# SLICING THE EARTH: A LAYER-STRIPPING METHOD EMPLOYING A CAUSALITY-BASED IMAGING CONDITION

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#### ABSTRACT

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We formulate the theory for a direct nonlinear seismic inversion method in the acoustic approximation. It is a completely data-driven method, aiming at the determination of subsurface properties directly from the data. The theory is presented for the full three-dimensional, laterally varying case. For this situation we derive a layer replacement method based on the reciprocity theorem and we derive an imaging condition based on causality. Next we simplify the theory for the one-dimensional case and give some synthetic results for this case. We explain how the limited bandwidth of seismic data influences our method. We propose a solution method to deal with the lack of low frequencies in the data. This method uses the absolute value of the data. Finally we present a synthetic inversion example for a laterally varying earth model using common-midpoint techniques.

KEY WORDS: inversion, imaging, layer replacement, reciprocity, causality.

#### INTRODUCTION

The different existing seismic inversion methods can roughly be divided into three classes: (1) direct approximate methods, (2) iterative (nonlinear) methods and (3) direct nonlinear methods. In direct approximate problems the nonlinear inverse problem is reduced to a linear problem by making approximations. The simplest linearization method is the Born approximation,

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as described by Morse and Feshbach (1953). There are many different linearized methods; examples are the migration methods described by Berkhout (1980) and the velocity inversion methods described by Cohen and Bleistein (1979). A slightly different approach is presented by Pratt and Worthington (1988), who present an imaging method using the Rytov approximation. The second class of inversion methods is the class of iterative nonlinear methods. With this type of method one tries to construct a solution of the inverse scattering problem by sequentially updating a model. Many different optimization methods can be applied to inverse scattering problems, such as the method described by Tarantola (1984), using the least squares criterion. Other examples are the seismic waveform inversion method in the frequency-space domain by Pratt and Worthington (1990) or the global optimization methods as described by Sen and Stoffa (1995). The third class of inversion methods is the class of direct nonlinear methods. This class contains methods that give an explicit expression for the unknown in terms of the data, as well as methods that use explicit algorithms to reconstruct the unknown in a finite number of steps. Most of these methods involve a transformation of the one-dimensional wave equation into the Schrödinger equation, which obviously cannot be applied directly to multidimensional problems. An overview of this class of methods for electromagnetic applications is given by Habashy and Mittra (1987). The most well-known direct nonlinear methods are 'layer-stripping' methods. In these methods, first the desired quantities are determined at the earth surface. Next, the measurements that would have been made if a thin layer below the surface had been absent are mathematically reconstructed. The desired quantities at this new surface are now determined and the method is repeated layer by layer. Examples of layerstripping methods are described by Yagle and Levy (1983) and Koster (1991). The advantage of layer-stripping methods is that they are generally fast and not constrained to small velocity perturbations. Since these methods are completely data driven, using no a priori information on the subsurface, measurement errors and noise may result in unstable results. Unlike the optimization methods, layer-stripping methods do not guarantee that the final reconstructed image is consistent with the measured data.

Even though many different seismic imaging and inversion methods exist, a direct nonlinear method determining the subsurface properties directly from the data, for any subsurface configuration, without making use of a priori information, still remains to be developed. The method described in this paper is a step towards the development of such a method. We introduce the theory for a layer-stripping method in the acoustic approximation for the full, laterally varying, three-dimensional case. The method is derived for a medium configuration with two half-spaces, the upper half-space consisting of a homogeneous background medium with known constant wave speed, corresponding to the configuration for marine data. The advantage of this method compared to linearized methods is the fact that multiples are treated

correctly and that there are no constraints on the wave speed properties of the subsurface. The advantage of our method compared to optimization methods, is that no a priori velocity model has to be included. The fact that the method described in this paper is derived for the 3D case is also the main advantage of this method over most other layer-stripping methods, such as the ones described by Yagle and Levy (1983), Koster (1991) and the work by Sylvester (1996). There have been attempts to solve the inverse problem without making restrictions on the dimensionality of space, such as the method for the electromagnetic case, described by Somersalo et al. (1991). The described method makes use of the fact that high frequent current does not penetrate very deeply into a body, and uses a Ricatti type differential equation for wavefield extrapolation. He et al. (1998) describe a wave-splitting approach. A stable solution however still remains to be found. The method described in this paper describes a new approach to tackle this problem. It combines layer replacement, based on the reciprocity theorem, with velocity contrast determination, using the causality principle.

The derivation of the underlying theory for the full three-dimensional case is given in the next section. After that, the theory is simplified for the one-dimensional case and an inversion result for this situation is given. The next part of the paper focusses on some practical considerations related to the finite bandwidth of seismic data. The paper concludes with the inversion result for synthetic data modeled in a laterally varying two-dimensional medium using common midpoint techniques.

### THEORY

Each layer-stripping step in our method consists of two actions: A layer replacement and a contrast determination. The layer replacement is performed by applying the reciprocity theorem. This theorem, as described by Fokkema and van den Berg (1993), relates two acoustic states to each other. It can be applied to many different acoustic problems such as wavefield decomposition, source deghosting, multiple removal (van Borselen, 1995) and seismic interferometry (Wapenaar and Fokkema, 2006). In our case, one state is the actual state and the other state has the same medium configuration as the actual state except for the fact that the top layer has been removed and replaced by a homogeneous layer with known wave speed. Given the data measured in the first state we can determine the wavefield in the second state using the reciprocity theorem.

The second action is the determination of the contrast between the top layer and the layer beneath it. In order to determine this contrast we apply an imaging condition which relates the up- and downgoing wavefields just above an interface to the wavefield just below this interface. The derivation of the imaging condition uses the boundary conditions over an interface and the causality principle. The causality principle is the well-known principle that there will always be a lapse of time for a wave to travel from one position in space to another position.

## Layer replacement

In this section we discuss how the wavefield just below a thin layer can be determined when the wavefield at the top of this layer is known. The use of the reciprocity theorem in the space-Laplace domain, as described by Fokkema and van den Berg (1993), is the foundation of the layer replacement method. The acoustic equations in the Laplace domain have the following form:

$$\partial_k \hat{\mathbf{p}}(\mathbf{x}, \mathbf{s}) + \mathbf{s} \rho \hat{\mathbf{v}}_k(\mathbf{x}, \mathbf{s}) = \hat{\mathbf{f}}_k(\mathbf{x}, \mathbf{s}) \quad , \tag{1}$$

$$\partial_k \hat{\mathbf{v}}_k(\mathbf{x}, \mathbf{s}) + \mathbf{s} \kappa \hat{\mathbf{p}}(\mathbf{x}, \mathbf{s}) = \hat{\mathbf{q}}(\mathbf{x}, \mathbf{s}) , \qquad (2)$$

where

 $\hat{p}(\mathbf{x},s) = acoustic pressure,$ 

 $v_k(\mathbf{x},s) = particle velocity,$ 

 $f_k(\mathbf{x}, \mathbf{s}) =$  volume source density of volume force,

 $q(\mathbf{x},s) =$  volume source density of volume injection rate,

 $\rho(\mathbf{x}) =$  volume density of mass,

 $\kappa(\mathbf{x}) = \text{compressibility}.$ 

Lower case Latin subscripts take on the values 1, 2 and 3; the summation convention applies to repeated subscripts. The  $t = t_0$  contributions of the Laplace transform are incorporated in the source terms. The Cartesian coordinate vector is denoted by  $\mathbf{x} = (x_1, x_2, x_3)$ , where the  $x_3$ -axis is pointing downward. The Laplace transform parameter is given by s. The reciprocity theorem relates two non-identical acoustic states in a three-dimensional domain D to each other. The domain D is bounded by boundary surface  $\partial D$ . The two different states inside the domain are referred to as state A and state B. Each state is characterized by the acoustic wavefield  $(\hat{p}, \hat{v}_k)$ , the constitutive parameters  $(\rho, \kappa)$ , and the source terms  $(\hat{f}_k, \hat{q})$ . Table 1 shows how the states are defined in the space-Laplace domain. Rayleigh's reciprocity theorem in global form, following Fokkema and van den Berg (1993), is given by

$$\begin{split} &\int_{\mathbf{x}\in\partial D} (\hat{p}^{A}\hat{\mathbf{v}}_{k}^{B} - \hat{p}^{B}\hat{\mathbf{v}}_{k}^{A})\mathbf{n}_{k}dA \\ &= \int_{\mathbf{x}\in D} [\mathbf{s}(\rho^{B} - \rho^{A})\hat{\mathbf{v}}_{k}^{A}\hat{\mathbf{v}}_{k}^{B} - \mathbf{s}(\kappa^{B} - \kappa^{A})\hat{p}^{A}\hat{p}^{B}]dV \\ &+ \int_{\mathbf{x}\in D} [\hat{f}_{k}^{A}\hat{\mathbf{v}}_{k}^{B} + \hat{q}^{B}\hat{p}^{A} - \hat{f}_{k}^{B}\hat{\mathbf{v}}_{k}^{A} - \hat{q}^{A}\hat{p}^{B}]dV \quad , \end{split}$$
(3)

where  $n_k$  is the unit vector normal to  $\partial D$  and oriented away from D. Fig. 1 shows the two states to which the reciprocity theorem is applied. These two states will from now on be referred to as state 0 and state 1. In both states we assume an upper  $(x_3 < x_3^0)$  and a lower half-space  $(x_3 \ge x_3^0)$ . The upper half-space consists of a homogeneous background medium with known constant wave speed  $c_0$ . Source and receiver are positioned in this upper half-space.



Fig. 1. Configuration of the two states in Rayleigh's reciprocity theorem for the derivation of layer replacement. Both states have an upper  $(x_3 < x_3^0)$  and a lower half-space  $(x_3 \ge x_3^0)$ . The upper half-space consists of a homogeneous background medium with known constant wave speed  $c_0$ . Source and receiver are positioned in the upper half-space. The lower half-space in both states is divided into thin horizontal layers with thickness  $\Delta x_3$ . The layers wave speed inside a layer does not vary in the vertical direction, but can be variable in the lateral direction. The horizontal coordinate vector is denoted by  $\mathbf{x}_T = (x_1, x_2)$ .

	State A	State B
Field State	$\{\hat{p}^{A},\hat{v}_{k}^{A}\}(\textbf{x},s)$	$\{\hat{p}^B, \hat{v}^B_k\}(\bm{x}, s)$
Material State	$\{\rho^{\mathrm{A}},\kappa^{\mathrm{A}}\}(\mathbf{x})$	$\{\rho^{\mathrm{B}},\kappa^{\mathrm{B}}\}(\mathbf{X})$
Source State	${\hat{\mathbf{q}}^{A}, \hat{\mathbf{f}}^{A}_{k}}(\mathbf{x}, s)$	${\hat{q}^{B}, \hat{f}^{B}_{k}}(\mathbf{x}, s)$
	Domain D	

Table 1. States in the field reciprocity theorem.

The lower half-space in both states is divided into thin horizontal virtual layers with thickness  $\Delta x_3$ . These layers are virtual, meaning that they do not necessarily coincide with the geological layering. The layers are thin enough to justify the assumption that the wave speed inside a layer does not vary in the vertical direction. The wave speed can however be variable in the lateral direction. The horizontal coordinate vector will be denoted by  $\mathbf{x}_{T} = (\mathbf{x}_{1}, \mathbf{x}_{2})$ . We will assume the density  $\rho$  to be constant and identical in both half-spaces. We will also assume that the data to which the layer-stripping method will be applied is preprocessed such that the described configuration with its two half-spaces is simulated. This means for example that surface-related multiples in the marine case should have been removed (van Borselen, 1995). State 0 represents the actual state in which the wavefield was measured. State 1 represents an almost identical medium configuration except for the top layer which is replaced by a layer with the same properties as the homogeneous background medium in the upper half-space. The properties of the acoustic states 0 and 1 are shown in Table 2. The dependency on s is omitted for simplicity of notation. Since the compressibility  $\kappa$  and the wave speed c are related through  $c = (\kappa \rho)^{-1/2}$  and the density  $\rho$  is a constant, we can use  $c_n(\mathbf{x}_T)$  to describe the material state in the n-th layer, where  $x_3^{n-1} < x_3 \le x_3^n$  and n can be any number between 1 and the desired number of stripped layers N. We will take for both states 0 and 1 a point source of volume injection (monopole source). Note the reversed source and receiver positions in state 1 with respect to state 0, see Fig. 1. Domain D contains both the upper and the lower half-space. Application of Rayleigh's reciprocity theorem in global form, eq. (3), to the states shown in Fig. 1 and Table 2 leads to:

$$\begin{split} \int_{\mathbf{x}\in\partial D} [\hat{p}^{0}(\mathbf{x} \,|\, \mathbf{x}^{S}) \hat{v}_{k}^{1}(\mathbf{x} \,|\, \mathbf{x}^{R}) \,-\, \hat{p}^{1}(\mathbf{x} \,|\, \mathbf{x}^{R}) \hat{v}_{k}^{0}(\mathbf{x} \,|\, \mathbf{x}^{S})] \mathbf{n}_{k} dA \\ &= \int_{\mathbf{x}\in D} -\mathbf{s}(\kappa^{1} \,-\, \kappa^{0}) \hat{p}^{0}(\mathbf{x} \,|\, \mathbf{x}^{S}) \hat{p}^{1}(\mathbf{x} \,|\, \mathbf{x}^{R}) dV \\ &+ \int_{\mathbf{x}\in D} [\hat{q}^{S} \delta(\mathbf{x} \,-\, \mathbf{x}^{R}) \hat{p}^{0}(\mathbf{x} \,|\, \mathbf{x}^{S}) \,-\, \hat{q}^{S} \delta(\mathbf{x} \,-\, \mathbf{x}^{S}) \hat{p}^{1}(\mathbf{x} \,|\, \mathbf{x}^{R})] dV \quad . \end{split}$$
(4)

Table 2. States in the field reciprocity theorem.

	State 0	State 1
Field State	$\{\hat{p}^0, \hat{v}^0_k\}(\bm{x} \bm{x}^S)$	$\{\hat{p}^{1},\hat{v}_{k}^{1}\}(\bm{x} \bm{x}^{R})$
Material State	$\{c_0, c_1(\mathbf{x}_T), c_2(\mathbf{x}_T), \dots, c_N(\mathbf{x}_T)\}$	$\{c_0, c_0, c_2(\mathbf{x}_T), \dots, c_N(\mathbf{x}_T)\}$
Source State	${\hat{q}^{s}\delta(\mathbf{x} - \mathbf{x}^{s}), 0}$	$\{\hat{q}^{s}\delta(\mathbf{x} - \mathbf{x}^{R}), 0\}$
	Domain <b>D</b>	

In this notation  $\{\hat{p}, \hat{v}_k\}(\mathbf{x} | \mathbf{x}^S)$  is the wavefield  $\{\hat{p}, \hat{v}_k\}$ , which was generated at source position  $\mathbf{x}^S$  and measured at receiver position  $\mathbf{x}$ . If we take domain Dto be a sphere with radius  $\Delta$ , the left hand side of eq. (4) vanishes in the limit  $\Delta \rightarrow \infty$ . In that case, the domain D resembles an unbounded domain. Note that  $\kappa^0$  and  $\kappa^1$  in the right hand side of eq. (4) are the compressibilities in states 0 and 1 respectively. They only differ in layer 1, where they are given by  $\kappa_1(\mathbf{x}_T)$ and  $\kappa_0$ , respectively (see Fig. 1). We define the laterally variable contrast  $K(\mathbf{x}_T)$ in layer 1 as:

$$K(\mathbf{x}_{\rm T}) = \rho(\kappa^1 - \kappa^0) = \rho[\kappa_0 - \kappa_1(\mathbf{x}_{\rm T})]$$
  
=  $\rho[(1/\rho c_0^2) - 1/\rho c_1^2(\mathbf{x}_{\rm T})] = (1/c_0^2) - 1/c_1^2(\mathbf{x}_{\rm T})$  (5)

Next, we substitute  $\hat{q}^{s}(s) = \hat{W}(s)/\rho s$ , where  $\hat{W}(s)$  is the wavelet spectrum, into the right hand side of eq. (4). Using the contrast function  $K(\mathbf{x}_{T})$ , this results in the following expression:

$$\begin{split} \int_{\mathbf{x}\in D^{1}} s^{2} K(\mathbf{x}_{T}) \hat{p}^{0}(\mathbf{x} \,|\, \mathbf{x}^{S}) \hat{p}^{1}(\mathbf{x} \,|\, \mathbf{x}^{R}) dV \\ &= \hat{W}[\hat{p}^{0}(\mathbf{x}^{R} \,|\, \mathbf{x}^{S}) \,-\, \hat{p}^{1}(\mathbf{x}^{S} \,|\, \mathbf{x}^{R})] , \end{split}$$
(6)

where  $D^1$  denotes layer 1.

In the upper half-space, the total wavefield in state 0,  $\hat{p}^0$ , can be decomposed into an incident and a reflected wavefield,  $\hat{p}^{0,i}$  and  $\hat{p}^{0,r}$ , respectively, according to:

$$\hat{p}^{0}(\mathbf{x} \mid \mathbf{x}^{S}) = \hat{p}^{0,i}(\mathbf{x} \mid \mathbf{x}^{S}) + \hat{p}^{0,r}(\mathbf{x} \mid \mathbf{x}^{S}) , \quad \mathbf{x}_{3} \le \mathbf{x}_{3}^{0}.$$
(7)

The incident field can be expressed as:

$$\hat{\mathbf{p}}^{0,i}(\mathbf{x} \,|\, \mathbf{x}^{S}) = \,\hat{\mathbf{W}} \exp[-(\mathbf{s}/\mathbf{c}_{0}) \,|\, \mathbf{x} - \mathbf{x}^{S} \,|\,]/4\pi \,|\, \mathbf{x} - \mathbf{x}^{S} \,|\,, \tag{8}$$

where we made use of the Laplace domain representation of the Green's function, see Fokkema and van den Berg (1993). The wavefield in state 1 can be decomposed into an incident and reflected part in a similar way:

$$\hat{p}^{1}(\mathbf{x} \mid \mathbf{x}^{R}) = \hat{p}^{1,i}(\mathbf{x} \mid \mathbf{x}^{R}) + \hat{p}^{1,r}(\mathbf{x} \mid \mathbf{x}^{R}), \quad \mathbf{x}_{3} \le \mathbf{x}_{3}^{1}.$$
(9)

Note that the decomposition in state 1 holds for the homogeneous upper half-space, extended with layer 1.

We rewrite the integral over layer 1 in the left hand side of eq. (6) as

$$\int_{x\in D^1} \{\cdot\} dV = \int_{x_{\tau}\in \mathbb{R}^2} dA \int_{x_3^0}^{x_3} \{\cdot\} dx_3 .$$

Making use of Parseval's theorem we thus find:

$$[1/(2\pi)^{2}] \int_{s\alpha_{T} \in \mathbb{R}^{2}} dA \int_{x_{3}^{0}}^{x_{3}^{1}} \bar{p}^{1}(-js\alpha_{T}, x_{3} | \mathbf{x}^{R}) s^{2} K\{\bar{p}^{0}\}(js\alpha_{T}, x_{3} | \mathbf{x}^{S})dx_{3}$$
$$= \hat{W}[\hat{p}^{0}(\mathbf{x}^{R} | \mathbf{x}^{S}) - \hat{p}^{1}(\mathbf{x}^{S} | \mathbf{x}^{R})] , \qquad (10)$$

where  $\alpha_T = (\alpha_1, \alpha_2)$ . A spatial Fourier transformation was performed with respect to the horizontal receiver positions  $(\mathbf{x}_T)$  only. These spatial Fourier transformations are defined as:

$$\bar{p}^{0}(js\alpha_{T}, x_{3} | \mathbf{x}^{S}) = \int_{\mathbf{x}_{T} \in \mathbb{R}^{2}} \exp(js\alpha_{T} \cdot \mathbf{x}_{T}) \hat{p}^{0}(\mathbf{x} | \mathbf{x}^{S}) dA \quad , \tag{11}$$

$$\bar{p}^{1}(-js\alpha_{T}, \mathbf{x}_{3} | \mathbf{x}^{R}) = \int_{\mathbf{x}_{T} \in \mathbb{R}^{2}} \exp(-js\alpha_{T} \cdot \mathbf{x}_{T}) \hat{p}^{1}(\mathbf{x} | \mathbf{x}^{R}) dA \quad .$$
(12)

 $K\{\bar{p}^0\}$  in eq. (10) is the spatial Fourier transform of  $K(\mathbf{x}_T)\hat{p}^0(\mathbf{x} | \mathbf{x}^S)$ . Hence, it is a compact way of writing the convolution operator in the transformed domain:

$$K\{\bar{\mathbf{p}}^{0}\}(\mathbf{js}\boldsymbol{\alpha}_{\mathrm{T}},\mathbf{x}_{3} | \mathbf{x}^{\mathrm{S}})$$
  
=  $[1/(2\pi)^{2}] \int_{\mathbf{s}\boldsymbol{\alpha}_{1}^{\prime} \in \mathbb{R}^{2}} \bar{\mathbf{K}}(\mathbf{js}\boldsymbol{\alpha}_{\mathrm{T}} - \mathbf{js}\boldsymbol{\alpha}_{\mathrm{T}}^{\prime})\bar{\mathbf{p}}^{0}(\mathbf{js}\boldsymbol{\alpha}_{\mathrm{T}}^{\prime},\mathbf{x}_{3} | \mathbf{x}^{\mathrm{S}})\mathrm{d}\mathbf{A}$  (13)

where

$$\bar{K}(js\alpha_{T}) = \int_{\mathbf{x}_{T} \in \mathbb{R}^{2}} \exp(js\alpha_{T} \cdot \mathbf{x}_{T}) K(\mathbf{x}_{T}) dA \quad . \tag{14}$$

Taking a closer look at eq. (10) it is clear that the wavefields in the integrand on the left hand side of the equation are not expressed in the same domain as the wave fields on the right hand side. In order to evaluate this equation the wavefields should all be expressed in the same domain. To accomplish this the following operator is applied:

$$\int_{\mathbf{x}_{T}^{R} \in \mathbb{R}^{2}} \exp(js\alpha_{T}^{R} \cdot \mathbf{x}_{T}^{R}) \int_{\mathbf{x}_{T}^{S} \in \mathbb{R}^{2}} \exp(-js\alpha_{T}^{S} \cdot \mathbf{x}_{T}^{S}) \{\cdot\} dAdA \quad .$$
(15)

Application of this operator to the fields on the right-hand side of eq. (10) corresponds to a transformation to the spatial Fourier domain for both source and receiver coordinates. To finally get all fields in the same domain, the transformation is applied to the left-hand side of the equation as well, and we find:

$$[1/(2\pi)^{2}] \int_{s\alpha_{T} \in \mathbb{R}^{2}} dA \int_{x_{3}^{0}}^{x_{3}^{1}} \dot{p}^{1}(-js\alpha_{T},x_{3}|js\alpha_{T}^{R},x_{3}^{R}) \times s^{2}K\{\dot{p}^{0}\}(js\alpha_{T},x_{3}| - js\alpha_{T}^{S},x_{3}^{S})dx_{3} = \hat{W}[\dot{p}^{0}(js\alpha_{T}^{R},x_{3}^{R}| - js\alpha_{T}^{S},x_{3}^{S}) - \dot{p}^{1}(-js\alpha_{T}^{S},x_{3}^{S}|js\alpha_{T}^{R},x_{3}^{R})] , \qquad (16)$$

where the asterisk \* indicates the double spatial Fourier transformation with respect to both source and receiver coordinates.

The physical reciprocity condition can now be applied. In the space domain, physical reciprocity for the wave field in state 1 is formulated as  $\hat{p}^1(\mathbf{x}^S | \mathbf{x}^R) = \hat{p}^1(\mathbf{x}^R | \mathbf{x}^S)$ . Applying the double Fourier transformation of equation (15) to both sides yields:

$$\mathbf{p}^{1}(-js\alpha_{T}^{S}, x_{3}^{S}|js\alpha_{T}^{R}, x_{3}^{R}) = \mathbf{p}^{1}(js\alpha_{T}^{R}, x_{3}^{R}| - js\alpha_{T}^{S}, x_{3}^{S}) \quad .$$
(17)

When we apply this to eq. (16) we find:

$$[1/(2\pi)^{2}] \int_{s\alpha_{T} \in \mathbb{R}^{2}} dA \int_{x_{3}^{0}}^{x_{3}^{i}} \dot{p}^{1}(js\alpha_{T}^{R}, x_{3}^{R}| - js\alpha_{T}, x_{3})$$

$$\times s^{2}K\{\dot{p}^{0}\}(js\alpha_{T}, x_{3}| - js\alpha_{T}^{S}, x_{3}^{S})dx_{3}$$

$$= \hat{W}[\dot{p}^{0}(js\alpha_{T}^{R}, x_{3}^{R}| - js\alpha_{T}^{S}, x_{3}^{S}) - \dot{p}^{1}(js\alpha_{T}^{R}, x_{3}^{R}| - js\alpha_{T}^{S}, x_{3}^{S})] .$$
(18)

The incident wavefield in the transformed domain in state 0 can be written  $\dot{p}^{0,i}(is\alpha_T, x_3| - is\alpha_T^S, x_3^S)$ 

$$= (2\pi)^2 (\hat{W}/2s\Gamma_0^S) \delta(s\alpha_T - s\alpha_T^S) \exp(-s\Gamma_0^S |x_3 - x_3^S|) , x_3 \le x_3^0 , \quad (19)$$

(24)

which is the spatial Fourier domain version of eq. (8) with the vertical slowness  $\Gamma_0^s$ :

$$\Gamma_0^{\rm S} = [(1/c_0^2) + \alpha_{\rm T}^{\rm S} \cdot \alpha_{\rm T}^{\rm S}]^{\frac{1}{2}} , \quad {\rm R}\{\Gamma_0^{\rm S}\} \ge 0.$$
<sup>(20)</sup>

A similar expression holds for the incident wave field in state 1 for  $x_3 \le x_3^1$ .

The reflected wavefield in state 0 at depth  $x_3^R$  can be expressed in terms of the reflected wavefield at depth  $x_3^0$  by conducting a simple extrapolation step:

$$\hat{p}^{0,r}(js\alpha_{T}^{R}, x_{3}^{R}| - js\alpha_{T}^{S}, x_{3}^{S})$$

$$= \exp[-s\Gamma_{0}^{R}(x_{3}^{0} - x_{3}^{R})]\hat{p}^{0,r}(js\alpha_{T}^{R}, x_{3}^{0}| - js\alpha_{T}^{S}, x_{3}^{S}), \quad x_{3}^{R} \le x_{3}^{0},$$

$$(21)$$

with the vertical slowness  $\Gamma_0^{\rm R}$ :

$$\Gamma_0^{\rm R} = \left[ (1/c_0^2) + \alpha_{\rm T}^{\rm R} \cdot \alpha_{\rm T}^{\rm R} \right]^{\frac{1}{2}} , \quad {\rm R} \{ \Gamma_0^{\rm R} \} \ge 0.$$
(22)

Analogously to eq. (21) the reflected wavefield in state 1 at depth  $x_3^R$  can be written in terms of the reflected wavefield at depth  $x_3^1$  by:

$$\dot{p}^{1,r}(js\alpha_{T}^{R},x_{3}^{R}| - js\alpha_{T}^{S},x_{3}^{S}) = \exp[-s\Gamma_{0}^{R}(x_{3}^{1} - x_{3}^{R})]\dot{p}^{1,r}(js\alpha_{T}^{R},x_{3}^{1}| - js\alpha_{T}^{S},x_{3}^{S}), \quad x_{3}^{R} \le x_{3}^{1}.$$
(23)

Now we rewrite eq. (18) by decomposing the wavefield in state 1 on the left-hand side into an incident and a reflected part following eq. (9). The wavefields on the right hand side are decomposed in a similar manner leading to:

$$\begin{split} s^{2}/(2\pi)^{2} \int_{s\alpha_{T} \in \mathbb{R}^{2}} dA \int_{x_{3}^{0}}^{x_{3}^{*}} [p^{1,i}(js\alpha_{T}^{R},x_{3}^{R}| - js\alpha_{T},x_{3}) \\ &+ p^{1,r}(js\alpha_{T}^{R},x_{3}^{R}| - js\alpha_{T},x_{3})] \\ &\times K\{p^{0}\}(js\alpha_{T},x_{3}| - js\alpha_{T}^{S},x_{3}^{S})dx_{3} \\ &= \hat{W}[p^{0,i}(js\alpha_{T}^{R},x_{3}^{R}| - js\alpha_{T}^{S},x_{3}^{S}) \\ &- p^{1,i}(js\alpha_{T}^{R},x_{3}^{R}| - js\alpha_{T}^{S},x_{3}^{S}) \\ &+ p^{0,r}(js\alpha_{T}^{R},x_{3}^{R}| - js\alpha_{T}^{S},x_{3}^{S}) \\ &- p^{1,r}(js\alpha_{T}^{R},x_{3}^{R}| - js\alpha_{T}^{S},x_{3}^{S}) \\ &- p^{1,r}(js\alpha_{T}^{R},x_{3}^{R}| - js\alpha_{T}^{S},x_{3}^{S}) \\ &- p^{1,r}(js\alpha_{T}^{R},x_{3}^{R}| - js\alpha_{T}^{S},x_{3}^{S})] \end{split}$$

Now we substitute the expression for the incident wavefield, eq. (19). Note that the incident wavefields on the right hand side will drop out because the incident field in state 1 is equal to the incident field in state 0:

$$\dot{\mathbf{p}}^{1,i}(\mathbf{x}^{R} \,|\, \mathbf{x}^{S}) \,=\, \dot{\mathbf{p}}^{0,i}(\mathbf{x}^{R} \,|\, \mathbf{x}^{S}) \quad.$$
(25)

This is the case since source and receiver are both positioned in the same homogeneous background medium  $(x_3^{(S,R)} \le x_3^0)$ . Next, we extrapolate the receiver position to the first interface,  $x_3^0$ , by applying eqs. (21) and (23) and we find:

$$s^{2}\hat{W} \int_{x_{3}^{0}}^{x_{3}^{i}} \exp[-s\Gamma_{0}^{R}(x_{3} - x_{3}^{0})]/2s\Gamma_{0}^{R} \times K\{p^{0}\}(js\alpha_{T}^{R}, x_{3}| - js\alpha_{T}^{S}, x_{3}^{S})dx_{3}$$

$$+ \exp(-s\Gamma_{0}^{R}\Delta x_{3})/(2\pi)^{2} \times \int_{s\alpha_{T} \in \mathbb{R}^{2}} p^{1,r}(js\alpha_{T}^{R}, x_{3}^{1}| - js\alpha_{T}, x_{3}^{S})dA$$

$$\times s^{2} \int_{x_{3}^{0}}^{x_{3}^{i}} \exp[s\Gamma_{0}(x_{3} - x_{3}^{S})] \times K\{p^{0}\}(js\alpha_{T}, x_{3}| - js\alpha_{T}^{S}, x_{3}^{S})dx_{3}$$

$$= \hat{W}[p^{0,r}(js\alpha_{T}^{R}, x_{3}^{0}| - js\alpha_{T}^{S}, x_{3}^{S})$$

$$- p^{1,r}(js\alpha_{T}^{R}, x_{3}^{1}| - js\alpha_{T}^{S}, x_{3}^{S})\exp(-s\Gamma_{0}^{R}\Delta x_{3})] , \qquad (26)$$

with

$$\Gamma_0 = [(1/c_0^2) + \alpha_{\rm T} \cdot \alpha_{\rm T}]^{\nu_2} , \quad {\rm R}\{\Gamma_0\} \ge 0,$$
(27)

and  $\Delta x_3 = x_3^1 - x_3^0$ . We will use this equation to calculate the reflected wavefield in state 1 measured at depth  $x_3^1$ . In order to do this we need to know the total wavefield in state 0 at the same depth. We now perform a similar procedure to eq. (18), by extrapolating the receiver to one thin layer below the first interface,  $x_3^1$ , this yields:

$$s^{2}\hat{W} \int_{x_{3}^{0}}^{x_{3}^{1}} \exp[-s\Gamma_{0}^{R}|x_{3} - x_{3}^{1}|]/2s\Gamma_{0}^{R} \times K\{p^{0}\}(js\alpha_{T}^{R}, x_{3}| - js\alpha_{T}^{S}, x_{3}^{S})dx_{3}$$

$$+ [1/(2\pi)^{2}] \int_{s\alpha_{T} \in \mathbb{R}^{2}} p^{1,r}(js\alpha_{T}^{R}, x_{3}^{1}| - js\alpha_{T}, x_{3}^{S})dA$$

$$\times s^{2} \int_{x_{3}^{0}}^{x_{3}^{1}} \exp[s\Gamma_{0}(x_{3} - x_{3}^{S})] \times K\{p^{0}\}(js\alpha_{T}, x_{3}| - js\alpha_{T}^{S}, x_{3}^{S})dx_{3}$$

$$= \hat{W}[\dot{p}^{0,r}(js\alpha_{T}^{R}, x_{3}^{1}| - js\alpha_{T}^{S}, x_{3}^{S}) - \dot{p}^{1,i}(js\alpha_{T}^{R}, x_{3}^{1}| - js\alpha_{T}^{S}, x_{3}^{S}) - \dot{p}^{1,r}(js\alpha_{T}^{R}, x_{3}^{1}| - js\alpha_{T}^{S}, x_{3}^{S})] .$$
(28)

In this case the incident fields on the right-hand side of the equation do not cancel, since in state 0 the incident field travels through a layer with wave speed  $c_1(\mathbf{x}_T)$ , and in state 1 it does not. Multiplying the right- and left-hand side of eq. (26) with  $\exp(s\Gamma_0^R\Delta x_3)$  and subtracting the result from eq. (28) we find:

$$\begin{split} \dot{p}^{0}(js\alpha_{T}^{R},x_{3}^{1}) &= js\alpha_{T}^{S},x_{3}^{S}) \\ &= \dot{p}^{0,r}(js\alpha_{T}^{R},x_{3}^{0}) - js\alpha_{T}^{S},x_{3}^{S})exp(s\Gamma_{0}^{R}\Delta x_{3}) \\ &+ \dot{p}^{0,i}(js\alpha_{T}^{R},x_{3}^{0}) - js\alpha_{T}^{S},x_{3}^{S})exp(-s\Gamma_{0}^{R}\Delta x_{3}) \\ &+ s^{2} \int_{x_{3}^{1}}^{x_{3}^{1}} sinh[s\Gamma_{0}^{R}(x_{3} - x_{3}^{1})]/s\Gamma_{0}^{R} \end{split}$$

 $\times K\{p^{0}\}(js\alpha_{T}^{R},x_{3}| - js\alpha_{T}^{S},x_{3}^{S})dx_{3}$ , (29)

where

$$\sinh(x) = [\exp(x) - \exp(-x)]/2.$$
 (30)

We call eqs. (26) and (28) the two basic equations, where eq. (28) is the consistence equation. This equation is used to determine  $p^0(js\alpha_T^R, x_3^1) - js\alpha_T^S, x_3^S)$ , the total wavefield in state 0 recorded at position  $x_3^1$  (see Fig. 1). In other words, this wavefield is the total wavefield below the top thin layer, before replacement of this layer by a layer with the homogeneous background medium properties. Once this wavefield is known, eq. (26) is used to determine  $p^{1.r}(js\alpha_T^R, x_3^1) - js\alpha_T^S, x_3^S)$ , the reflected field in state 1 (after replacement of the layer by a layer with the homogeneous background medium properties) at the same depth  $x_3^1$ . When the top layer has been replaced, the source can be moved downwards over  $\Delta x_3$  and state 1 can become the new actual state (state 0) for a next layer-stripping step. This process will be described in the section on the layer-stripping method. In both basic equations, Eqs. (26) and (28), the velocity we use an imaging condition based on the causality principle. The derivation of this imaging condition is shown in the next section.

## Derivation of imaging condition

In this section we derive an imaging condition which relates the up- and downgoing waves just above an interface to the velocity contrast over this interface. The term imaging condition is a term known from migration, where in the imaging condition the (t = 0) - value is used to determine the reflectivity. This principle was first introduced by Claerbout (1971). Our imaging condition resembles the imaging condition in migration in the sense that we also use the (t = 0) - value of the wavefields. A difference is however the fact that we use the imaging condition to directly determine the medium parameter wave speed as opposed to determining the reflectivity.

We consider a medium consisting of two half-spaces, a homogeneous upper half-space and a heterogeneous lower half-space. The lower half-space is heterogeneous in the horizontal direction but not in the vertical direction. The medium configuration is shown in Fig. 2.

When a wave travels downwards the two boundary conditions across the interface between the two half-spaces are:

$$\lim_{\epsilon \neq 0} \partial_3 \bar{p} \big|_{x_1^0 - \epsilon} = \lim_{\epsilon \neq 0} \partial_3 \bar{p} \big|_{x_1^0 + \epsilon} \quad , \tag{31}$$

$$\lim_{\epsilon \neq 0} \bar{p} \Big|_{x_{\tau}^{0} - \epsilon} = \lim_{\epsilon \neq 0} \bar{p} \Big|_{x_{\tau}^{0} + \epsilon} , \qquad (32)$$

where the wavefields are written in the single spatial Fourier domain and  $\bar{p}$  is shorthand for  $\bar{p}(js\alpha_T, x_3 | \mathbf{x}^s)$ . The first boundary condition states that the component of the particle velocity normal to the interface is continuous across this interface. We will write this boundary condition in a short hand notation:

$$\partial_{3}^{\dagger}\bar{p}|_{x_{1}^{0}} = \partial_{3}^{\dagger}\bar{p}|_{x_{1}^{0}} , \qquad (33)$$

where the term  $\partial_3^{\dagger} \bar{p} |_{x_3^0}$  denotes the vertical derivative of the pressure wavefield very close to the interface when approaching the interface from the side of the upper half-space (from small  $x_3$  to large  $x_3$ ). A solution for the partial derivative in the  $x_3$  direction approaching the interface from above is known:



Fig. 2. Medium configuration for derivation of imaging condition.

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$$\partial_{3}^{\dagger}\bar{p}|_{x^{0}} = s\Gamma_{0}^{R}(\bar{p}^{r} - \bar{p}^{i}) , \qquad (34)$$

where  $\bar{p} = \bar{p}^i + \bar{p}^r$ . This solution accounts for the opposite direction of propagation of the upgoing wavefield  $\bar{p}^r$  and the downgoing wavefield  $\bar{p}^i$  by the minus sign. From the Helmholtz equation it follows that the second derivative in the vertical direction over the pressure wavefield shows a jump across the interface which is proportional to the velocity contrast over this interface. The Helmholtz equation for constant density is defined as:

$$\partial_3^2 \hat{\mathbf{p}} = [(\mathbf{s}^2/\mathbf{c}^2) - \partial_\gamma, \partial_\gamma] \hat{\mathbf{p}} \quad . \tag{35}$$

From this it follows directly that:

$$[(\partial_3^{\dagger})^2 - (\partial_3^{\dagger})^2]\hat{\mathbf{p}} = [(\mathbf{s}^2/\mathbf{c}_0^2) - \mathbf{s}^2/\mathbf{c}_1^2(\mathbf{x}_T)]\hat{\mathbf{p}} = \mathbf{s}^2 \mathbf{K} \hat{\mathbf{p}} \quad . \tag{36}$$

In the spatial Fourier domain this corresponds to:

$$[(\partial_{3}^{\dagger})^{2} - (\partial_{3}^{\dagger})^{2}]\bar{p} = s^{2}K\bar{p} \quad . \tag{37}$$

Note that K is a convolutional operator similar to the one in eq. (13). Eq. (37) can be rewritten as:

$$(\partial_3^{\dagger})^2 \overline{\mathbf{p}} = (\partial_3^{\dagger})^2 \overline{\mathbf{p}} - \mathbf{s}^2 K \overline{\mathbf{p}} \quad . \tag{38}$$

The second derivative of the wavefield in a homogeneous medium is known, so we can write for the upper half-space:

$$(\partial_3^{\dagger})^2 \overline{\mathbf{p}} = \mathbf{s}^2 (\Gamma_0^{\mathsf{R}})^2 \overline{\mathbf{p}} \quad . \tag{39}$$

Substituting this in eq. (38) we find for the lower half-space:

$$(\partial_{3}^{*})^{2}\bar{\mathbf{p}} = \mathbf{s}^{2}[(\Gamma_{0}^{R})^{2} - K]\bar{\mathbf{p}} \quad .$$
(40)

When there are only downgoing waves below the interface we can write:

$$(\partial_3^{\downarrow})\overline{\mathbf{p}} = -\mathbf{s}\sqrt{[(\Gamma_0^{\mathsf{R}})^2 - K]} \ \overline{\mathbf{p}} \quad . \tag{41}$$

Note that the term  $\sqrt{[(\Gamma_0^R)^2 - K]}$  is a pseudo-differential operator. Substituting this term in eq. (33) together with eq. (34) yields:

$$\Gamma_0^{\rm R}(\bar{p}^{\rm r} - \bar{p}^{\rm i}) = -\sqrt{[(\Gamma_0^{\rm R})^2 - K]} \bar{p} \quad . \tag{42}$$

Now write, multiplying  $\bar{p} = \bar{p}^i + \bar{p}^r$  by  $\Gamma_0^{R}$ :

$$\Gamma_0^{\mathsf{R}}\bar{\mathsf{p}}^{\mathsf{r}} + \Gamma_0^{\mathsf{R}}\bar{\mathsf{p}}^{\mathsf{i}} = \Gamma_0^{\mathsf{R}}\bar{\mathsf{p}} \quad , \tag{43}$$

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and add these equations:

$$2\Gamma_0^{\mathsf{R}}\bar{p}^{\mathsf{r}} = \{-\sqrt{[(\Gamma_0^{\mathsf{R}})^2 - K]} + \Gamma_0^{\mathsf{R}}\}\bar{p} \quad .$$
(44)

Subtracting the same two equations results in:

$$2\Gamma_0^{\mathsf{R}}\bar{p}^{\mathsf{i}} = \{\sqrt{[(\Gamma_0^{\mathsf{R}})^2 - K]} + \Gamma_0^{\mathsf{R}}\}\bar{p} \quad .$$
(45)

In the next step we will make use of the inner product which is defined as:

$$\langle \bar{\mathbf{f}}, \bar{\mathbf{g}} \rangle = (1/2\pi) \int_{-\infty}^{\infty} \bar{\mathbf{f}}(\alpha) \bar{\mathbf{g}}(-\alpha) d\alpha \quad .$$
 (46)

For the inner product between the terms for the up- and downgoing wavefield we can write:

$$\langle 2\Gamma_0^{\mathsf{R}}\bar{p}^{\mathsf{r}}, 2\Gamma_0^{\mathsf{R}}\bar{p}^{\mathsf{i}} \rangle$$

$$= \left[ \langle \Gamma_0^{\mathsf{R}}\bar{p}, \Gamma_0^{\mathsf{R}}\bar{p} \rangle + \langle \Gamma_0^{\mathsf{R}}\bar{p}, \sqrt{[(\Gamma_0^{\mathsf{R}})^2 - K]} \bar{p} \rangle \right]$$

$$- \langle \sqrt{[(\Gamma_0^{\mathsf{R}})^2 - K]} \bar{p}, \Gamma_0^{\mathsf{R}}\bar{p} \rangle$$

$$- \langle \sqrt{[(\Gamma_0^{\mathsf{R}})^2 - K]} \bar{p}, \sqrt{[(\Gamma_0^{\mathsf{R}})^2 - K]} \bar{p} \rangle$$

$$(47)$$

We will now make use of the fact that the pseudo-differential operator is symmetric, as shown by Poot (2004). Note that the operator is only proven to be symmetric when we meet the condition  $s = j\omega$ . The two middle terms on the right hand side drop out and we can write for the remaining two terms:

$$[\langle \Gamma_0^{\mathsf{R}}\bar{p}, \Gamma_0^{\mathsf{R}}\bar{p}\rangle] - \langle \sqrt{[(\Gamma_0^{\mathsf{R}})^2 - K]} \,\bar{p}, \sqrt{[(\Gamma_0^{\mathsf{R}})^2 - K]} \,\bar{p}\rangle = [\langle \bar{p}, (\Gamma_0^{\mathsf{R}})^2 \bar{p}\rangle - \langle \bar{p}, [(\Gamma_0^{\mathsf{R}})^2 - K] \bar{p}\rangle] = \langle \bar{p}, K \bar{p}\rangle .$$
(48)

We have now found the following result:

$$4\langle \Gamma_0^{\mathsf{R}}\bar{p}^{\mathsf{r}}, \Gamma_0^{\mathsf{R}}\bar{p}^{\mathsf{i}}\rangle = \langle \bar{p}, K\bar{p}\rangle \quad . \tag{49}$$

Note that this equation is only valid when there is no upgoing wavefield below the interface. If this is not the case, eq. (40) does not hold. When the lower half-space is heterogeneous in the vertical direction as well as in horizontal direction, as is would be in a realistic earth model, there will be an upgoing wavefield below the interface. We now use the causality principle: since it will always take a lapse of time for a wave to travel from one position in space to another position in space, there will always be a time interval where the upgoing waves in the lower half-space have not reached the area just below the interface yet. During this short time interval there are only downgoing waves just below the interface and the imaging condition is valid. In order to meet this condition we define our imaging condition to be valid for time t = 0only. This leaves us the desired imaging condition:

$$\mathscr{F}_{t}^{-1}[4\langle \Gamma_{0}^{R}\bar{p}^{r}, \Gamma_{0}^{R}\bar{p}^{i}\rangle] = \mathscr{F}_{t}^{-1}[\langle \bar{p}, K\bar{p}\rangle] \quad , \quad t = 0,$$

$$\tag{50}$$

where  $\mathscr{F}_t^{-1}$  stands for the inverse temporal Fourier transform, where we use  $s = j\omega$ . The imaging condition shows that the contrast over an interface can be determined once the up- and downgoing parts of the pressure wavefield are known

## Layer-stripping method

The layer-stripping procedure combines the wavefield extrapolation method explained in the section on layer replacement and the causality based imaging condition as explained in the section on derivation of imaging condition. Our starting point is a medium which is divided in thin horizontal layers. As stated once before in the section on layer replacement, these layers are virtual, meaning that they do not necessarily coincide with the actual interfaces in the subsurface. The horizontal layers are defined to be thin enough to justify the assumption that they are homogeneous in the vertical direction (but not necessarily in the lateral direction). On top of this medium, wavefields are measured. When both pressure and velocity wavefields are measured, the pressure wavefield can be decomposed in an upgoing and a downgoing part. This is shown by Fokkema and van den Berg (1993). Using the decomposed pressure wavefield, the contrast over the first virtual horizontal interface can be defined by applying the imaging condition described in the previous section [eq. (50)]. The imaging condition is solved by determining the inner product of the wavefields per frequency in the frequency domain, performing an inverse Laplace transform on the inner products, and solving for the velocity contrast K in the time domain. Once the contrast over the first interface is known and therefore also the propagation velocity in the top layer, we can determine the total wavefield just below this layer using the basic 'consistence' eq. (28). The integral over  $x_3$  in eq. (28) is solved using the trapezoidal rule. The determined total wavefield,  $p^0(s\alpha_T^R, x_3^1) - s\alpha_T^S, x_3^S)$ , is then used in the determination of the wavefield  $\dot{p}^{1,r}(js\alpha_T^R, x_3^1) - js\alpha_T^S, x_3^S)$ . This is the upgoing pressure wavefield measured at depth  $x_3^1$  when the top layer is replaced by a layer with the same properties as the homogeneous background medium (state 1 in Rayleigh's reciprocity theorem). This wavefield after 'stripping' the top layer is calculated using the second basic equation, eq. (26). After discretization in the frequency and wavenumber domain we can write this integral equation as a matrix

equation of the following form:

$$\overset{*}{\mathbf{P}}^{\mathbf{1},\mathbf{r}}(\overset{*}{\mathbf{I}} + \overset{*}{\mathbf{K}}) = \overset{*}{\mathbf{T}}^{\mathrm{data}} , \qquad (51)$$

where  $\mathbf{P}^{1,r}$  is the matrix to be solved,  $\mathbf{T}^{data}$  is a term containing known wavefields,  $\mathbf{I}$  is the unit matrix and  $\mathbf{K}$  is the discretized kernel. In order to solve this matrix equation, the sampling rate in the source and receiver direction must be equal and the equation has to be solved for each frequency. The matrix equations can be solved using a standard matrix inversion method. A solution method for the integral equations using Taylor series and Neumann iterations is given by Poot (2004).

Once the wavefield  $\dot{p}^{1,r}(js\alpha_T^R, x_3^1| - js\alpha_T^S, x_3^S)$  is known, the entire procedure can be repeated. The imaging condition is used to determine the contrast over the next (virtual) interface and the wavefield as determined in the second basic equation [eq. (26)] becomes the wavefield in the new state 0, see Fig. 1. The entire procedure can be repeated until the desired depth is reached and a velocity profile of the lower half-space is determined. The procedure is referred to as a 'layer-stripping'-procedure since the medium is evaluated layer by layer and once the information concerning one layer is used, this layer is replaced and not considered anymore in the further evaluation of the medium. An overview of the total procedure is shown in Fig. 3.



Fig. 3. Flow diagram of the layer-stripping algorithm.

## The layer-stripping method in the one-dimensional case

The theory can be simplified to the theory for a one-dimensional wavefield in a one-dimensional medium. The two basic equations, Eqs. (26) and (28), can now be written as:

$$s^{2}K\hat{W} \int_{x_{3}^{0}}^{x_{3}^{1}} (c_{0}/2s) \exp[-(s/c_{0})(x_{3} - x_{3}^{0})]\hat{p}^{0}(x_{3} | x_{3}^{S}) dx_{3}$$

$$+ \exp[-(s/c_{0})\Delta x_{3}]\hat{p}^{1,r}(x_{3}^{1} | x_{3}^{S})$$

$$\times s^{2}K \int_{x^{0}}^{x^{1}} \exp[(s/c_{0})(x_{3} - x_{3}^{S})]\hat{p}^{0}(x_{3} | x_{3}^{S}) dx_{3}$$

$$= \hat{W}\{\hat{p}^{0,r}(x_{3}^{0} | x_{3}^{S}) - \hat{p}^{1,r}(x_{3}^{1} | x_{3}^{S}) \exp[-(s/c_{0})\Delta x_{3}]\} , \qquad (52)$$

and:

$$\hat{p}^{0}(\mathbf{x}_{3}^{1} | \mathbf{x}_{3}^{S}) = \hat{p}^{0,r}(\mathbf{x}_{3}^{0} | \mathbf{x}_{3}^{S}) \exp[(s/c_{0})\Delta \mathbf{x}_{3}] + \hat{p}^{0,i}(\mathbf{x}_{3}^{0} | \mathbf{x}_{3}^{S}) \exp[-(s/c_{0})\Delta \mathbf{x}_{3}] + s^{2}K \int_{\mathbf{x}^{0}}^{\mathbf{x}^{1}} (c_{0}/s) \sinh[(s/c_{0})(\mathbf{x}_{3} - \mathbf{x}_{3}^{1})] \hat{p}^{0}(\mathbf{x}_{3} | \mathbf{x}_{3}^{S}) d\mathbf{x}_{3} .$$
(53)

These equations can be solved easily by using the trapezoidal rule for the integrals over the area from  $x_3^0$  to  $x_3^1$ . The one-dimensional version of the imaging condition [eq. (50)] is given by:

$$\mathscr{F}_{t}^{-1}[\hat{p}^{t}\hat{p}^{i}] = (c_{0}^{2}/4)K \times \mathscr{F}_{t}^{-1}[(\hat{p})^{2}] , \quad t = 0,$$
(54)

where  $\mathscr{F}_t^{-1}$  stands for the inverse temporal Fourier transform. We assume  $s = j\omega$ . Note that the convolutional operator  $K\{\bar{p}\}$  is now reduced to a scalar K. The imaging condition, eq. (54) is valid only for the short period of time that there are only downgoing waves below the interface over which the contrast is calculated. In practice, taking only the value of the wavefields at time t = 0 when calculating the contrast from the imaging condition, does not give accurate results. So, in order to improve the accuracy of the method, we use a shifted causal wavelet. We assume that the sources that generate the acoustic wavefields are switched on at time instant  $t_0$ , where  $t_0 < 0$ . The energy of the wavelet between  $t = t_0$  and t = 0 is now taken into account in the evaluation of the imaging condition. Hence, we modify eq. (54) to:

$$\int_{-ws}^{0} \mathscr{F}_{t}^{-1}[\hat{p}^{r}\hat{p}^{i}]dt = (c_{0}^{2}/4)K \times \int_{-ws}^{0} \mathscr{F}_{t}^{-1}[(\hat{p})^{2}]dt , \qquad (55)$$

where ws is the window size used for calculating the energy of the wavefield. An imaging result for synthetic one-dimensional data is given in Fig. 4. Note that the imaging result matches the velocity model very well. An investigation of the performance of this method compared to the Schur method described by Yagle and Levy (1983) is described by Poot (2004). Fig. 5 shows that our method performs well for noisy data in comparison to this Schur method. It is also shown in Poot (2004) that the use of the energy of the wavelet results is a stabilization of the layer-stripping, which is missing in for example the Schur method.

In order to investigate how our layer-stripping method deals with the internal multiple reflections in the data take a closer look at the first of our basic equations, eq. (26):



Fig. 4. One-dimensional imaging result together with the earth model. The earth velocity model is denoted by the solid line, the imaging result by the dotted line.



Fig. 5. Imaging result for noisy data. Top left is the synthetically generated upgoing field used as input for the Schur method. The imaging result for this method is shown bottom left. The actual velocities are denoted by the dashed line. Top right is the upgoing wavefield used as input for the causality-based imaging method, the imaging result and the actual velocities for this method are shown bottom right.

$$s^{2}K\hat{W} \int_{x_{3}^{0}}^{x_{3}^{1}} (c_{0}/2s) \exp[-(s/c_{0})(x_{3} - x_{3}^{0})]\hat{p}^{0}(x_{3} | x_{3}^{S}) dx_{3}$$

$$+ \exp[-(s/c_{0})\Delta x_{3}]\hat{p}^{1,r}(x_{3}^{1} | x_{3}^{S})$$

$$\times s^{2}K \int_{x^{0}}^{x^{1}} \exp[(s/c_{0})(x_{3} - x_{3}^{S})]\hat{p}^{0}(x_{3} | x_{3}^{S}) dx_{3}$$

$$= \hat{W}\{\hat{p}^{0,r}(x_{3}^{0} | x_{3}^{S}) - \hat{p}^{1,r}(x_{3}^{1} | x_{3}^{S})\exp[-(s/c_{0})\Delta x_{3}]\} .$$
(56)

The second and third term on the left-hand side can be recognized as a multiple generator term as described by van Borselen (1995). This term takes care of the correct handling of the multiples in the data. To demonstrate this, we take a look at a velocity model with two interfaces, shown together with the imaging result in Fig. 6. The behaviour of the multiple reflection while stripping



Fig. 6. Velocity model with two reflecting interfaces, one at 300 m and one at 900 m depth (dashed line) together with the imaging result (solid line). The small event at 600 m depth is explained in Fig. 7.



Fig. 7. The upgoing wavefield at different source and receiver depths. At the top the velocity model. At the beginning of the layer-stripping procedure the multiple is visible at about 1.4 s. At a source and receiver depth of 600 m the multiple is at 0.5 s. At 900 m depth the multiple and its primary event reach the (t = 0)-axis at the same time. For means of visibility the amplitude of the multiple is enlarged. The size of the multiple in the separate traces for source and receiver depths at 0, 600 and 840 m, shown below the wiggle plot, is the actual size. The small event at 600 m depth in Fig. 6 is caused by miscalculations in the imaging procedure starting at about 300 m depth, when the first interface has reached the (t = 0)-axis. Part of this miscalculation is visible above, starting from a depth of 300 m at the negative side of the (t = 0)-axis.

the layers is made visible in Fig. 7. This figure shows intermediate results of the calculated upgoing wavefield after an increasing amount of layer stripping steps. The more layers stripped, the larger the virtual source and receiver depths, and the closer both the multiple and the primary event that caused it move towards the (t = 0)-axis. The multiple and its primary event reach the (t=0)-axis at the same moment. As noted above, the energy of the wavelet between  $t = t_0$  and t = 0 is taken into account in the evaluation of the imaging condition. Therefore, had multiple and primary not reached the (t = 0)-axis at the same time, the multiple would have been treated as a separate event when applying the imaging condition.

We would like to note that in case of a very smoothly varying medium, there will be no detectable events in the data. Since the method we present here is completely data-driven, this smooth variation in the medium parameters will therefore not be detected.

## THE LIMITED SEISMIC BANDWIDTH

In the previous examples we used a Gaussian wavelet, containing all low frequencies, to model and image our data. Such a wavelet is however not realistic for seismic data since the lowest frequencies cannot be produced by the seismic source. It is well known that these low frequencies are extremely important to the quantitative interpretation of seismic data. This is for example described by Pao et al. (1984). The effect of lack of low frequencies on our method is illustrated in Figs. 8 and 9. Fig. 8 shows the imaging result for a model with two velocity interfaces. This is the imaging result of the reflection response modelled with a Gaussian wavelet, containing all low frequencies. The imaging result resembles the velocity model well. Fig. 9 shows the same velocity model together with the imaging result of the reflection response modelled with the derivative of a Gaussian wavelet. This wavelet does not contain all low frequencies. Now, the imaging result is not satisfactory since even though the velocity contrasts cause a response in the imaging result, the imaged velocity between the contrasts is the same as the background velocity. The reason for failure of the method when the lowest frequencies are missing can be found in the spectral information of the desired velocity function. This spectral information contains a broad range of wavenumbers, including small wavenumber components. It is not possible to recover these small wavenumber components from the data when the data does not contain low frequency information. Some theoretical inversion methods ignore this problem, such as the method described by Raz (1981). In those cases it is assumed that the data can be properly deconvolved in order to recover the reflectivity sequence. This is explained by Treitel, Lines and Ruckgaber (1993). In the next section a solution method for dealing with the lack of low frequencies is proposed.



Fig. 8. Imaging result for synthetic data modelled using a wavelet that contains all low frequencies (solid line). The true velocity model is given by the dash-dotted line. The window size used to find these results is 1.4 s.



Fig. 9. Imaging result for synthetic data modelled using a wavelet that does not contain all low frequencies (solid line). A very small window (0.02 s) was used to evaluate the imaging condition. The true velocity model is given by the dash-dotted line.

## The absolute value method

In this section we propose a method to add low-frequency content to the data without using a background velocity model. To do this, we use the absolute value of the data in the imaging procedure. The absolute value of the measured wavefields is taken at the beginning of the layer-stripping procedure. Both the extrapolation of the wavefields and the calculation of the contrast with the imaging condition is performed with the absolute value of the wavefields. The frequency spectrum of the absolute value of a wavelet that does not contain low frequencies, does contain the desired dc-component. Testing of the absolute value method on wavelets such as a Ricker wavelet gave good results. Fig. 10 shows the imaging result using the absolute value method for the same earth model as in Figs. 8 and 9. As we stated before, the reason for failure of the imaging method when the lowest frequencies are missing is that the spectral information of the desired velocity profile contains small wavenumber components. This means that the desired low-frequency information somehow has to be added to the data or to the imaging result. The common method in seismics is to add this information to the imaging result by using background velocity information. The absolute-value method however adds low-frequency information directly to the data. The underlying assumption is that the events in the data are separate events. For this reason, the method will fail for overlapping events.



Fig. 10. Results for the absolute value-method, calculated velocities (solid) together with the actual velocity model (dashed). The derivative of a Gaussian was used as input wavelet. The absolute value method was used.

## Negative velocity contrasts

Because the absolute value method results in a loss of the sign of the events in the data, this method fails for negative velocity contrasts. In this section we propose a solution method for this problem.

After the absolute value of the wavefield is taken the method 'sees' a negative contrast as a positive one. The imaging result for a velocity model which has a negative contrast is shown in Fig. 11. The inversion result was obtained using the absolute value method and as the figure makes clear, the negative velocity contrast was interpreted as a positive one. A solution method for this problem is to determine the sign of the events before taking the absolute value, and then to apply this sign to the resulting data. We use a median filter, as described by Marion (1991), to determine the sign of the events. Before applying the median filter we transform the wavefield such that the events are symmetric (zero-phasing). After median filtering, a time series containing the sign of the events in the data is computed by assigning the value -1 or +1 to every negative or positive point in the filtered wavefield, respectively. The absolute-value data are now multiplied by the time series containing the sign of the original data in order to assign the correct sign to the events. The choice of



Fig. 11. Imaging result (solid line) for the absolute value-method together with the actual velocity model (dashed) which contains a negative velocity contrast. The true velocity varies from 1500 m/s to 1800 m/s and 2000 m/s and then back to 1800 m/s. The imaged velocity however varies from 1500 m/s to 1800 m/s to about 2230 m/s. The derivative of a Gaussian was used as input wavelet.

the width of the median filter window is important. If the window is chosen too wide, the events will overlap and if the window is chosen too small, the result will almost be the same as the original data. The time series containing the sign of the original data together with the result of application of this time series to the absolute value of the data is shown in Fig. 12. The absolute-value data after manipulation with the correct sign give the imaging result shown in Fig. 13.



Fig. 12. In the middle the median filtering result, containing the sign of the events in the wavefield shown on the left. This wavefield on the left was modelled with the velocities shown in Fig. 11. On the right the corresponding reflected wavefield after taking the absolute value and manipulation with the time series shown in the middle, as explained in the section on negative velocity contrasts.



Fig. 13. Imaging result for the absolute value-method (solid line) together with the actual velocity model (dashed) which contains a negative velocity contrast. The reflected wavefields were filtered to find a time series containing the sign of the events in the data, which was applied to the absolute value of the data, as described in the section on negative velocity contrasts. The derivative of a Gaussian was used as the input wavelet.

Fig. 14 shows the synthetic upgoing wavefield for the multi-layered depth model from the previous section, after manipulation with the time series containing the sign of the original data. Fig. 15 shows the imaging result for the same depth model. The result is slightly less accurate than the result shown in Fig. 4. This is the consequence of very small events or artifacts being given the wrong sign. Note that the method to determine the sign of each event in the data will fail when events with an opposite sign are overlapping.

## 2D IMAGING RESULTS

So far, we have examined the behaviour and characteristics of the method for the one-dimensional case. In this section we use common midpoint gathers to invert a 2D example.



Fig. 14. The figure on the left shows the time series containing the sign of the events in the data modelled using the velocity model shown in Fig. 4. This time series was found by median filtering. On the right shows the reflected wavefield after taking the absolute value and manipulation with the median filtered time series.

The imaging result for this example is shown in Fig. 16. In order to reach this result, the synthetic data were sorted to common midpoint gathers. A comprehensive review of the use of common midpoint data is given by Diebold and Stoffa (1981). Every 20th CMP-gather was then transformed to the intercept time and ray parameter ( $\tau$ -p) domain. An overview of  $\tau$ -p mapping of seismic data is given by Stoffa et al. (1981). Since the trace for p = 0 s/m after transformation to the  $\tau$ -p domain corresponds to a plane wave at normal incidence, we were able to apply our imaging procedure to this trace. The imaging result for several CMP-positions is shown in Fig. 16 (bottom). The imaging result closely resembles the velocity model. This method, where we use plane-wave decomposition of CMP-gathers is known to be applicable only to media varying smoothly in the horizontal direction. When the theory for the full



Fig. 15. Imaging result (solid line) for the absolute-value method after sign-correcting by median filtering as shown in Fig. 14. The dotted line denotes the true velocities.

three-dimensional case as described in the first section of this paper will be implemented, this restriction on variations in the horizontal direction will not have to be made.

### CONCLUSIONS

In the first part of this paper, we have derived the theory for a layer-stripping method for the three-dimensional acoustic case. We have tested the one-dimensional version of this theory on synthetic data, obtaining very accurate results. We have also inverted a two-dimensional earth model making use of CMP-gathers.

As an overall conclusion we can state that promising results have been obtained by designing and partly implementing a completely data driven inversion method which is theoretically applicable to laterally varying media.

We have described how the limited bandwidth of seismic data affects our method. In this paper we have proposed a solution method for this problem using the absolute value of the data, which gives good results as long as the events in the data are not overlapping.



Fig. 16. Imaging result (bottom) per CMP-position for laterally varying velocity model (top). A 2D finite-difference code was used to model 126 shots with 126 receivers at the surface, each 20 meters apart. A bandlimited Ricker wavelet was used as input wavelet. The absolute value method was applied. The gray scale on the right defines the velocity in m/s.

In the final section of the paper we have shown that good imaging results can be obtained for laterally varying subsurface models using common midpoint gathers. This technique is applicable only to media varying smoothly in the horizontal direction. The imaging result for the two-dimensional case shows the potential of the method for this kind of problems. By fully implementing our layer-stripping method for the multi-dimensional case we expect to find more accurate imaging results for laterally varying media. Other problems still to be addressed before being able to apply the multi-dimensional method to geophysical exploration are for example the stability of the method, the fact that events in the data are often overlapping and the expected large amount of computation time that will be involved in solving the large matrix equations.

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