathbf{Z}}_A^{-1}\hat{\mathbf{Z}}_B\right)\mathbf{M}\hat{\mathbf{u}}_B\,{\rm d}^2\mathbf{x} = \int_{\mathcal{V}}\left(\hat{\mathbf{u}}_A^T\mathbf{K}\hat{\mathbf{s}}_B - \hat{\mathbf{s}}_A^T\mathbf{K}\hat{\mathbf{u}}_B\right){\rm d}^3\mathbf{x} + \int_{\mathcal{V}}\hat{\mathbf{u}}_A^T\mathbf{K}\left(j\omega(\mathbf{A}_A - \mathbf{A}_B) + (\mathbf{B}_A - \mathbf{B}_B)\right)\hat{\mathbf{u}}_B\,{\rm d}^3\mathbf{x} \tag{50}$$

and for the correlation-type reciprocity theorem

$$\oint_{\partial\mathcal{V}}\hat{\mathbf{u}}_A^H\mathbf{N_x}\hat{\mathbf{u}}_B\,{\rm d}^2\mathbf{x} + \int_{\Sigma_{\rm int}}\hat{\mathbf{u}}_A^H\mathbf{M}^H\mathbf{J}\left(\mathbf{I} - \left(\hat{\mathbf{Z}}_A'\right)^{-1}\hat{\mathbf{Z}}_B\right)\mathbf{M}\hat{\mathbf{u}}_B\,{\rm d}^2\mathbf{x} = \int_{\mathcal{V}}\left(\hat{\mathbf{u}}_A^H\hat{\mathbf{s}}_B + \hat{\mathbf{s}}_A^H\hat{\mathbf{u}}_B\right){\rm d}^3\mathbf{x} + \int_{\mathcal{V}}\hat{\mathbf{u}}_A^H\left(j\omega(\mathbf{A}_A - \mathbf{A}_B) - \left(\mathbf{B}_A^H + \mathbf{B}_B\right)\right)\hat{\mathbf{u}}_B\,{\rm d}^3\mathbf{x}. \tag{51}$$

In the internal interface integrals, $\hat{\mathbf{u}}_A$ and $\hat{\mathbf{u}}_B$ stand for $\hat{\mathbf{u}}_{1,A}$ and $\hat{\mathbf{u}}_{1,B}$, respectively (similar as M stands for $\mathbf{M}_1$). It is arbitrary which side of the interface is designated "side 1." All that matters is that $\hat{\mathbf{u}}_A$, $\hat{\mathbf{u}}_B$ and $\mathbf{M}$ all refer to the same side of the interface.

[23] In general the interface matrices $\hat{\mathbf{Z}}_A$ and $\hat{\mathbf{Z}}_B$ need not be the same. For example, when state A is a Green's state in a background medium with perfect interfaces, then $\hat{\mathbf{Z}}_A = \mathbf{I}$. In this case equations (50) and (51) can be cast as representations for the actual wave field $\hat{\mathbf{u}}_B$ in a medium with imperfect interfaces, represented by $\hat{\mathbf{Z}}_B$. A further discussion of representations is beyond the scope of this paper.

[24] Note that when the boundary parameters contained in matrix $\hat{\mathbf{Y}}$ are small, we may approximate $\hat{\mathbf{Z}}$ by $\hat{\mathbf{Z}} \approx \mathbf{I} - j\omega\hat{\mathbf{Y}}$. For this situation the internal interface integrals in equations (50) and (51) can be written in such a way that they contain explicitly the contrasts of the boundary parameter matrices in states A and B, according to

$$-j\omega\int_{\Sigma_{\rm int}}\hat{\mathbf{u}}_A^T\mathbf{M}^T\mathbf{N}\left(\hat{\mathbf{Y}}_A - \hat{\mathbf{Y}}_B\right)\mathbf{M}\hat{\mathbf{u}}_B\,{\rm d}^2\mathbf{x} \tag{52}$$

and

$$-j\omega\int_{\Sigma_{\rm int}}\hat{\mathbf{u}}_A^H\mathbf{M}^H\mathbf{J}\left(\hat{\mathbf{Y}}_A^* - \hat{\mathbf{Y}}_B\right)\mathbf{M}\hat{\mathbf{u}}_B\,{\rm d}^2\mathbf{x}, \tag{53}$$

respectively.

[25] We conclude this section by discussing a special situation for each of the reciprocity theorem equations (50) and (51).

4.1. Source-Receiver Reciprocity

[26] Consider the convolution-type reciprocity theorem, given by equation (50). It is easily seen that the internal interface integral vanishes when the boundary parameters of the imperfect interfaces in states A and B are identical. Hence for this situation the reciprocity theorem of the convolution type reduces to the original theorem, defined in equation (32), but with $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cdots \cup \mathcal{V}_L$. When, apart from the boundary parameters, the medium parame-

ters are identical in both states and the medium outside $\partial\mathcal{V}$ is source-free, this theorem reduces to equation (33), with $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cdots \cup \mathcal{V}_L$. By choosing spatial delta functions for the source distributions in different regions $\mathcal{V}_l$ it follows that, for each of the wave phenomena treated in this paper, source-receiver reciprocity remains valid when the source and receiver are separated by imperfect interfaces. Of course the underlying assumption is that the boundary conditions can be expressed by equation (1) and that the boundary parameter matrix $\hat{\mathbf{Y}}$ obeys the first of the symmetry relations expressed by equation (8).

4.2. Power Balance

[27] Consider the correlation-type reciprocity theorem, given by equation (51). When the medium and boundary parameters, sources and wave fields are identical in both states, this reciprocity theorem reduces to the power balance of equation (35), with in the right-hand side the extra term

$$\int_{\Sigma_{\rm int}} \hat{\mathbf{u}}^H \mathbf{M}^H \mathbf{J}\left(\mathbf{I} - \left(\hat{\mathbf{Z}}'\right)^{-1}\hat{\mathbf{Z}}\right)\mathbf{M}\hat{\mathbf{u}}\, \mathrm{d}^2\mathbf{x}, \tag{54}$$

which represents the power dissipated by the internal imperfect interfaces. Note that this term vanishes when $\hat{\mathbf{Z}} = \hat{\mathbf{Z}}'$. In the linear slip model of *Schoenberg* [1980], the parameters in matrix $\hat{\mathbf{Y}}$ are real valued. This implies $\hat{\mathbf{Z}} = \hat{\mathbf{Z}}'$ (see equation (49)), hence, in this model there is no power dissipation at the interfaces. For many other situations, like the extended linear slip model of *Pyrak-Nolte et al.* [1990], the permeable boundary model of *Gurevich and Schoenberg* [1999] and the conductive interface model of *Kaufman and Keller* [1983], the parameters in matrix $\hat{\mathbf{Y}}$ are complex valued, hence, in these cases the integral of equation (54) quantifies the power loss at the imperfect interfaces.

5. Conclusions

[28] We have postulated general boundary conditions at imperfect interfaces for acoustic waves in fluids, elastodynamic waves in solids, electromagnetic waves in matter, poroelastic waves in porous solids and seismoelectric waves in porous solids, in such a way that they cover the linear slip model of *Schoenberg* [1980], the extended linear slip models of *Pyrak-Nolte et al.* [1990] and *Nakagawa et al.* [2000], the permeable boundary model of *Gurevich and Schoenberg* [1999], and the conductive interface model of *Kaufman and Keller* [1983]. These boundary conditions are expressed by the general matrix-vector equation $[\mathbf{M}\hat{\mathbf{u}}] = -j\omega\hat{\mathbf{Y}}\langle\mathbf{M}\hat{\mathbf{u}}\rangle$, where matrix $\hat{\mathbf{Y}}$ contains the boundary parameters. Using this equation, we have extended two unified reciprocity theorems (one of the convolution type and one of the correlation type) with an extra integral over the imperfect interfaces (equations (50) and (51)). It appears that the extra integral in the convolution-type reciprocity theorem vanishes when the boundary parameters in both states are identical. This implies that source-receiver reciprocity remains valid when the source and receiver are separated by imperfect interfaces. When the medium and boundary parameters, sources and wave fields are identical in both states then the extra integral in the correlation-type reciprocity theorem quantifies the power dissipation at the imperfect interfaces. This integral vanishes when the boundary parameters contained in matrix $\hat{\mathbf{Y}}$ are real valued.

Appendix A: Matrices in the Symmetry Relations and Reciprocity Theorems

A1. Acoustic Waves

[29] For acoustic waves, the matrices involved in the symmetry relations of equation (8) and in the reciprocity theorem equations (32) and (34) are given by

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{A1}$$

$$\mathbf{N_x} = \begin{pmatrix} 0 & n_1 & n_2 & n_3 \\ n_1 & 0 & 0 & 0 \\ n_2 & 0 & 0 & 0 \\ n_3 & 0 & 0 & 0 \end{pmatrix}, \mathbf{K} = \mathrm{diag}(1, -1, -1, -1). \tag{A2}$$

Note that equations (38) and (39) are obeyed, with $\mathbf{M}$ defined in equation (7).

A2. Elastodynamic Waves

[30] For elastodynamic waves, the matrices involved in the symmetry relations of equation (8) and in the reciprocity theorem equations (32) and (34) are given by

$$\mathbf{N} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{pmatrix}, \mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}, \tag{A3}$$

$$\mathbf{N_x} = \begin{pmatrix} \mathbf{O} & n_1\mathbf{I} & n_2\mathbf{I} & n_3\mathbf{I} \\ n_1\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ n_2\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ n_3\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}, \mathbf{K} = \mathrm{diag}(\mathbf{1}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}), \tag{A4}$$

with $\mathbf{1} = (1, 1, 1)$. Note that equations (38) and (39) are obeyed, with $\mathbf{M}$ defined in equation (11).

A3. Electromagnetic Waves

[31] For electromagnetic waves, the matrices involved in the symmetry relations of equation (8) and in the reciprocity theorem equations (32) and (34) are given by

$$\mathbf{N} = \begin{pmatrix} \mathbf{O} & -\mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}, \mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}, \tag{A5}$$

$$\mathbf{N_x} = \begin{pmatrix} \mathbf{O} & \mathbf{N}_0^T \\ \mathbf{N}_0 & \mathbf{O} \end{pmatrix}, \quad \mathbf{N}_0 = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}, \quad \mathbf{K} = \mathrm{diag}(-\mathbf{1}, \mathbf{1}). \tag{A6}$$

Note that equations (38) and (39) are obeyed, with $\mathbf{M}$ defined in equation (21).

A4. Poroelastic Waves

[32] For poroelastic waves, the matrices involved in the symmetry relations of equation (8) and in the reciprocity theorem equations (32) and (34) are given by

$$\mathbf{N} = \begin{pmatrix} \mathbf{O} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{O} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & 0 & 1 \\ \mathbf{0}^T & \mathbf{0}^T & -1 & 0 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{O} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & 0 & 1 \\ \mathbf{0}^T & \mathbf{0}^T & 1 & 0 \end{pmatrix}, \quad \text{(A7)}$$

$$\mathbf{N_x} = \begin{pmatrix} \mathbf{O} & n_1\mathbf{I} & n_2\mathbf{I} & n_3\mathbf{I} & \mathbf{O} & \mathbf{0} \\ n_1\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} \\ n_2\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} \\ n_3\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{n} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{n}^T & 0 \end{pmatrix}, \quad \text{(A8)}$$

$$\mathbf{K} = \mathrm{diag}(\mathbf{1}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, -1).$$

Note that equations (38) and (39) are obeyed, with $\mathbf{M}$ defined in equation (29).

A5. Seismoelectric Waves

[33] For seismoelectric waves, the matrices involved in the symmetry relations of equation (8) and in the reciprocity theorem equations (32) and (34) are given by

$$\mathbf{N} = \begin{pmatrix} \mathbf{O} & -\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} & \mathbf{0} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{O} & \mathbf{O} & -\mathbf{I} & \mathbf{O} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & 0 & 1 \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & -1 & 0 \end{pmatrix},$$

$$\mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} & \mathbf{0} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & 0 & 1 \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & 1 & 0 \end{pmatrix}, \quad \text{(A9)}$$

$$\mathbf{N_x} = \begin{pmatrix} \mathbf{O} & \mathbf{N}_0^T & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} \\ \mathbf{N}_0 & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & n_1\mathbf{I} & n_2\mathbf{I} & n_3\mathbf{I} & \mathbf{O} & \mathbf{0} \\ \mathbf{O} & \mathbf{O} & n_1\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} \\ \mathbf{O} & \mathbf{O} & n_2\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} \\ \mathbf{O} & \mathbf{O} & n_3\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{0} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{n} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{n}^T & 0 \end{pmatrix}, \quad \text{(A10)}$$

$$\mathbf{K} = \mathrm{diag}(-\mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, -1).$$

Note that equations (38) and (39) are obeyed, with $\mathbf{M}$ defined in equation (31).

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References

Allard, J. F., B. Brouard, and D. Lafarge (1993), Reciprocity and antireciprocity in sound transmission through layered materials including elastic and porous media, *Wave Motion*, *17*, 329–335.
Belinskiy, B. P. (2001), On some general mathematical properties of the system: Elastic plate–acoustic medium, in *Acoustic Interactions With Submerged Elastic Structures*, part II, *Propagation, Ocean Acoustics, and Scattering*, *Ser. Stability Vibration Control Syst. B*, vol. 5, edited by A. Guran et al., pp. 193–218, World Sci., River Edge, N. J.
Biot, M. A. (1956), Theory of propagation of elastic waves in a fluid-saturated porous solid: I. Low frequency range, *J. Acoust. Soc. Am.*, *28*, 168–178.
Bojarski, N. N. (1983), Generalized reaction principles and reciprocity theorems for the wave equations, and the relationship between the time-advanced and time-retarded fields, *J. Acoust. Soc. Am.*, *74*, 281–285.
de Hoop, A. T. (1995), *Handbook of Radiation and Scattering of Waves*, Academic, San Diego, Calif.
de Hoop, A. T., and H. J. Stam (1988), Time-domain reciprocity theorems for elastodynamic wave fields in solids with relaxation and their application to inverse problems, *Wave Motion*, *10*, 479–489.
Deresiewicz, H., and R. Skalak (1963), On uniqueness in dynamic poroelasticity, *Bull. Seismol. Soc. Am.*, *53*, 783–788.
Fisher, M., and K. J. Langenberg (1984), Limitations and defects of certain inverse scattering theories, *IEEE Trans. Antennas Propag.*, *32*, 1080–1088.
Fokkema, J. T., and P. M. van den Berg (1993), *Seismic Applications of Acoustic Reciprocity*, Elsevier Sci., New York.
Gangi, A. F. (1970), A derivation of the seismic representation theorem using seismic reciprocity, *J. Geophys. Res.*, *75*, 2088–2095.
Gangi, A. F. (2000), Constitutive equations and reciprocity, *Geophys. J. Int.*, *143*, 311–318.
Gurevich, B., and M. Schoenberg (1999), Interface conditions for Biot's equations of poroelasticity, *J. Acoust. Soc. Am.*, *105*, 2585–2589.
Kaufman, A. A., and G. V. Keller (1983), *Frequency and Transient Soundings*, *Methods Geochem. Geophys.*, vol. 16, Elsevier Sci., New York.
Nakagawa, S., K. T. Nihei, and L. R. Myer (2000), Shear-induced conversion of seismic waves across single fractures, *Int. J. Rock Mech. Min. Sci.*, *37*, 203–218.
Pride, S. (1994), Governing equations for the coupled electromagnetics and acoustics of porous media, *Phys. Rev. B*, *50*, 15,678–15,696.
Pride, S. R., and M. W. Haartsen (1996), Electroseismic wave properties, *J. Acoust. Soc. Am.*, *100*, 1301–1315.
Pyrak-Nolte, L. J., L. R. Myer, and N. G. W. Cook (1990), Transmission of seismic waves across single natural fractures, *J. Geophys. Res.*, *95*, 8617–8638.
Schoenberg, M. (1980), Elastic wave behaviour across linear slip interfaces, *J. Acoust. Soc. Am.*, *68*, 1516–1521.
Smeulders, D. M. J., R. L. G. M. Eggels, and M. E. H. van Dongen (1992), Dynamic permeability: Reformulation of theory and new experimental and numerical data, *J. Fluid Mech.*, *245*, 211–227.
Wapenaar, C. P. A., and J. T. Fokkema (2004), Reciprocity theorems for diffusion, flow and waves, *J. Appl. Mech.*, *71*, 145–150.

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