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Reciprocity theorems for electromagnetic or acoustic one-way wave fields in dissipative inhomogeneous media

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Abstract. Reciprocity theorems have proven their usefulness in the study of forward and inverse scattering problems. Most reciprocity theorems in the literature apply to the total wave field and are thus not compatible with one-way wave theory, which is often applied in situations in which there is a clear preferred direction of propagation, like in electromagnetic or acoustic wave guides and in seismic exploration. In this paper we review the theory for one-way wave fields (or bidirectional beams), and we extensively discuss the symmetry properties of the square root operator appearing in the one-way wave equation. Using these symmetry properties, it appears to be possible to derive reciprocity theorems of the convolution type and of the correlation type for electromagnetic or acoustic one-way wave fields in dissipative inhomogeneous media along the same lines as the usual derivation of the reciprocity theorems for the total wave field. The one-way reciprocity theorem of the convolution type provides a basis for representations of scattered one-way wave fields in terms of generalized Bremmer series expansions or generalized primaries. The one-way reciprocity theorem of the correlation type finds its application in reflection imaging based on inverse one-way wave field propagators.

1. Introduction

Reciprocity theorems for electromagnetic and acoustic wave fields have been formulated at the end of the nineteenth century by H. A. Lorentz and J. W. S. Rayleigh, respectively. A reciprocity theorem interrelates the sources and wave fields in two admissable physical and/or computational states in one and the same domain. It can be obtained by inserting the appropriate wave equation into an extended version of Green's theorem [*Morse and Feshbach*, 1953]. In the modern literature, reciprocity theorems also account for possible differences between the medium parameters in both states. One can distinguish between convolution-type and correlation-type reciprocity theorems [*Bojarski*, 1983]. These two types of reciprocity theorems have proven their usefulness in the study of forward and inverse scattering problems, respectively. For a discussion on general reciprocity theorems for electromagnetic and acoustic wave fields in dissipative media we refer to *de Hoop* [1987, 1988]. An extensive overview of reciprocity and its applications in seismic exploration is given by *Fokkema and van den Berg* [1993].

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In many wave propagation problems one can define a "preferred direction of propagation." Electromagnetic and acoustic waveguides are obvious examples, but also in laterally unbounded media it is often advantageous to define a preferred propagation direction, such as in seismic exploration. In all those situations it is useful to decompose the wave equation into a system of coupled equations for oppositely propagating waves. In the literature on electromagnetic wave theory these oppositely propagating waves are known as "bidirectional beams" [*Hoekstra*, 1997; *van Stralen*, 1997]; in the acoustic literature one usually speaks of "one-way wave wave fields" [*Claerbout*, 1971; *Fishman et al.*, 1987]. In this paper we adopt the latter terminology.

The reciprocity theorems discussed above apply to the total electromagnetic or acoustic wave field. Obviously, these reciprocity theorems are not compatible with one-way wave theory. In a recent paper we derived reciprocity theorems for acoustic one-way wave fields in lossless inhomogeneous fluids [*Wapenaar and Grimbergen*, 1996]. With some minor modifications these theorems are also applicable to transverse electric or transverse magnetic one-way wave fields in lossless inhomogeneous media. The aim of the current paper is to extend those one-way reciprocity theorems to electromagnetic or acoustic wave fields in dissipative inhomogeneous media. To this end, we first formulate a general equation for the total electromagnetic or acoustic wave field in dissipative inhomogeneous media. Next we decompose this equation into a coupled system of equations for oppositely propagating one-way wave fields. The central part of the paper is dedicated to deriving the symmetry properties of the operators that appear in these equations. This derivation is essentially different from our former derivation for the lossless situation, for which we made use of the well-established theory of self-adjoint operators. Using the obtained symmetry properties, in the last part of the paper we derive reciprocity theorems of the convolution type and of the correlation type for electromagnetic or acoustic one-way wave fields in dissipative inhomogeneous media.

2. General Wave Equation

In this section we discuss a general equation for electromagnetic or acoustic wave fields in dissipative inhomogeneous media. First we give the basic scalar equations. Next we recast these equations into a matrix-vector form that will be useful as a starting point for the decomposition into one-way wave equations.

2.1. Basic Scalar Equations

We consider a two-dimensional (2-D) configuration, that is, we assume that the medium parameters and the source distributions (and, consequently, the wave fields) are functions of two spatial coordinates [$\mathbf{x} = (x_1, x_3)$] only. With this assumption the vectorial wave equations reduce to scalar wave equations for the following situations: electromagnetic waves (transverse electric waves (TE waves) and transverse magnetic waves (TM waves)) and acoustic waves (compressional waves (P waves) in fluids and horizontally polarized shear waves (SH waves) in solids). We define the Fourier transform with respect to time t of a real-valued function as

$$U(\omega) = \int_{-\infty}^{\infty} \exp(-j\omega t)u(t)\, dt \tag{1}$$

and its inverse as

$$u(t) = \frac{1}{\pi} \operatorname{Re} \int_{-\infty}^{\infty} \exp(j\omega t)\chi(\omega)U(\omega)\, d\omega, \tag{2}$$

where ω denotes the angular frequency, j is the imaginary unit, Re denotes that the real part is taken, and $\chi(\omega)$ is the characteristic function, defined as

$$\chi(\omega) = \begin{cases} 0 & \omega < 0 \\ \frac{1}{2} & \omega = 0 \\ 1 & \omega > 0. \end{cases} \tag{3}$$

In this paper the angular frequency is chosen nonnegative and real-valued, with the remark that its vanishing imaginary part is chosen negative when $u(t)$ is a causal function of time and positive when $u(t)$ is anticausal (i.e., Im $(\omega) \uparrow 0$ when $u(t) = 0$ for $t < 0$ and Im $(\omega) \downarrow 0$ when $u(t) = 0$ for $t > 0$).

In the space-frequency $(\mathbf{x}, \omega)$ domain the general form of the 2-D basic equations is given by

$$j\omega\alpha P + \partial_i Q_i = B, \tag{4}$$

$$j\omega\beta Q_i + \partial_i P = C_i, \tag{5}$$

where i only takes on the values 1 and 3 and Einstein's summation convention applies to repeated subscripts. After elimination of Q_i, (4) and (5) lead to the 2-D scalar wave equation

$$\beta\partial_i\left(\frac{1}{\beta}\,\partial_i P\right) + \alpha\beta\omega^2 P = \beta\partial_i\left(\frac{1}{\beta}\,C_i\right) - j\omega\beta B. \tag{6}$$

$P(\mathbf{x}, \omega)$, $Q_1(\mathbf{x}, \omega)$, and $Q_3(\mathbf{x}, \omega)$ represent the wave fields, $B(\mathbf{x}, \omega)$, $C_1(\mathbf{x}, \omega)$, and $C_3(\mathbf{x}, \omega)$ are the source distributions, and $\alpha(\mathbf{x}, \omega)$ and $\beta(\mathbf{x}, \omega)$ denote the medium parameters (which are assumed to be infinitely differentiable). These functions are further specified in Table 1 for the various wave types discussed above. In a lossless medium the parameters $\alpha(\mathbf{x}, \omega)$ and $\beta(\mathbf{x}, \omega)$ are real valued, positive, and frequency independent. In dissipative media, however, they are complex-valued frequency-dependent functions. For example, for TE waves, α is given by $\varepsilon + \sigma/j\omega$. Note that ε and σ in themselves may be complex valued and frequency-dependent as well. The conditions for the behavior of α and β follow from physics. Causality requires that the time domain counterparts of α and β are zero for $t < 0$ [*Landau and Lifshitz*, 1960; *Boltzmann*, 1876; *de Hoop*, 1987, 1988]. Moreover, the sign of the imaginary parts of α and β in the frequency domain follows from energy considerations. To this end, we analyze the quantity $\partial_i(P^* Q_i + P Q_i^*)$, where the asterisk denotes complex conjugation. Applying the product rule for dif-

Table 1. Overview of Electromagnetic and Acoustic Field Quantities, Medium Parameters, and Source Functions[a]

	P	Q_1	Q_3	α	β	B	C_1	C_3
TE	E_2	H_3	$-H_1$	$\varepsilon + \sigma/j\omega$	$\mu + \Gamma/j\omega$	$-J_2^e$	$-J_3^m$	J_1^m
TM	H_2	$-E_3$	E_1	$\mu + \Gamma/j\omega$	$\varepsilon + \sigma/j\omega$	$-J_2^m$	J_3^e	$-J_1^e$
P (fluids)	$-T_{33}$	V_1	V_3	κ	ϱ	$D_{11} + D_{33}$	F_1	F_3
SH (solids)	V_2	$-T_{21}$	$-T_{23}$	ϱ	$1/\mu$	F_2	$D_{12} + D_{21}$	$D_{23} + D_{32}$

[a]For electromagnetic waves, field quantities are E_i (electric field strength) and H_i (magnetic field strength), medium parameters are ε (permittivity), μ (permeability), σ (conductivity), and Γ (magnetic hysteresis loss term), and source functions are J_i^e (electric current density) and J_i^m (magnetic current density). For acoustic waves, field quantities are V_i (particle velocity) and T_{ij} (stress), medium parameters are κ (compressibility), μ (shear modulus), and ϱ (mass density) and source functions are F_i (force density) and D_{ij} (deformation rate density).

ferentiation, substituting (4) and (5), integrating the result over some 2-D domain $\mathfrak{D}$ with boundary $\partial\mathfrak{D}$ and outward pointing normal vector $\mathbf{n} = (n_1, n_3)$, and applying the theorem of Gauss yields

$$\mathrm{Re}\int_{\mathfrak{D}} (B^*P + C_i^*Q_i)\, d^2\mathbf{x} = \mathrm{Re}\oint_{\partial\mathfrak{D}} P^*Q_in_i\, d\mathbf{x}$$

$$+ \mathrm{Re}\int_{\mathfrak{D}} (j\omega\alpha P^*P + j\omega\beta Q_i^*Q_i)\, d^2\mathbf{x}. \tag{7}$$

The term on the left-hand side represents the power, generated by the sources in $\mathfrak{D}$. The first term on the right-hand side represents the power flux propagating outward through $\partial\mathfrak{D}$ whereas the second term on the right-hand side represents the dissipated power in $\mathfrak{D}$. Hence, in order to comply with physics, the latter term should be nonnegative, so the imaginary parts of α and β should be either zero (for lossless media) or negative (for dissipative media).

For the derivation of the reciprocity theorems in section 5 it is useful to distinguish between dissipative and effectual media. A wave propagating through an effectual medium gains energy. Effectual media will usually be associated with a computational state. For effectual media the time domain counterparts of α and β are anticausal functions of time (i.e., they are zero for $t > 0$), and the imaginary parts of α and β in the frequency domain are positive.

Also for later use, we introduce time-reverse adjoint medium parameters α' and β', according to

$$\alpha'(\mathbf{x}, \omega) = \alpha^*(\mathbf{x}, \omega), \tag{8}$$

$$\beta'(\mathbf{x}, \omega) = \beta^*(\mathbf{x}, \omega). \tag{9}$$

Note that when a medium is dissipative, its time-reverse adjoint medium is effectual and vice versa. In light of the discussion below equations (1) and (2), it should be noted that the primes involve a change of sign not only of the imaginary parts of α and β but also of the vanishing imaginary part of ω.

2.2. Wave Equation in Matrix-Vector Form

From here onward we assume that the preferred propagation direction is along the x_3 axis; see Figure 1. Hence x_1 and x_3 will be referred to as the lateral and axial coordinates, respectively. In the lateral direction the medium may be bounded or unbounded, hence $x_{1,a}$ and $x_{1,b}$ in Figure 1 are finite (with $x_{1,a} < x_{1,b}$) or infinite (with $x_{1,a} \to -\infty$ and $x_{1,b} \to +\infty$). In the former case, homogeneous Dirichlet or Neumann boundary conditions will be imposed at $x_1 = x_{1,a}$ and $x_1 = x_{1,b}$. In the latter case it will be assumed that the medium is laterally homogeneous, lossless, and source-free beyond some finite $|x_1|$ value; moreover, the wave fields are assumed to lie in the appropriate Sobolev space; that is, they are assumed to have "sufficient decay" for $x_1 \to \pm\infty$ [*Rudin*, 1973; *de Hoop*, 1992].

We reorganize the general wave equation into a

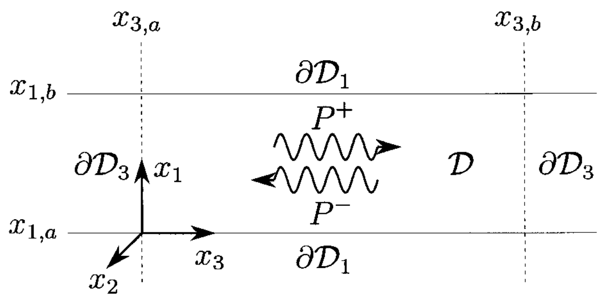

Figure 1. Configuration for which we will derive a reciprocity theorem for one-way wave fields. The direction of preference is chosen parallel to the x_3 axis. The coupled one-way wave fields P^+ and P^- will be defined in section 3.3.

form that acknowledges the direction of preference. By eliminating Q_1 from the system of equations (4) and (5) we obtain

$$\partial_3 \mathbf{Q} = \hat{\mathbf{A}}\mathbf{Q} + \mathbf{D}, \tag{10}$$

where the wave field vector $\mathbf{Q}$ and the source vector $\mathbf{D}$ are given by

$$\mathbf{Q} = \begin{pmatrix} P \\ Q_3 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} C_3 \\ B - \frac{1}{j\omega}\partial_1\left(\frac{1}{\beta}C_1\right) \end{pmatrix} \tag{11}$$

and the operator matrix $\hat{\mathbf{A}}$ is given by

$$\hat{\mathbf{A}} = \begin{pmatrix} 0 & -j\omega\beta \\ -j\omega\hat{A} & 0 \end{pmatrix}, \tag{12}$$

with

$$\hat{A} = \alpha + \frac{1}{\omega^2}\partial_1\left(\frac{1}{\beta}\partial_1 \cdot\right). \tag{13}$$

The circumflex denotes an operator containing the lateral differentiation operator ∂_1. Finally, we introduce an operator $\hat{A}'$ for the time-reverse adjoint medium, defined by the parameters α' and β'. According to (8), (9), and (13) this operator is given by

$$\hat{A}' = \hat{A}^*. \tag{14}$$

Note that an "operator for the time-reverse adjoint medium" is not by definition the same as the "adjoint operator for the original medium." We come back to this in section 4.3.

3. One-Way Wave Equation

In this section we derive the matrix-vector form of the coupled system of equations for electromagnetic or acoustic one-way wave fields in dissipative inhomogeneous media (which we will call for short the "one-way wave equation"). First we introduce the square root operator that plays a central role in the one-way wave equation. Next we use this operator to diagonalize the operator matrix $\hat{\mathbf{A}}$. Finally, we decompose the wave field vector $\mathbf{Q}$ into one-way wave fields and derive the one-way wave equation, employing the diagonalized form of the operator matrix $\hat{\mathbf{A}}$.

3.1. Introduction of the Square Root Operator

We start by introducing an operator $\hat{\mathcal{H}}_2$ via [*Brekhovskikh*, 1960; *Wapenaar and Berkhout*, 1989, Appendix B]

$$\hat{\mathcal{H}}_2 = \omega^2\beta^{1/2}(\hat{A}\beta^{1/2} \cdot), \tag{15}$$

or, using (13),

$$\hat{\mathcal{H}}_2 = k^2 + \partial_1^2, \tag{16}$$

where

$$k^2 = \left(\frac{\omega}{c}\right)^2 = \alpha\beta\omega^2 - \frac{3(\partial_1\beta)^2}{4\beta^2} + \frac{\partial_1^2\beta}{2\beta}. \tag{17}$$

Note that $\hat{\mathcal{H}}_2$, as defined in (16), represents the Helmholtz operator; (17) has the form of the Klein-Gordon dispersion relation, known from relativistic quantum mechanics [*Messiah*, 1962; *Anno et al.*, 1992], with $c(\mathbf{x}, \omega)$ being the (complex-valued) propagation velocity and $k(\mathbf{x}, \omega)$ being the wave number.

We introduce an operator $\hat{\mathcal{H}}_1$ as the square root of the Helmholtz operator $\hat{\mathcal{H}}_2$, according to [*Claerbout*, 1971; *Berkhout*, 1982; *Fishman et al.*, 1987; *de Hoop*, 1992, 1996; *Wapenaar and Grimbergen*, 1996]

$$\hat{\mathcal{H}}_1\hat{\mathcal{H}}_1 = \hat{\mathcal{H}}_2, \quad \text{or} \quad \hat{\mathcal{H}}_1 = \hat{\mathcal{H}}_2^{1/2}. \tag{18}$$

Unlike $\hat{\mathcal{H}}_2$, the square root operator $\hat{\mathcal{H}}_1$ cannot be written as a polynomial in ∂_1. Therefore $\hat{\mathcal{H}}_1$ is a so-called pseudodifferential operator [*Kumano-go*, 1974; *Treves*, 1980; *Fishman*, 1992]. The square root of an operator is not uniquely defined. In the following we assume that the square root is taken such that the imaginary part of the eigenvalue spectrum of $\hat{\mathcal{H}}_1$ is negative for dissipative media and positive for effectual media, analogous to the imaginary parts of α and β. An example of the eigenvalue spectra of $\hat{\mathcal{H}}_2$ and $\hat{\mathcal{H}}_1$ for dissipative and effectual media is shown in Figure 2.

Given the Helmholtz and square root operators in a specific medium (either dissipative or effectual), we obtain for the operators in the time-reverse adjoint medium, analogous to (14),

$$\hat{\mathcal{H}}_2' = \hat{\mathcal{H}}_2^* \quad \text{and} \quad \hat{\mathcal{H}}_1' = \hat{\mathcal{H}}_1^*. \tag{19}$$

For the limiting case of a lossless medium, $\hat{\mathcal{H}}_2'$ reduces to $\hat{\mathcal{H}}_2$. However, for the square root operator we have $\hat{\mathcal{H}}_1' \neq \hat{\mathcal{H}}_1$: The prime changes the sign of the vanishing imaginary part of the frequency ω and, consequently, of the imaginary part of the eigenvalue spectrum; see Figure 3.

3.2. Diagonalization of the Operator Matrix

Using (15), we may reformulate the operator matrix $\hat{\mathbf{A}}$, as defined in (12), as follows:

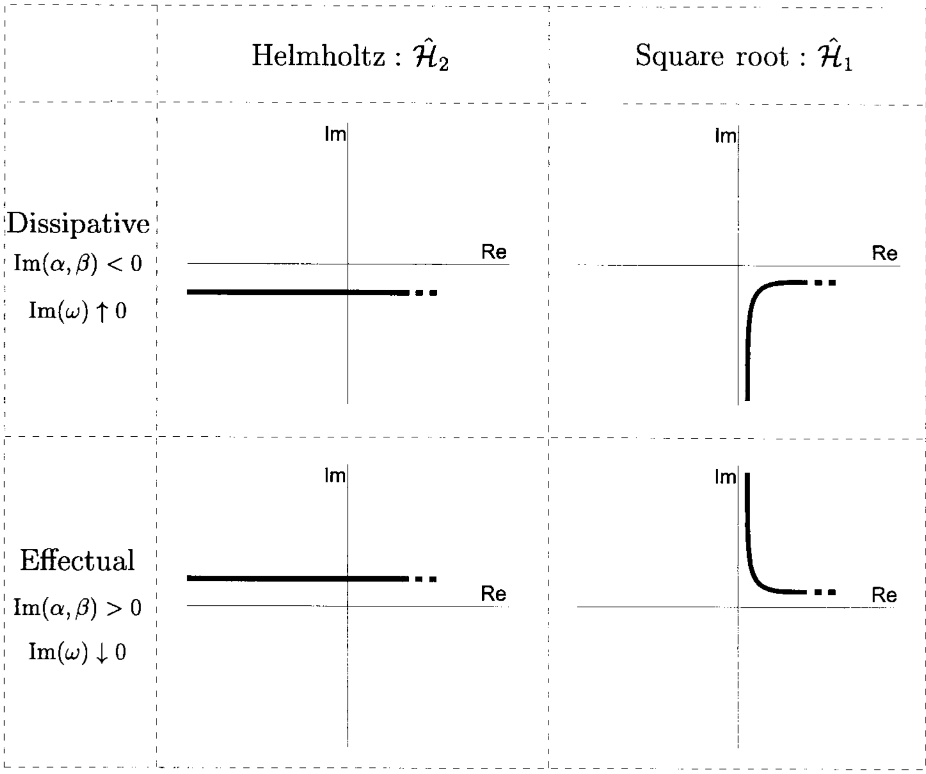

Figure 2. Example of eigenvalue spectra (in the complex plane) of the Helmholtz operator $\hat{\mathcal{H}}_2$ and the square root operator $\hat{\mathcal{H}}_1$, for a dissipative and an effectual medium. The solid lines denote the continuous (or essential) part of the spectrum; the dotted lines denote the discrete eigenvalues, usually associated with guided modes [*Grimbergen et al.*, 1998].

$$\hat{\mathbf{A}} = \begin{pmatrix} 0 & -j\omega\beta \\ \frac{1}{j\omega\beta^{1/2}}(\hat{\mathcal{H}}_2\beta^{-1/2}\cdot) & 0 \end{pmatrix}. \tag{20}$$

Using the square root operator introduced in (18), we may thus write for the operator matrix $\hat{\mathbf{A}}$

$$\hat{\mathbf{A}} = \hat{\mathbf{L}}\hat{\mathbf{H}}\hat{\mathbf{L}}^{-1}, \tag{21}$$

where $\hat{\mathbf{H}}$ is a diagonal operator matrix, containing the square root operator on its diagonal, according to

$$\hat{\mathbf{H}} = \begin{pmatrix} -j\hat{\mathcal{H}}_1 & 0 \\ 0 & j\hat{\mathcal{H}}_1 \end{pmatrix} \tag{22}$$

and where

$$\hat{\mathbf{L}} = \begin{pmatrix} \hat{L}_1 & \hat{L}_1 \\ \hat{L}_2 & -\hat{L}_2 \end{pmatrix}, \qquad \hat{\mathbf{L}}^{-1} = \frac{1}{2}\begin{pmatrix} \hat{L}_1^{-1} & \hat{L}_2^{-1} \\ \hat{L}_1^{-1} & -\hat{L}_2^{-1} \end{pmatrix}, \tag{23}$$

with

$$\hat{L}_1 = \left(\frac{\omega\beta}{2}\right)^{1/2}\hat{\mathcal{H}}_1^{-1/2}, \qquad \frac{1}{2}\hat{L}_1^{-1} = [\hat{\mathcal{H}}_1^{1/2}(2\omega\beta)^{-1/2}\cdot\,], \tag{24}$$

$$\hat{L}_2 = (2\omega\beta)^{-1/2}\hat{\mathcal{H}}_1^{1/2}, \qquad \frac{1}{2}\hat{L}_2^{-1} = \left[\hat{\mathcal{H}}_1^{-1/2}\left(\frac{\omega\beta}{2}\right)^{1/2}\cdot\right]. \tag{25}$$

Note that $\hat{\mathcal{H}}_1^{1/2}$ represents the square root of the square root operator. Similarly to the above, we assume that this square root is taken such that the imaginary part of the eigenvalue spectrum of $\hat{\mathcal{H}}_1^{1/2}$ is negative for dissipative media and positive for effectual media. Analogous to (19), we have the following relations for the operators $\hat{L}'_1$ and $\hat{L}'_2$ in the time-reverse adjoint medium

$$\hat{L}'_1 = \hat{L}_1^* \quad \text{and} \quad \hat{L}'_2 = \hat{L}_2^*. \tag{26}$$

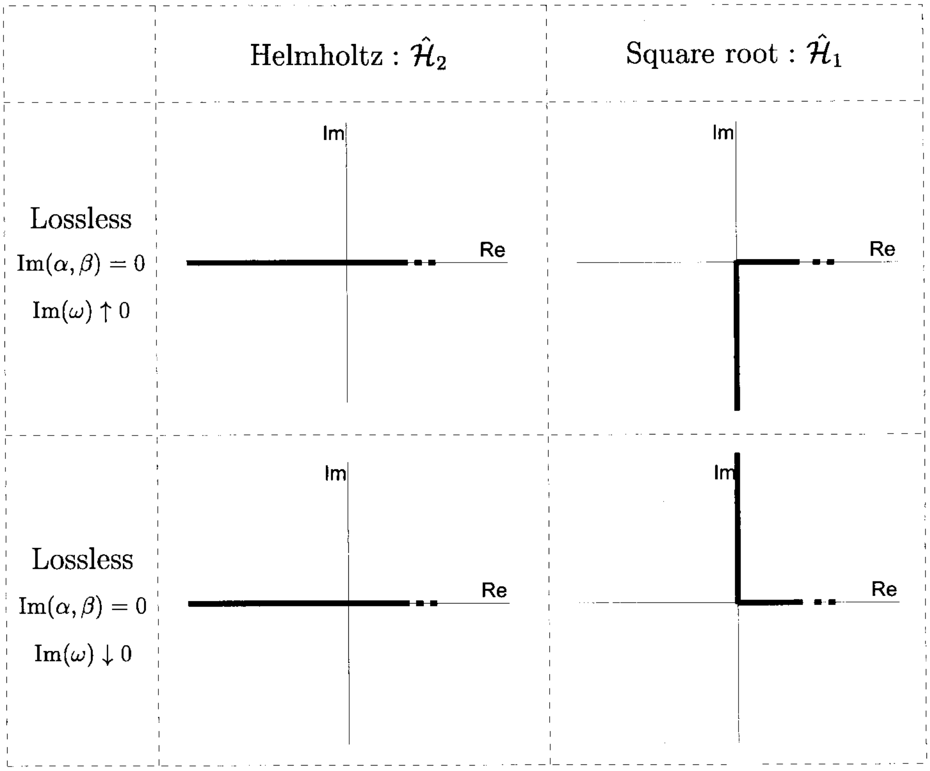

Figure 3. Example of eigenvalue spectra (in the complex plane) of the Helmholtz operator $\hat{\mathcal{H}}_2$ and the square root operator $\hat{\mathcal{H}}_1$, for a lossless medium. Note that the sign of the imaginary part of the eigenvalue spectrum of the square root operator depends on the sign of the vanishing imaginary part of ω.

3.3. One-Way Wave Equation in Matrix-Vector Form

We introduce a "one-way wave field vector" $\mathbf{P}$, according to

$$\mathbf{P}=\begin{pmatrix}P^+\\P^-\end{pmatrix}, \tag{27}$$

where P^+ and P^- are the one-way wave fields that propagate in the positive and negative x_3 direction, respectively; see Figure 1. In a homogeneous medium, P^+ and P^- propagate independently; hence, for this situation, $\mathbf{P}$ satisfies an equation of the same form as (10), with operator matrix $\hat{\mathbf{A}}$ replaced by the diagonal operator matrix $\hat{\mathbf{H}}$. Our aim is to find an equation for $\mathbf{P}$ in an inhomogeneous medium, such that the leading term of the operator matrix is again given by the diagonal operator matrix $\hat{\mathbf{H}}$ [*Corones*, 1975; *Fishman et al.*, 1987]. To this end, we substitute (21) into (10) and multiply both sides by $\hat{\mathbf{L}}^{-1}$, which gives

$$\hat{\mathbf{L}}^{-1}\partial_3\mathbf{Q}=\hat{\mathbf{H}}\,\hat{\mathbf{L}}^{-1}\mathbf{Q}+\hat{\mathbf{L}}^{-1}\mathbf{D}. \tag{28}$$

This suggests that we define $\mathbf{P}$ as

$$\mathbf{P}=\hat{\mathbf{L}}^{-1}\mathbf{Q}\quad\text{or, equivalently,}\quad\mathbf{Q}=\hat{\mathbf{L}}\mathbf{P}, \tag{29}$$

in order to arrive at the desired equation for $\mathbf{P}$. Note that $\hat{\mathbf{L}}^{-1}$ is then by definition a decomposition operator, and consequently, $\hat{\mathbf{L}}$ is a composition operator. After some straightforward manipulations we obtain the one-way wave equation

$$\partial_3\mathbf{P}=\hat{\mathbf{B}}\mathbf{P}+\mathbf{S}, \tag{30}$$

where the one-way operator matrix $\hat{\mathbf{B}}$ is defined as

$$\hat{\mathbf{B}}=\hat{\mathbf{H}}-\mathbf{L}^{-1}\partial_3\hat{\mathbf{L}} \tag{31}$$

and the one-way source vector $\mathbf{S}$ is defined as

$$\mathbf{S}=\hat{\mathbf{L}}^{-1}\mathbf{D}\quad\text{or, equivalently,}\quad\mathbf{D}=\hat{\mathbf{L}}\mathbf{S}. \tag{32}$$

In analogy to (27) we write

$$\mathbf{S} = \begin{pmatrix} S^+ \\ S^- \end{pmatrix}, \tag{33}$$

where S^+ and S^- are the sources for the one-way wave fields P^+ and P^-, respectively. Using (22) and (23), we may write for $\hat{\mathbf{B}}$, as defined in (31),

$$\hat{\mathbf{B}} = \begin{pmatrix} -j\hat{\mathscr{H}}_1 & 0 \\ 0 & j\hat{\mathscr{H}}_1 \end{pmatrix} + \begin{pmatrix} \hat{T} & -\hat{R} \\ -\hat{R} & \hat{T} \end{pmatrix}, \tag{34}$$

where $\hat{R}$ and $\hat{T}$ are reflection and transmission operators, respectively, given by

$$\hat{R} = (\hat{L}_1^{-1}\partial_3\hat{L}_1 - \hat{L}_2^{-1}\partial_3\hat{L}_2)/2, \tag{35}$$

$$\hat{T} = -(\hat{L}_1^{-1}\partial_3\hat{L}_1 + \hat{L}_2^{-1}\partial_3\hat{L}_2)/2. \tag{36}$$

It appears that unlike the operator matrix $\hat{\mathbf{A}}$, the one-way operator matrix $\hat{\mathbf{B}}$ distinguishes explicitly between propagation (the diagonal matrix in (34)) and scattering (the full matrix in (34); note that this latter operator matrix vanishes in any region where the medium parameters do not vary in the axial direction).

On account of (26) we have

$$\hat{R}' = \hat{R}^* \quad \text{and} \quad \hat{T}' = \hat{T}^*. \tag{37}$$

Hence, given the one-way operator matrix $\hat{\mathbf{B}}$ in a specific medium (either dissipative or effectual), we obtain for the one-way operator matrix in the time-reverse adjoint medium, using (19) and (37),

$$\hat{\mathbf{B}} = \begin{pmatrix} -j\hat{\mathscr{H}}_1^* & 0 \\ 0 & j\hat{\mathscr{H}}_1^* \end{pmatrix} + \begin{pmatrix} \hat{T}^* & -\hat{R}^* \\ -\hat{R}^* & \hat{T}^* \end{pmatrix}. \tag{38}$$

Note that for the limiting case of a lossless medium we have $\hat{\mathbf{B}}' \neq \hat{\mathbf{B}}$; see the discussion below (19).

4. Symmetry Properties of the One-Way Operator Matrix

The one-way wave equation (30) that we derived in section 3.3 will be used in section 5 for the derivation of reciprocity theorems for one-way wave fields. For this purpose we first need to establish a number of symmetry properties of the one-way operator matrix $\hat{\mathbf{B}}$, defined in (31). This is the subject of the current section. First we briefly review the concept of transposed and adjoint operators. Next we derive symmetry properties of square root operators. Finally, we employ the results to derive the symmetry properties of the one-way operator matrix $\hat{\mathbf{B}}$.

4.1. Transposed and Adjoint Operators

For two scalar functions $f(x_1)$ and $g(x_1)$ we define the bilinear form as [*Rudin*, 1973; *Reddy*, 1986]

$$\langle f, g\rangle_b = \int_{x_{1,a}}^{x_{1,b}} f(x_1)g(x_1)\,dx_1 \tag{39}$$

and the sesquilinear form as

$$\langle f, g\rangle_s = \int_{x_{1,a}}^{x_{1,b}} f^*(x_1)g(x_1)\,dx_1. \tag{40}$$

The integration limits $x_{1,a}$ and $x_{1,b}$ may be either finite or infinite; in the latter case, $f(x_1)$ and $g(x_1)$ are assumed to lie in the appropriate Sobolev space. Consider a scalar operator $\hat{\mathscr{U}} = \hat{\mathscr{U}}(x_1, \partial_1)$. We introduce the transposed operator $\hat{\mathscr{U}}^t$ via

$$\langle \hat{\mathscr{U}}f, g\rangle_b = \langle f, \hat{\mathscr{U}}^t g\rangle_b \tag{41}$$

and the adjoint operator $\hat{\mathscr{U}}^\dagger$ via

$$\langle \hat{\mathscr{U}}f, g\rangle_s = \langle f, \hat{\mathscr{U}}^\dagger g\rangle_s. \tag{42}$$

Note that the following relation holds between the transposed and the adjoint operators

$$\hat{\mathscr{U}}^\dagger = (\hat{\mathscr{U}}^t)^*. \tag{43}$$

An operator is called symmetric when it obeys the relation

$$\hat{\mathscr{U}}^t = \hat{\mathscr{U}}. \tag{44}$$

On the other hand, it is self-adjoint when

$$\hat{\mathscr{U}}^\dagger = \hat{\mathscr{U}}. \tag{45}$$

For two vector functions $\mathbf{f}(x_1)$ and $\mathbf{g}(x_1)$ we define the bilinear form as

$$\langle \mathbf{f}, \mathbf{g}\rangle_b = \int_{x_{1,a}}^{x_{1,b}} \mathbf{f}^t(x_1)\mathbf{g}(x_1)\,dx_1 \tag{46}$$

and the sesquilinear form as

$$\langle \mathbf{f}, \mathbf{g}\rangle_s = \int_{x_{1,a}}^{x_{1,b}} \mathbf{f}^\dagger(x_1)\mathbf{g}(x_1)\,dx_1. \tag{47}$$

Note that we use the superscript t for transposed operators (equation (41)) as well as for transposed vectors or matrices (equation (46)). Similarly, the superscript dagger is used to denote adjoint operators (equation (42)) as well as adjoint (i.e., complex conjugate transposed) vectors or matrices (equation

(47)). In the following we use these superscripts also for transposed and adjoint matrices containing operators. Consider an operator matrix

$$\hat{\mathbf{U}} = \hat{\mathbf{U}}(x_1, \partial_1) = \begin{pmatrix} \hat{\mathcal{U}}_{11} & \hat{\mathcal{U}}_{12} \\ \hat{\mathcal{U}}_{21} & \hat{\mathcal{U}}_{22} \end{pmatrix}. \tag{48}$$

We introduce the transposed operator matrix $\hat{\mathbf{U}}^t$ via

$$\langle \hat{\mathbf{U}}\, \mathbf{f}, \mathbf{g} \rangle_b = \langle \mathbf{f}, \hat{\mathbf{U}}^t\, \mathbf{g} \rangle_b. \tag{49}$$

From the relations above it easily follows that

$$\hat{\mathbf{U}}^t = \begin{pmatrix} \hat{\mathcal{U}}_{11}^t & \hat{\mathcal{U}}_{21}^t \\ \hat{\mathcal{U}}_{12}^t & \hat{\mathcal{U}}_{22}^t \end{pmatrix}; \tag{50}$$

hence $\hat{\mathbf{U}}^t$ is a transposed matrix, containing transposed operators. The adjoint operator matrix $\hat{\mathbf{U}}^\dagger$ is introduced via

$$\langle \hat{\mathbf{U}}\, \mathbf{f}, \mathbf{g} \rangle_s = \langle \mathbf{f}, \hat{\mathbf{U}}^\dagger \mathbf{g} \rangle_s. \tag{51}$$

Note that the following relation holds between the transposed and the adjoint operator matrices

$$\hat{\mathbf{U}}^\dagger = (\hat{\mathbf{U}}^t)^*. \tag{52}$$

4.2. Symmetry of Square Root Operators

We demonstrate the symmetry of square root operators, using a generic notation. The approach we follow is modified after *Dillen* [2000]. Let $\hat{\mathcal{U}} = \hat{\mathcal{U}}(x_1, \partial_1)$ be an arbitrary symmetric operator (in the sense of equations (39), (41), and (44)), and let $\hat{\mathcal{V}} = \hat{\mathcal{V}}(x_1, \partial_1)$ be its square root: $\hat{\mathcal{V}} = \hat{\mathcal{U}}^{1/2}$. In order to derive the symmetry properties of $\hat{\mathcal{V}}$ we construct the following pseudodifferential equation:

$$\partial_z f = -j\hat{\mathcal{V}} f, \tag{53}$$

with $f = f(x_1, z)$ (note that z is a new variable, which bears no relation to x_3). We assume that the square root is taken such that the imaginary part of the eigenvalue spectrum of $\hat{\mathcal{V}}$ is either negative (for all eigenvalues) or positive (again for all eigenvalues). This is equivalent to stating that either

$$\lim_{z\to\infty} f = 0 \quad \text{or} \quad \lim_{z\to-\infty} f = 0, \tag{54}$$

for any f obeying (53). Obviously, when $\hat{\mathcal{U}}$ is the Helmholtz operator for a dissipative medium, (53) (together with the first condition of (54)) has the form of a one-way wave equation for P^+ waves in the (x_1, z) coordinate system. Throughout this analysis, however, $\hat{\mathcal{U}}$ is an arbitrary symmetric operator, so in general we cannot assign a physical interpretation to (53). Note that since $\hat{\mathcal{V}}$ is not a function of z, (53) implies

$$\partial_z^2 f = -\hat{\mathcal{U}} f. \tag{55}$$

Let f and g be two linearly independent solutions of (53). We introduce an interaction quantity $\mathcal{I}$, according to

$$\mathcal{I} = \langle f, \partial_z g \rangle_b - \langle \partial_z f, g \rangle_b. \tag{56}$$

On account of (54) we have either

$$\lim_{z\to\infty} \mathcal{I} = 0 \quad \text{or} \quad \lim_{z\to-\infty} \mathcal{I} = 0. \tag{57}$$

We evaluate the z derivative of $\mathcal{I}$, which yields

$$\begin{aligned}\partial_z \mathcal{I} &= \langle \partial_z f, \partial_z g \rangle_b + \langle f, \partial_z^2 g \rangle_b - \langle \partial_z f, \partial_z g \rangle_b - \langle \partial_z^2 f, g \rangle_b \\ &= \langle \hat{\mathcal{U}} f, g \rangle_b - \langle f, \hat{\mathcal{U}} g \rangle_b. \end{aligned}\tag{58}$$

Since $\hat{\mathcal{U}}$ is assumed to be a symmetric operator, we obtain

$$\partial_z \mathcal{I} = 0. \tag{59}$$

From (57) and (59) we obtain $\mathcal{I} = 0$ for all z, or, using (56),

$$\langle f, \partial_z g \rangle_b = \langle \partial_z f, g \rangle_b. \tag{60}$$

This implies, together with (53),

$$\langle f, \hat{\mathcal{V}} g \rangle_b = \langle \hat{\mathcal{V}} f, g \rangle_b, \tag{61}$$

or, according to (41),

$$\hat{\mathcal{V}}^t = \hat{\mathcal{V}}. \tag{62}$$

Hence, under the assumptions made above, the square root of a symmetric operator is symmetric. Using induction it follows that the operator $\hat{\mathcal{U}}^{1/2^n}$ is symmetric for any $n \geq 0$:

$$(\hat{\mathcal{U}}^{1/2^n})^t = \hat{\mathcal{U}}^{1/2^n}. \tag{63}$$

4.3. Symmetry Properties of the One-Way Operator Matrix

First we show that the Helmholtz operator is symmetric. For fixed ω and fixed x_3 the Helmholtz operator defined in (16) is of the form $\hat{\mathcal{H}}_2 = \hat{\mathcal{H}}_2(x_1, \partial_1)$. Substituting this operator in the left-hand side of (41) and applying integration by parts twice yields

$$\langle \hat{\mathcal{H}}_2 f, g\rangle_b = \int_{x_{1,a}}^{x_{1,b}} [k^2 fg - (\partial_1 f)(\partial_1 g)]\, dx_1 + (\partial_1 f)g|_{x_{1,a}}^{x_{1,b}}$$

$$= \langle f, \hat{\mathcal{H}}_2 g\rangle_b + (\partial_1 f)g|_{x_{1,a}}^{x_{1,b}} - f(\partial_1 g)|_{x_{1,a}}^{x_{1,b}}. \quad (64)$$

When $x_{1,a}$ and $x_{1,b}$ are finite, the last two terms on the right-hand side vanish by imposing homogeneous Dirichlet or Neumann boundary conditions for f and g at $x_{1,a}$ and $x_{1,b}$. When $x_{1,a}$ and $x_{1,b}$ are infinite, these terms vanish when f and g lie in the appropriate Sobolev space. In both cases we have $\langle \hat{\mathcal{H}}_2 f, g\rangle_b = \langle f, \hat{\mathcal{H}}_2 g\rangle_b$, or according to (41),

$$\hat{\mathcal{H}}_2^t = \hat{\mathcal{H}}_2. \quad (65)$$

Note that this symmetry relation is valid for any fixed x_3 value, no matter whether the medium parameters vary in the x_3 direction or not. Hence, for any fixed x_3 value in an inhomogeneous medium we may use the derivation of section 4.2 to show that the square root operators $\hat{\mathcal{H}}_1 = \hat{\mathcal{H}}_2^{1/2}$, etc., are symmetric, according to

$$\hat{\mathcal{H}}_1^t = \hat{\mathcal{H}}_1, \qquad (\hat{\mathcal{H}}_1^{1/2})^t = \hat{\mathcal{H}}_1^{1/2}, \quad \text{etc.} \quad (66)$$

The condition for this derivation is that each square root is taken such that the imaginary part of its entire eigenvalue spectrum is either negative or positive, which corresponds to a dissipative or an effectual medium, respectively. Hence this derivation excludes the limiting situation of a lossless medium, because in that case a part of the eigenvalue spectrum of the square root operator is real valued. However, for the lossless situation the symmetry relations have been established previously by modal decomposition of the operator kernels [*Wapenaar and Grimbergen*, 1996]. Hence we may conclude that (66) is valid for dissipative, lossless, and effectual media. Note that according to (43), (65), (66), and (19) the adjoint Helmholtz and square root operators are given by

$$\hat{\mathcal{H}}_2^\dagger = \hat{\mathcal{H}}'_2, \quad \hat{\mathcal{H}}_1^\dagger = \hat{\mathcal{H}}'_1, \quad (\hat{\mathcal{H}}_1^{1/2})^\dagger = (\hat{\mathcal{H}}_1^{1/2})', \quad \text{etc.} \quad (67)$$

For the limiting case of a lossless medium the Helmholtz operator is self-adjoint (i.e., $\hat{\mathcal{H}}_2^\dagger = \hat{\mathcal{H}}_2$), but the square root operators are not (see the discussion below equation (19)).

Using the fact that the inverse of a symmetric operator is symmetric as well, we have $(\hat{\mathcal{H}}_1^{-1/2})^t = \hat{\mathcal{H}}_1^{-1/2}$. Consequently, for the transposed operators $\hat{L}_1^t$ and $\hat{L}_2^t$ we obtain

$$\hat{L}_1^t = \frac{1}{2}\hat{L}_2^{-1} \quad \text{and} \quad \hat{L}_2^t = \frac{1}{2}\hat{L}_1^{-1}; \quad (68)$$

see (24) and (25). With these results we find for the transposed reflection and transmission operators, defined in (35) and (36),

$$\hat{R}^t = [(\partial_3 \hat{L}_2^{-1})\hat{L}_2 - (\partial_3 \hat{L}_1^{-1})\hat{L}_1]/2, \quad (69)$$

$$\hat{T}^t = -[(\partial_3 \hat{L}_2^{-1})\hat{L}_2 + (\partial_3 \hat{L}_1^{-1})\hat{L}_1]/2. \quad (70)$$

Using the property $(\partial_3 \hat{L}_1^{-1})\hat{L}_1 + \hat{L}_1^{-1}\partial_3\hat{L}_1 = \partial_3(\hat{L}_1^{-1}\hat{L}_1) = 0$, or $(\partial_3 \hat{L}_1^{-1})\hat{L}_1 = -\hat{L}_1^{-1}\partial_3\hat{L}_1$ (and a similar property for $\hat{L}_2$), we obtain

$$\hat{R}^t = \hat{R} \quad \text{and} \quad \hat{T}^t = -\hat{T}. \quad (71)$$

Note that according to (43), (71), and (37) the adjoint reflection and transmission operators are given by

$$\hat{R}^\dagger = \hat{R}' \quad \text{and} \quad \hat{T}^\dagger = -\hat{T}'. \quad (72)$$

The latter result illustrates the fact that an operator for a time-reverse adjoint medium ($\hat{T}'$) is not necessarily the same as the adjoint operator for the original medium ($\hat{T}^\dagger$).

We have now derived everything we need to find the symmetry properties of the one-way operator matrix $\hat{\mathbf{B}}$, defined in (34). From (50), (66), and (71) it follows that the transposed one-way operator matrix is given by

$$\hat{\mathbf{B}}^t = \begin{pmatrix} -j\hat{\mathcal{H}}_1 - \hat{T} & -\hat{R} \\ -\hat{R} & j\hat{\mathcal{H}}_1 - \hat{T} \end{pmatrix}, \quad (73)$$

or

$$\hat{\mathbf{B}}^t\mathbf{N} = -\mathbf{N}\hat{\mathbf{B}}, \quad (74)$$

with

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (75)$$

From (52) and (73) it follows that the adjoint one-way operator matrix is given by

$$\hat{\mathbf{B}}^\dagger = \begin{pmatrix} j\hat{\mathcal{H}}_1^* - \hat{T}^* & -\hat{R}^* \\ -\hat{R}^* & -j\hat{\mathcal{H}}_1^* - \hat{T}^* \end{pmatrix}, \quad (76)$$

or

$$\hat{\mathbf{B}}^\dagger\mathbf{J} = -\mathbf{J}\hat{\mathbf{B}}', \quad (77)$$

Table 2. States in the One-Way Reciprocity Theorems

	State A	State B
Wave field	$\mathbf{P}_A$	$\mathbf{P}_B$
Operator	$\hat{\mathbf{B}}_A$	$\hat{\mathbf{B}}_B$
Source	$\mathbf{S}_A$	$\mathbf{S}_B$

with $\hat{\mathbf{B}}'$ being the one-way operator matrix in the time-reverse adjoint medium (see equation (38)) and

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{78}$$

5. Reciprocity Theorems for One-Way Wave Fields

We derive reciprocity theorems that interrelate the one-way wave vectors, operator matrices, and source vectors in two different states. These states will be distinguished by the subscripts A and B; see Table 2. The domain $\mathcal{D}$ for which we derive the reciprocity theorems is defined as $\mathcal{D} = \{\mathbf{x}|x_{1,a} \le x_1 \le x_{1,b} \wedge x_{3,a} \le x_3 \le x_{3,b}\}$; see Figure 1. For the boundary $\partial\mathcal{D}$ of this domain we write $\partial\mathcal{D} = \partial\mathcal{D}_1 \cup \partial\mathcal{D}_3$, where $\partial\mathcal{D}_1 = \{\mathbf{x}|x_1 = x_{1,a} \wedge x_{3,a} \le x_3 \le x_{3,b}\} \cup \{\mathbf{x}|x_1 = x_{1,b} \wedge x_{3,a} \le x_3 \le x_{3,b}\}$ and $\partial\mathcal{D}_3 = \{\mathbf{x}|x_3 = x_{3,a} \wedge x_{1,a} \le x_1 \le x_{1,b}\} \cup \{\mathbf{x}|x_3 = x_{3,b} \wedge x_{1,a} \le x_1 \le x_{1,b}\}$; see Figure 1. The wave field conditions that have been discussed above equation (10) are assumed to apply to $\mathbf{P}_A$ as well as to $\mathbf{P}_B$.

5.1. Convolution-Type Reciprocity Theorem for One-Way Wave Fields

We define a convolution-type "interaction quantity" between oppositely propagating waves in both states, according to

$$\partial_3\{P_A^+ P_B^- - P_A^- P_B^+\}, \tag{79}$$

or, using a more compact notation,

$$\partial_3\{\mathbf{P}_A^t \mathbf{N} \mathbf{P}_B\}. \tag{80}$$

We speak of "convolution type" since the products in the frequency domain ($P_A^+ P_B^-$, etc.) correspond to convolutions in the time domain. Applying the product rule for differentiation, substituting the one-way wave equation (30) for states A and B, integrating the result over domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$, and applying the theorem of Gauss, yields

$$\int_{\partial\mathcal{D}_3} \mathbf{P}_A^t \mathbf{N} \mathbf{P}_B n_3 \, dx_1 = \int_{\mathcal{D}} \{\mathbf{P}_A^t \mathbf{N} \hat{\mathbf{B}}_B \mathbf{P}_B + (\hat{\mathbf{B}}_A \mathbf{P}_A)^t \mathbf{N} \mathbf{P}_B\} \, d^2\mathbf{x} + \int_{\mathcal{D}} \{\mathbf{P}_A^t \mathbf{N} \mathbf{S}_B + \mathbf{S}_A^t \mathbf{N} \mathbf{P}_B\} \, d^2\mathbf{x}, \tag{81}$$

where the component n_3 of the outward pointing normal vector on $\partial\mathcal{D}_3$ is defined as $n_3 = -1$ for $x_3 = x_{3,a}$ and $n_3 = +1$ for $x_3 = x_{3,b}$. Note that the domain integrals could be written as

$$\int_{\mathcal{D}} \{\quad\} \, d^2\mathbf{x} = \int_{x_{3,a}}^{x_{3,b}} dx_3 \int_{x_{1,a}}^{x_{1,b}} \{\quad\} \, dx_1.$$

Hence, on account of equations (46), (49), and (74) we obtain for the x_1 integral in the second term on the right-hand side of (81)

$$\langle \hat{\mathbf{B}}_A \mathbf{P}_A, \mathbf{N}\mathbf{P}_B \rangle_b = \langle \mathbf{P}_A, \hat{\mathbf{B}}_A^t \mathbf{N} \mathbf{P}_B \rangle_b = -\langle \mathbf{P}_A, \mathbf{N} \hat{\mathbf{B}}_A \mathbf{P}_B \rangle_b. \tag{82}$$

Using this result, we may rewrite (81) as

$$\int_{\partial\mathcal{D}_3} \mathbf{P}_A^t \mathbf{N} \mathbf{P}_B n_3 \, dx_1 = \int_{\mathcal{D}} \mathbf{P}_A^t \mathbf{N} (\hat{\mathbf{B}}_B - \hat{\mathbf{B}}_A) \mathbf{P}_B \, d^2\mathbf{x} + \int_{\mathcal{D}} \{\mathbf{P}_A^t \mathbf{N} \mathbf{S}_B + \mathbf{S}_A^t \mathbf{N} \mathbf{P}_B\} \, d^2\mathbf{x}. \tag{83}$$

Equation (83) formulates a reciprocity theorem of the convolution type for one-way wave fields. It relates the one-way wave field vectors at the boundary $\partial\mathcal{D}_3$ to the one-way sources and the contrast between the one-way operators in both states in domain $\mathcal{D}$. Equation (83) applies to lossless as well as dissipative and effectual media. In its general form, (83) provides a basis for, amongst others, representations of scattered one-way wave fields in terms of generalized Bremmer series expansions or in terms of generalized primaries [*Corones*, 1975; *de Hoop*, 1996; *Wapenaar*, 1996; *van Stralen et al.*, 1996]. Note that the contrast term in the right-hand side of (83) vanishes when the medium parameters in both states are identical. Upon substituting (27), (33), and (75) in the remaining terms we may rewrite (83) as

$$\int_{\partial\mathcal{D}_3} \{P_A^+ P_B^- - P_A^- P_B^+\} n_3 \, dx_1 = \int_{\mathcal{D}} \{P_A^+ S_B^- - P_A^- S_B^+ + S_A^+ P_B^- - S_A^- P_B^+\} \, d^2\mathbf{x}. \tag{84}$$

For comparison, for the same situation the convolution-type reciprocity theorem for the total wave fields defined in Table 1 reads [*de Hoop*, 1987, 1988; *Fokkema and van den Berg*, 1993]

$$\int_{\partial\mathcal{D}_3} \{P_A Q_{3,B} - Q_{3,A} P_B\} n_3 \, dx_1 = \int_{\mathcal{D}} \{P_A B_B - Q_{i,A} C_{i,B} + C_{i,A} Q_{i,B} - B_A P_B\} \, d^2\mathbf{x}. \quad (85)$$

Note that when there are only outgoing waves at $\partial\mathcal{D}_3$ (i.e., $P_A^+ = P_B^+ = 0$ at $x_3 = x_{3,a}$ and $P_A^- = P_B^- = 0$ at $x_3 = x_{3,b}$), then the term on the left-hand side of (84) vanishes as well. Finally, for the special case of one-way point sources defined as $S_A^+ = \delta(\mathbf{x} - \mathbf{x}_A)$, $S_B^+ = \delta(\mathbf{x} - \mathbf{x}_B)$, and $S_A^- = S_B^- = 0$, with $\mathbf{x}_A \in \mathcal{D}$ and $\mathbf{x}_B \in \mathcal{D}$, we obtain

$$P_A^-(\mathbf{x}_B, \omega) = P_B^-(\mathbf{x}_A, \omega). \quad (86)$$

The left-hand side represents the one-way wave field P_A^- at observation point $\mathbf{x}_B$, due to the one-way source S_A^+ at source point $\mathbf{x}_A$; the right-hand side represents the one-way wave field P_B^- at observation point $\mathbf{x}_A$, due to the one-way source S_B^+ at source point $\mathbf{x}_B$. Hence (86) is a one-way version of the well-known principle that an electromagnetic or acoustic response remains the same when source and receiver are interchanged.

5.2. Correlation-Type Reciprocity Theorem for One-Way Wave Fields

We define a correlation-type interaction quantity, according to

$$\partial_3\{(P_A^+)^* P_B^+ - (P_A^-)^* P_B^-\}, \quad (87)$$

or, using a more compact notation,

$$\partial_3\{\mathbf{P}_A^\dagger \mathbf{J} \mathbf{P}_B\}. \quad (88)$$

We speak of "correlation type" since the products in the frequency domain ($(P_A^+)^* P_B^+$, etc.) correspond to correlations in the time domain. Applying the product rule for differentiation, substituting the one-way wave equation (30) for states A and B, integrating the result over domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$, and applying the theorem of Gauss, yields

$$\int_{\partial\mathcal{D}_3} \mathbf{P}_A^\dagger \mathbf{J} \mathbf{P}_B n_3 \, dx_1 = \int_{\mathcal{D}} \{\mathbf{P}_A^\dagger \mathbf{J} \hat{\mathbf{B}}_B \mathbf{P}_B + (\hat{\mathbf{B}}_A \mathbf{P}_A)^\dagger \mathbf{J} \mathbf{P}_B\} \, d^2\mathbf{x} + \int_{\mathcal{D}} \{\mathbf{P}_A^\dagger \mathbf{J} \mathbf{S}_B + \mathbf{S}_A^\dagger \mathbf{J} \mathbf{P}_B\} \, d^2\mathbf{x}. \quad (89)$$

On account of (47), (51), and (77) we obtain for the x_1 integral in the second term on the right-hand side of (89)

$$\langle \hat{\mathbf{B}}_A \mathbf{P}_A, \mathbf{J}\mathbf{P}_B \rangle_s = \langle \mathbf{P}_A, \hat{\mathbf{B}}_A^\dagger \mathbf{J} \mathbf{P}_B \rangle_s = -\langle \mathbf{P}_A, \mathbf{J} \hat{\mathbf{B}}'_A \mathbf{P}_B \rangle_s. \quad (90)$$

Using this result, we may rewrite (89) as

$$\int_{\partial\mathcal{D}_3} \mathbf{P}_A^\dagger \mathbf{J} \mathbf{P}_B n_3 \, dx_1 = \int_{\mathcal{D}} \mathbf{P}_A^\dagger \mathbf{J} (\hat{\mathbf{B}}_B - \hat{\mathbf{B}}'_A) \mathbf{P}_B \, d^2\mathbf{x} + \int_{\mathcal{D}} \{\mathbf{P}_A^\dagger \mathbf{J} \mathbf{S}_B + \mathbf{S}_A^\dagger \mathbf{J} \mathbf{P}_B\} \, d^2\mathbf{x}. \quad (91)$$

Equation (91) formulates a reciprocity theorem of the correlation type for one-way wave fields. It applies to lossless as well as dissipative and effectual media. Equation (91) provides a basis for reflection imaging based on inverse one-way wave field propagators [*Wapenaar*, 1996]. Note that the contrast term in the right-hand side of (91) vanishes when the medium parameters in one state are the time-reverse adjoint versions of the medium parameters in the other state. Upon substituting (27), (33), and (78) in the remaining terms we may rewrite (91) as

$$\int_{\partial\mathcal{D}_3} \{(P_A^+)^* P_B^+ - (P_A^-)^* P_B^-\} n_3 \, dx_1 = \int_{\mathcal{D}} \{(P_A^+)^* S_B^+ - (P_A^-)^* S_B^- + (S_A^+)^* P_B^+ - (S_A^-)^* P_B^-\} \, d^2\mathbf{x}. \quad (92)$$

For comparison, for the same situation the correlation-type reciprocity theorem for the total wave fields defined in Table 1 reads [*de Hoop*, 1987, 1988; *Fokkema and van den Berg*, 1993]

$$\int_{\partial\mathcal{D}_3} \{P_A^* Q_{3,B} + Q_{3,A}^* P_B\} n_3 \, dx_1 = \int_{\mathcal{D}} \{P_A^* B_B + Q_{i,A}^* C_{i,B} + C_{i,A}^* Q_{i,B} + B_A^* P_B\} \, d^2\mathbf{x}. \quad (93)$$

Equation (92) applies, for example, to the situation where state B corresponds to a physical wave field in an actual dissipative medium and state A corresponds to a computational wave field in an effectual medium that is the time-reverse adjoint of the actual dissipative medium. Of course, the computational state A should be treated with care because of the instabilities inherent to wave propagation in an effectual medium. In numerical implementations, stability may be obtained by suppressing a part of the eigenvalue spectrum of the square root operator for the effectual medium (Figure 2).

For the limiting case of a lossless medium, (92) is valid when the medium parameters in both states are identical and the square root operator in one state (for example, the physical state B) is defined via the limit Im $(\omega) \uparrow 0$ and that in the other state (for example, the computational state A) is defined via the limit Im $(\omega) \downarrow 0$; see Figure 3. Again, the latter choice will lead to instabilities in numerical implementations. A simple robust way of stabilizing the numerical implementation of (92) for lossless media is by defining the square root operator via the limit Im $(\omega) \uparrow 0$ in both states, which of course involves an approximation [*Wapenaar and Grimbergen*, 1996].

6. Conclusions

We have derived reciprocity theorems of the convolution type (equation (83)) and of the correlation type (equation (91)), both for electromagnetic or acoustic one-way wave fields in dissipative inhomogeneous media. The same results were derived before for lossless inhomogeneous acoustic media. Both theorems honor the natural separation between propagation and scattering in the one-way wave equations (see equations (30) and (34)). They are particularly suited for wave propagation problems in which there is a "preferred direction of propagation," like in electromagnetic or acoustic wave guides and in seismic exploration. Both theorems relate the one-way wave field vectors at the boundary $\partial\mathcal{D}_3$ of some domain $\mathcal{D}$ (Figure 1) to the one-way sources and the contrast between the one-way operators in both states in the domain $\mathcal{D}$. The contrast term in the reciprocity theorem of the convolution type vanishes when the medium parameters in both states are identical; the contrast term in the reciprocity theorem of the correlation type vanishes when the medium parameters in one state are the time-reverse adjoint versions of the medium parameters in the other state. The one-way reciprocity theorem of the convolution type provides a basis for representations of scattered one-way wave fields in terms of generalized Bremmer series expansions or in terms of generalized primaries. The one-way reciprocity theorem of the correlation type finds its application in reflection imaging based on inverse one-way wave field propagators.

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Erratum: see eq 38

$$\mathbf{S} = \begin{pmatrix} S^+ \\ S^- \end{pmatrix}, \tag{33}$$

where S^+ and S^- are the sources for the one-way wave fields P^+ and P^-, respectively. Using (22) and (23), we may write for $\hat{\mathbf{B}}$, as defined in (31),

$$\hat{\mathbf{B}} = \begin{pmatrix} -j\hat{\mathcal{H}}_1 & 0 \\ 0 & j\hat{\mathcal{H}}_1 \end{pmatrix} + \begin{pmatrix} \hat{T} & -\hat{R} \\ -\hat{R} & \hat{T} \end{pmatrix}, \tag{34}$$

where $\hat{R}$ and $\hat{T}$ are reflection and transmission operators, respectively, given by

$$\hat{R} = (\hat{L}_1^{-1}\partial_3\hat{L}_1 - \hat{L}_2^{-1}\partial_3\hat{L}_2)/2, \tag{35}$$

$$\hat{T} = -(\hat{L}_1^{-1}\partial_3\hat{L}_1 + \hat{L}_2^{-1}\partial_3\hat{L}_2)/2. \tag{36}$$

It appears that unlike the operator matrix $\hat{\mathbf{A}}$, the one-way operator matrix $\hat{\mathbf{B}}$ distinguishes explicitly between propagation (the diagonal matrix in (34)) and scattering (the full matrix in (34); note that this latter operator matrix vanishes in any region where the medium parameters do not vary in the axial direction).

On account of (26) we have

$$\hat{R}' = \hat{R}^* \quad \text{and} \quad \hat{T}' = \hat{T}^*. \tag{37}$$

Hence, given the one-way operator matrix $\hat{\mathbf{B}}$ in a specific medium (either dissipative or effectual), we obtain for the one-w[illegible] Missing prime: B' [illegible]e time-reverse adjoint medium, using (19) and (37),

$$\hat{\mathbf{B}} = \begin{pmatrix} -j\hat{\mathcal{H}}_1^* & 0 \\ 0 & j\hat{\mathcal{H}}_1^* \end{pmatrix} + \begin{pmatrix} \hat{T}^* & -\hat{R}^* \\ -\hat{R}^* & \hat{T}^* \end{pmatrix}. \tag{38}$$

Note that for the limiting case of a lossless medium we have $\hat{\mathbf{B}}' \neq \hat{\mathbf{B}}$; see the discussion below (19).

4. Symmetry Properties of the One-Way Operator Matrix

The one-way wave equation (30) that we derived in section 3.3 will be used in section 5 for the derivation of reciprocity theorems for one-way wave fields. For this purpose we first need to establish a number of symmetry properties of the one-way operator matrix $\hat{\mathbf{B}}$, defined in (31). This is the subject of the current section. First we briefly review the concept of transposed and adjoint operators. Next we derive symmetry properties of square root operators. Finally, we employ the results to derive the symmetry properties of the one-way operator matrix $\hat{\mathbf{B}}$.

4.1. Transposed and Adjoint Operators

For two scalar functions $f(x_1)$ and $g(x_1)$ we define the bilinear form as [*Rudin*, 1973; *Reddy*, 1986]

$$\langle f, g\rangle_b = \int_{x_{1,a}}^{x_{1,b}} f(x_1)g(x_1)\,dx_1 \tag{39}$$

and the sesquilinear form as

$$\langle f, g\rangle_s = \int_{x_{1,a}}^{x_{1,b}} f^*(x_1)g(x_1)\,dx_1. \tag{40}$$

The integration limits $x_{1,a}$ and $x_{1,b}$ may be either finite or infinite; in the latter case, $f(x_1)$ and $g(x_1)$ are assumed to lie in the appropriate Sobolev space. Consider a scalar operator $\hat{\mathcal{U}} = \hat{\mathcal{U}}(x_1, \partial_1)$. We introduce the transposed operator $\hat{\mathcal{U}}^t$ via

$$\langle \hat{\mathcal{U}}f, g\rangle_b = \langle f, \hat{\mathcal{U}}^t g\rangle_b \tag{41}$$

and the adjoint operator $\hat{\mathcal{U}}^\dagger$ via

$$\langle \hat{\mathcal{U}}f, g\rangle_s = \langle f, \hat{\mathcal{U}}^\dagger g\rangle_s. \tag{42}$$

Note that the following relation holds between the transposed and the adjoint operators

$$\hat{\mathcal{U}}^\dagger = (\hat{\mathcal{U}}^t)^*. \tag{43}$$

An operator is called symmetric when it obeys the relation

$$\hat{\mathcal{U}}^t = \hat{\mathcal{U}}. \tag{44}$$

On the other hand, it is self-adjoint when

$$\hat{\mathcal{U}}^\dagger = \hat{\mathcal{U}}. \tag{45}$$

For two vector functions $\mathbf{f}(x_1)$ and $\mathbf{g}(x_1)$ we define the bilinear form as

$$\langle \mathbf{f}, \mathbf{g}\rangle_b = \int_{x_{1,a}}^{x_{1,b}} \mathbf{f}^t(x_1)\mathbf{g}(x_1)\,dx_1 \tag{46}$$

and the sesquilinear form as

$$\langle \mathbf{f}, \mathbf{g}\rangle_s = \int_{x_{1,a}}^{x_{1,b}} \mathbf{f}^\dagger(x_1)\mathbf{g}(x_1)\,dx_1. \tag{47}$$

Note that we use the superscript t for transposed operators (equation (41)) as well as for transposed vectors or matrices (equation (46)). Similarly, the superscript dagger is used to denote adjoint operators (equation (42)) as well as adjoint (i.e., complex conjugate transposed) vectors or matrices (equation