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Field Fluctuations, Imaging with Backscattered Waves, a Generalized Energy Theorem, and the Optical Theorem*

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Abstract. We show the connection between four aspects of wave propagation: the autocorrelation of field fluctuations, imaging with backscattered waves, a theorem for energy flow, and the generalized optical theorem. The autocorrelation of field fluctuations can be used to extract the imaginary component of the Green's function at the source. The Green's function usually is singular at the source, but the imaginary component is not. The imaginary component of the Green's function at the source can thus be retrieved from the autocorrelation of field fluctuations, and can be used to image the medium using backscattered fields. We also show for general linear systems, which may be open or closed and may be dissipative, that the imaginary component of the Green's function at the source accounts for the loss of generalized energy by dissipation and/or propagation of the fields away from the source. Finally we show that the expressions for the extraction of the Green's function for scalar waves has the same mathematical structure as the generalized optical theorem. The theory presented here is shown to be applicable to damped acoustic waves, quantum mechanics, and diffusion.

Key words. Green's function extraction, interferometry, scattering, diffusion, imaging, optical theorem

AMS subject classifications. 70Sxx, 81Uxx, 86A15

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1. Introduction.

The extraction of the Green's function from ambient fluctuations within a diffuse field has received considerable attention for different research areas [1, 2, 3] that include ultrasound [4, 5, 6], ocean acoustics [7, 8], crustal seismology [9, 10], hazard monitoring [11, 12, 13], exploration seismology [14, 15, 16, 17], medical diagnostics [18], and engineering [19, 20, 21, 22]. It has been shown that the extraction of the Green's function can be carried out for a wide variety of linear systems [23, 24, 25]. The derivation of Green's function extraction has been derived using a representation in normal modes [26], surface waves [27], incident plane waves [28], and refracted waves [29]. Other derivations are based on time-reversal [30] and representation theorems [31]. For scalar systems that are invariant for time-reversal, theory relates the imaginary part of the Green's function to an integration over a closed surface ∂V

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that encloses two points $\mathbf{r}_A$ and $\mathbf{r}_B$ [24]:

$$G(\mathbf{r}_A,\mathbf{r}_B) - G^*(\mathbf{r}_B,\mathbf{r}_A) = \oint_{\partial V} L\left(G^*(\mathbf{r}_B,\mathbf{r}), G(\mathbf{r}_A,\mathbf{r})\right) dS, \tag{1}$$

where L is a bilinear operator. For example, for acoustic waves, $L\left(G^*(\mathbf{r}_B,\mathbf{r}), G(\mathbf{r}_A,\mathbf{r})\right) = \rho^{-1}\left(\partial G(\mathbf{r}_A,\mathbf{r})/\partial n\, G^*(\mathbf{r}_B,\mathbf{r}) - G(\mathbf{r}_A,\mathbf{r})\,\partial G^*(\mathbf{r}_B,\mathbf{r})/\partial n\right)$, with $\partial/\partial n$ the normal outward derivative to the surface and ρ the mass-density. A similar expression holds for systems that are not invariant for time-reversal, but for such systems a volume integral with the same functional form as the surface integral should be added to the right-hand side of this expression [24]. Equation (1) forms the basis for extraction of the Green's function from ambient fluctuations. When spatially uncorrelated noise with power spectrum $|S(\omega)|^2$ excites a field $u(\mathbf{r})$, then the Green's function follows from [2, 24, 32]

$$\omega^{-1}\mathrm{Im}\left(G(\mathbf{r}_A,\mathbf{r}_B)\right) = \frac{\langle u(\mathbf{r}_A)u^*(\mathbf{r}_B)\rangle}{|S(\omega)|^2}, \tag{2}$$

where $\langle\cdots\rangle$ denotes a source-average. In practice this source-average is replaced by an average over a set of nonoverlapping windows [2]. In order to extract the Green's function from field fluctuations, it is important that the noise sources provide an equipartitioned field [33, 5, 24], that is, a field where the energy current is independent of direction and location.

The right-hand side of expression (2) corresponds, in the frequency domain, to the cross-correlation of the field measured at locations $\mathbf{r}_A$ and $\mathbf{r}_B$, respectively. By setting $\mathbf{r}_A = \mathbf{r}_B$, the right-hand side reduces to the autocorrelation of the field fluctuations, while the left-hand side gives the imaginary component of the Green's function at the source. This quantity characterizes the return of fields to the source location. The backscattered fields thus obtained can be used for imaging, but there is also a connection with generalized energy principles.

The energy flow in a system depends critically on whether the system is open or closed and whether there is dissipation. The theory presented here covers all cases. For a closed system that conserves energy, the response can be expressed in the normal modes $\psi_n(\mathbf{r})$ and real angular frequencies ω_n. For example, in quantum mechanics the Green's function for such a system is given by [34]

$$G(\mathbf{r},\mathbf{r}') = \hbar^{-1}\sum_n \frac{\psi_n(\mathbf{r})\psi_n^*(\mathbf{r}')}{\omega-\omega_n}, \tag{3}$$

where $\hbar$ is Planck's constant divided by 2π. The eigenfunctions $\psi_n(\mathbf{r})$ are normalized so that $\int|\psi_n(\mathbf{r})|^2\,dV = 1$. The trace of the Green's function is given by [34]

$$\mathrm{tr}\,G = \int G(\mathbf{r},\mathbf{r})dV = \hbar^{-1}\sum_n \frac{1}{\omega-\omega_n} = P\frac{1}{E-E_n} - i\pi\delta(E-E_n), \tag{4}$$

where the energy $E=\hbar\omega$ and P denotes the principal value. From this relation, $-\pi^{-1}\mathrm{Im}\,(\mathrm{tr}\,G) = \delta(E-E_n)$, which establishes the relation between the imaginary component of the trace of the Green's function and spectral properties of the system such as the level spacing. (This quantity denotes the difference in energy of adjacent eigenstates.) The spectral properties of a system with normal modes have also been related to the Green's function $G(\mathbf{k})$ in the wave number ($\mathbf{k}$) domain for $\mathbf{k} = 0$ [35]. Since $G(\mathbf{k} = 0) = \int G(\mathbf{r}, \mathbf{r})dV = \operatorname{tr} G$, this connection also involves the trace of the Green's function. Note that the connection between $\operatorname{Im}(\operatorname{tr} G)$ and the density of states is relevant only for closed systems without dissipation, because only in that case are the eigenfrequencies real.

The relationship between the imaginary part of the Green's function at the source and radiated energy has been shown earlier for the special case of elastic waves in a homogeneous space [36, 37]. The presence of a free surface and of scatterers has been explored for both scalar and elastic systems [37]. In this work we generalize this analysis by studying a general scalar linear system that may be closed and conservative, but that may also be open and/or dissipative. In the latter case, energy is lost. We show in section 2 that the imaginary part of the Green's function at the source ($\operatorname{Im} G(\mathbf{r}, \mathbf{r})$) is proportional to a quadratic measure of the fields leaving the source. Note that we do not integrate the Green's function over the volume to take the trace; hence the obtained relation holds locally for every point $\mathbf{r}$. For acoustic waves (section 3) this quadratic function is the energy leaving the source, for quantum mechanics (section 5) it is the probability density, while for diffusion (section 6) it is a quadratic form whose physical interpretation depends on the character of the fields involved. In all cases we refer to this quadratic form as *generalized energy* and refer to the loss of this quantity as *dissipation*. We show in section 4 how the Green's function can be retrieved from field fluctuations in systems that are open and/or dissipative.

It is no surprise that for dissipative systems the imaginary part of the Green's function at the source describes the energy loss. Consider, as an example, the damped oscillator: $m\ddot{x}+\gamma\dot{x}+sx = f$, with m, γ, and s the mass, damping, and stiffness parameters, respectively. Using the Fourier convention $f(t) = (2\pi)^{-1}\int f(\omega)\exp(-i\omega t)d\omega$, the Green's function satisfies the relation $G = 1/\left(s - i\gamma\omega - m\omega^2\right)$ in the frequency domain, which implies that

$$\operatorname{Im}(G) = \gamma\omega|G|^2. \tag{5}$$

The imaginary part of the Green's function is thus proportional to the rate of dissipation by this one-degree-of-freedom system. Next we derive a general expression that relates the imaginary part of the Green's function at the source to both the radiated and dissipated power lost by a unit harmonic source.

2. The imaginary part of Green's function at the source.

Consider the following scalar partial differential equation for a field u that is of Nth order in the time derivatives:

$$\left(a_N(\mathbf{r},t) * \frac{\partial^N}{\partial t^N} + \cdots + a_2(\mathbf{r},t) * \frac{\partial^2}{\partial t^2} + a_1(\mathbf{r},t) * \frac{\partial}{\partial t}\right) u(\mathbf{r},t) = H(\mathbf{r},t) * u(\mathbf{r},t) + q(\mathbf{r},t). \tag{6}$$

The asterisk denotes convolution in time, while the operator H contains the space derivatives of the field equation. The field is excited by the forcing $q(\mathbf{r}, t)$. The Green's function $G(\mathbf{r}, \mathbf{r}_0, t)$ is defined as the causal response of this system to a point source in space and time; i.e., it is the response to the excitation $q(\mathbf{r}, t) = \delta(\mathbf{r} - \mathbf{r}_0)\delta(t)$ that is nonzero only for positive time. Using the above Fourier convention, (6) corresponds, in the frequency domain, to

$$\sum_{n=1}^{N} a_n(\mathbf{r},\omega)\,(-i\omega)^n\, u(\mathbf{r},\omega) = H(\mathbf{r},\omega)u(\mathbf{r},\omega) + q(\mathbf{r},\omega). \tag{7}$$

Henceforth the derivation is in the frequency domain, and we suppress the frequency dependence. The Green's function $G(\mathbf{r}, \mathbf{r}_0)$ is, in the frequency domain, given by the response to the excitation $q(\mathbf{r}, \omega) = \delta(\mathbf{r} - \mathbf{r}_0)$. Because we consider the Green's function to be causal in the time domain, its real and imaginary parts are related by the Kramers–Kronig relation [38, 39]. The operator H is not necessarily self-adjoint over the volume V under consideration, and, following [24], we define the bilinear form L by

$$\int_V \left(f\left(Hg\right) - \left(Hf\right)g\right) dV \equiv \oint_{\partial V} L(f,g) dS, \tag{8}$$

where the surface integral is over the surface ∂V that bounds the volume V under scrutiny. This volume can be the total volume through which the field propagates, but may also be an arbitrary subvolume. Now assume that the $q(\mathbf{r})$ are spatial delta functions $\delta(\mathbf{r} - \mathbf{r}_{A,B})$ at locations $\mathbf{r}_A$ and $\mathbf{r}_B$. The fields u_A and u_B are then the frequency domain Green's functions $G(\mathbf{r}, \mathbf{r}_{A,B})$. Using reciprocity $(G(\mathbf{r}_A, \mathbf{r}_B) = G(\mathbf{r}_B, \mathbf{r}_A))$ and setting $\mathbf{r}_A = \mathbf{r}_B = \mathbf{r}_0$, it follows from expression (16) of [24] that for the general system (6)

$$\begin{aligned} \mathrm{Im}\left(G(\mathbf{r}_0, \mathbf{r}_0)\right) = &- \sum_{n\ odd} (-1)^{(n+1)/2} \omega^n \int_V \mathrm{Re}\left(a_n\right) |G(\mathbf{r}_0, \mathbf{r})|^2 dV \\ &- \sum_{n\ even} (-1)^{n/2} \omega^n \int_V \mathrm{Im}\left(a_n\right) |G(\mathbf{r}_0, \mathbf{r})|^2 dV \\ &- \frac{i}{2} \oint_{\partial V} L\left(G^*(\mathbf{r}_0, \mathbf{r}), G(\mathbf{r}_0, \mathbf{r})\right) dS \\ &+ \int_V G(\mathbf{r}_0, \mathbf{r})\, \mathrm{Im}\left(H\right) G^*(\mathbf{r}_0, \mathbf{r}) dV, \end{aligned} \tag{9}$$

where Re denotes the real part. In order to see the meaning of this general expression we next present three examples.

3. Damped acoustic waves. These waves satisfy the following partial differential equation:

$$\kappa * \frac{\partial^2 p}{\partial t^2} = \nabla \cdot \left(\frac{1}{\rho} \nabla p\right) + q, \tag{10}$$

where p is pressure, ρ mass density, and κ compressibility. For attenuating media, κ is a time-dependent relaxation function in the time domain and is, in the frequency domain, complex-valued and frequency-dependent. In the notation of the general equation (6), $a_2 = \kappa$, and $H = \nabla \cdot (\rho^{-1} \nabla)$; all other a_n vanish. Using Green's theorem, the bilinear form L for this example is given by $L(f,g) = \rho^{-1}\left(f(\partial g/\partial n) - (\partial f/\partial n) g\right)$. Using Newton's law for the particle velocity ($i\omega\rho\mathbf{v} = \nabla p$), one obtains $-(i/2) \oint L(p^*, p) dS = (\omega/2) \oint (p^* \mathbf{v} + \mathbf{v}^* p) \cdot d\mathbf{S}$. For an acoustic medium the power flux is given by [40]

$$\mathbf{J}_P = \frac{1}{2}(p^* \mathbf{v} + \mathbf{v}^* p), \tag{11}$$

hence the general equation (9) reduces to

$$\omega^{-1} \mathrm{Im}\left(G(\mathbf{r}_0, \mathbf{r}_0)\right) = \omega \int_V \mathrm{Im}\left(\kappa\right) |G(\mathbf{r}_0, \mathbf{r})|^2 dV + \oint_{\partial V} \mathbf{J}_P(\mathbf{r}, \mathbf{r}_0) \cdot d\mathbf{S}, \tag{12}$$

where $\mathbf{J}_P(\mathbf{r},\mathbf{r}_0)$ is the power flux at location $\mathbf{r}$ generated by a point source at $\mathbf{r}_0$. The first term in the right-hand side has the same form as expression (5) for the energy loss of the damped harmonic oscillator and gives the energy dissipated within the volume V, while the last term accounts for the energy flux through the boundary ∂V. The imaginary part of the Green's function at the source thus accounts for the two ways in which energy injected into the medium by the source leaves the volume V.

Note that the volume V need not be the volume of all space; it can be an arbitrary subvolume, and its boundary ∂V need not be a physical boundary. When V is a volume with radius much less than the extinction length $l_{ext} = \left[\mathrm{Im}\left(\sqrt{\rho\kappa}\right)\omega\right]^{-1}$, most of the energy is lost through radiation through the boundary ∂V, and the second term in the right-hand side dominates. In contrast, where the volume defines a region much larger than the extinction length, the first term in the right-hand side of expression (12) dominates.

4. Green's function retrieval from ambient noise. In this section we show explicitly for damped acoustic waves how the Green's function can be extracted from random field fluctuations. Expression (12) can also be written as

(13)
$$\mathrm{Im}\left(G(\mathbf{r}_0,\mathbf{r}_0)\right) = \omega^2 \int_V \mathrm{Im}\left(\kappa\right)|G(\mathbf{r}_0,\mathbf{r})|^2 dV - \frac{i}{2}\oint_{\partial V}\frac{1}{\rho}\left(G^*(\mathbf{r}_0,\mathbf{r})\frac{G(\mathbf{r}_0,\mathbf{r})}{\partial n} - \frac{G^*(\mathbf{r}_0,\mathbf{r})}{\partial n}G(\mathbf{r}_0,\mathbf{r})\right)dS.$$

We assume that the boundary ∂V is a large sphere and that there are only outgoing waves on this surface. In that case $\partial G(\mathbf{r}_0,\mathbf{r})/\partial n = ikG(\mathbf{r}_0,\mathbf{r})$, with the wave number $k = \omega/c$, and expression (13) is given by

$$\mathrm{Im}\left(G(\mathbf{r}_0,\mathbf{r}_0)\right) = \omega^2 \int_V \mathrm{Im}\left(\kappa\right)|G(\mathbf{r}_0,\mathbf{r})|^2 dV + \omega\oint_{\partial V}\frac{1}{\rho c}|G(\mathbf{r}_0,\mathbf{r})|^2 dS. \tag{14}$$

We consider the special case that volume sources Q in V and surface sources q on ∂V are present and that these sources are uncorrelated and satisfy

$$\langle q(\mathbf{r}_1)Q(\mathbf{r}_2)\rangle = 0, \tag{15}$$

$$\langle Q(\mathbf{r}_1)Q(\mathbf{r}_2)\rangle = \omega^2\,\mathrm{Im}\left(\kappa\left(\mathbf{r}_1\right)\right)\delta(\mathbf{r}_1-\mathbf{r}_2)|S(\omega)|^2, \tag{16}$$

$$\langle q(\mathbf{r}_1)q(\mathbf{r}_2)\rangle = \frac{\omega}{\rho c(\mathbf{r}_1)}\delta(\mathbf{r}_1-\mathbf{r}_2)|S(\omega)|^2, \tag{17}$$

where $|S(\omega)|^2$ is the power spectrum of the excitation and $\langle\cdots\rangle$ denotes an ensemble average. In practice this average is replaced by averaging over nonoverlapping times windows [2].

The volume integral of expression (14), multiplied with $|S(\omega)|^2$, can be written as

$$\begin{aligned}
&\omega^2|S(\omega)|^2\int_V \mathrm{Im}\left(\kappa\right)|G(\mathbf{r}_0,\mathbf{r})|^2 dV \\
&\quad = \iint \omega^2|S(\omega)|^2\,\mathrm{Im}\left(\kappa(\mathbf{r}_1)\right)\delta(\mathbf{r}_1-\mathbf{r}_2)G(\mathbf{r}_0,\mathbf{r}_1)G^*(\mathbf{r}_0,\mathbf{r}_2)dV_1 dV_2 \\
&\quad = \left\langle \int G(\mathbf{r}_0,\mathbf{r}_1)Q(\mathbf{r}_1)dV_1 \int G^*(\mathbf{r}_0,\mathbf{r}_2)Q^*(\mathbf{r}_2)dV_2\right\rangle,
\end{aligned} \tag{18}$$

where expression (16) has been used in the last identity. A similar analysis can be applied to the surface term, and expression (14) can be written as

$$
\begin{aligned}
|S(\omega)|^2 \,\mathrm{Im}\,(G(\mathbf{r}_0,\mathbf{r}_0)) \quad &= \left\langle \int G(\mathbf{r}_0,\mathbf{r}_1)Q(\mathbf{r}_1)dV_1 \int G^*(\mathbf{r}_0,\mathbf{r}_2)Q^*(\mathbf{r}_2)dV_2 \right\rangle \\
&\quad + \left\langle \oint G(\mathbf{r}_0,\mathbf{r}_1)q(\mathbf{r}_1)dS_1 \oint G^*(\mathbf{r}_0,\mathbf{r}_2)q^*(\mathbf{r}_2)dS_2 \right\rangle .
\end{aligned}
\tag{19}
$$

It follows from Green's theorem and the employed boundary condition on ∂V that the field in V is related to the sources by

$$
u(\mathbf{r}) = \int G(\mathbf{r},\mathbf{r}')Q(\mathbf{r}')dV' + \oint G(\mathbf{r},\mathbf{r}')q(\mathbf{r}')dS'. \tag{20}
$$

Using that the sources in the volume and on the surface are uncorrelated (expression (15)), it follows from (19) and (20) that

$$
\mathrm{Im}\,(G(\mathbf{r}_0,\mathbf{r}_0)) = \frac{1}{|S(\omega)|^2}\langle |u(\mathbf{r}_0,\mathbf{r}_0)|^2\rangle. \tag{21}
$$

This means that the Green's function $\mathrm{Im}\,(G(\mathbf{r}_0,\mathbf{r}_0))$ follows from the autocorrelation of the field fluctuations. The reasoning used here is equally applicable for the Green's function $G(\mathbf{r}_A,\mathbf{r}_B)$ with general arguments. Note that in order to extract the Green's function from field fluctuations one must, in general, have surface sources and volume sources that are balanced and uncorrelated in such a way that expressions (15)–(17) are satisfied. In the case of strong attenuation the volume integral dominates, while for weakly attenuating systems the surface integral dominates. In all cases the volume sources in expression (16) compensate for the energy loss due to attenuation, while the surface sources of (17) compensate for the radiation loss through the bounding surface. These sources act together to create a state of equipartitioning, i.e., a state of field fluctuations where the energy current is independent of direction and location.

In practical situations, the volume sources and surface sources may not be balanced in such a way that expressions (16) and (17) are satisfied. This is the case, for example, in the Green's function extraction in crustal seismology [9]. Earth is attenuating, and the field fluctuations are excited mainly by ocean waves near Earth's surface [41]. Such incompleteness in the source distribution leads to an extracted Green's function that displays spurious arrivals [42, 43] or that lacks waves that are present in the Green's function. Indeed, the Green's functions extracted in crustal seismology often are deficient in the body wave amplitude [9, 44].

5. Quantum mechanics. For this application, we consider the simplest case of a field governed by Schrödinger's equation:

$$
i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi. \tag{22}
$$

In the notation of (6), $H = -(\hbar^2/2m)\nabla^2 + V$, $a_1 = i\hbar$, and all other a_n are equal to zero. Using Green's theorem it follows that $L(\psi^*,\psi) = -(\hbar^2/2m)\,(\psi^*(\partial\psi/\partial n) - (\partial\psi^*/\partial n)\psi)$. This quantity is related to the probability density current $\mathbf{J}_Q = (\hbar/2mi)(\psi^*\nabla\psi - \psi\nabla\psi^*)$ [45]. Inserting these results into expression (9) gives

$$-\frac{2}{\hbar}\mathrm{Im}\,(G(\mathbf{r}_0,\mathbf{r}_0)) = \oint_{\partial V} \mathbf{J}_Q(\mathbf{r},\mathbf{r}_0)\cdot d\mathbf{S}, \tag{23}$$

where $\mathbf{J}_Q$ is the probability density current associated with the Green's function $G(\mathbf{r},\mathbf{r}_0)$. The right-hand side gives the probability that the particle leaves the volume V. Here, the imaginary part of the Green's function is not equal to the energy loss, but instead corresponds to a loss in probability flowing through the surface ∂V.

When waves with power spectrum $|S(\omega)|^2$ excite the system, the imaginary component of the Green's function follows from the autocorrelation of the field fluctuations [24]:

$$-\frac{2}{\hbar^2}\mathrm{Im}\,(G(\mathbf{r}_0,\mathbf{r}_0)) = \frac{k}{m|S(\omega)|^2}\left\langle|\psi(\mathbf{r}_0,\omega)|^2\right\rangle, \tag{24}$$

where $k=\sqrt{2m\omega/\hbar}$ is the wave number. The combination of expressions (23) and (24) shows that the probability that particles leave the volume V is related to the fluctuations in the probability density $\left\langle|\psi(\mathbf{r}_0,\omega)|^2\right\rangle$. Note that this quantity denotes in quantum mechanics the intensity, which is an observable quantity, whereas the phase of the wave function $\psi(\mathbf{r}_0,\omega)$ cannot be measured. This is one of the reasons why we study the autocorrelation of the wave function in this work. It can be measured and, as shown here, is related to the probability of particles leaving the volume and to backscattered waves that can be used for imaging.

6. Diffusion equation.

Diffusion problems satisfy the following equation:

$$\frac{\partial u(\mathbf{r},t)}{\partial t} = \nabla\cdot(D(\mathbf{r})\nabla u(\mathbf{r},t)) + q(\mathbf{r},t), \tag{25}$$

where the diffusion parameter $D(\mathbf{r})$ may vary in space. Diffusion problems occur in heat transport, fluid transport in porous media, in low-frequency electromagnetic waves in conductive media, and in the propagation of waves in strongly scattering media [46]. The general equation (6) has for the case of the diffusion equation the following parameters: $a_1 = 1$, all other a_n vanish, and $H = \nabla\cdot D\nabla$. Using Green's theorem, the bilinear form L is for this example given by $L(f,g) = D\,(f\partial g/\partial n - \partial f/\partial n g)$. In diffusion problems, the current associated with a field u is given by $\mathbf{J}_D = -D\nabla u$. Inserting these results in the general expression (9) gives

$$\mathrm{Im}\,(G(\mathbf{r}_0,\mathbf{r}_0)) = \omega\int_V |G(\mathbf{r}_0,\mathbf{r})|^2\,dV + \oint_{\partial V}\mathrm{Im}\,(G(\mathbf{r}_0,\mathbf{r})\mathbf{J}_D^*(\mathbf{r}_0,\mathbf{r}))\cdot d\mathbf{S}, \tag{26}$$

where $\mathbf{J}_D(\mathbf{r}_0,\mathbf{r})$ is the current associated with the Green's function $G(\mathbf{r}_0,\mathbf{r})$.

Just as in expression (12) the first term accounts for the loss of the field in the volume V, while the second term accounts for the flow through the boundary ∂V. One should be cautious, though, in interpreting expression (26) as an energy equation. The field u could be a variety of different fields such as temperature in a heat conduction problem, the concentration of a chemical in a diffusive dispersion problem, or the pore pressure in the case of flow in porous media. In the first case, the thermal energy is proportional to the temperature rather than to the quadratic combination of the fields in expression (26) that do not account for energy. This shows that the imaginary component of the Green's function at the source is quadratic in the field variables, but it is not necessarily the energy; for this reason we use the phrase *generalized energy* instead.

7. Connection to imaging and a simple acoustic model. According to expression (2), the imaginary part of the Green's function can, in general, be extracted from field fluctuations using the relation

$$\omega^{-1}\mathrm{Im}\left(G(\mathbf{r}_0,\mathbf{r}_0)\right)=\frac{\left\langle |u(\mathbf{r}_0)|^2\right\rangle}{|S(\omega)|^2}. \tag{27}$$

The right-hand side of this expression gives the power spectrum of the field fluctuations. This quantity is the Fourier transform of the autocorrelation of field fluctuations measured in the time domain and either can be computed from recorded field fluctuations or, in the case of quantum mechanics, follows from recorded intensity fluctuations. The left-hand side is proportional to $G(\mathbf{r}_0,\mathbf{r}_0)-G^*(\mathbf{r}_0,\mathbf{r}_0)$ and gives the superposition of the causal and acausal backscattered waves. These backscattered waves can be used for imaging purposes.

As a prototype of potential imaging applications, consider the problem of acoustic waves propagating in three dimensions within a layer of constant thickness h. At the upper and lower edges, the pressure gradient is assumed to vanish ($\partial p/\partial n=0$); hence the reflection coefficient is equal to $+1$ at the boundaries. It follows from the method of images that the wave field at the source location for a source placed at the top of the layer is given by

$$G(0,0)=\frac{2\rho}{4\pi}\left(\lim_{r\to 0}\frac{e^{ikr}}{r}+2\frac{e^{2ikh}}{2h}+2\frac{e^{4ikh}}{4h}+\cdots\right), \tag{28}$$

with $k=\omega/c$ the wave number. The overall factor of 2 results because the source radiates only into the plate rather than in all directions, while the factor 2 for the backscattered waves arises from the interaction of the waves with the surface (where the receiver is assumed to be located). The first term in the right-hand side gives the direct wave at the source in infinite space, corrected with a factor 2 that accounts for the boundary condition. Since this wave is singular, it is written in the form of a limit. The subsequent terms in expression (28) come from the successive waves that bounce back and forth within the layer. The first term is twice the Green's function in a homogeneous space $G_0(r)=\rho\exp(ikr)/4\pi r$. While at the source ($r=0$) this Green's function is singular, the imaginary part of this Green's function, given by $\mathrm{Im}\left(G_0(r)\right)=\rho\sin(kr)/4\pi r$, tends to the finite value $\rho k/4\pi$ as $r\to 0$ [47].

For the layer model, the imaginary component of the Green's function is given by

$$\mathrm{Im}\left(G(0,0)\right)=\frac{2\rho k}{4\pi}\left(1+2\frac{\sin(2kh)}{2kh}+2\frac{\sin(4kh)}{4kh}+\cdots\right). \tag{29}$$

This function, normalized to the corresponding value $\mathrm{Im}\left(G_0(0,0)\right)=\rho k/4\pi$ for the free-space Green's function, is shown in Figure 1 as a function of normalized frequency $f\tau$, where $\tau=2h/c$ is the two-way vertical travel time in the layer. The peaks in the response curve correspond to the resonances of the layer at frequencies $f_n=n/\tau=nc/2h$, with n an integer. Since

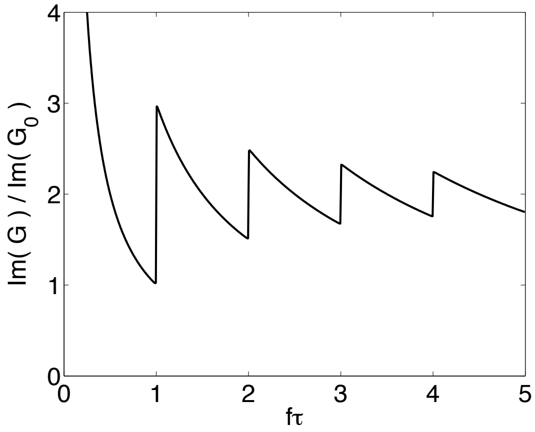

Figure 1. *The imaginary part of the Green's function of the acoustic layer at the source, normalized with the corresponding value for the free-space Green's function, as a function of normalized frequency $f\tau$, where $\tau = 2h/c$.*

energy is radiated sideways in the layer, the peaks have nonzero width. These eigenfrequencies can be used to measure the travel time h/c in the layer. Alternatively, the time-domain version of expression (28) corresponds to a series of waves arriving at time $t = \pm 2nh/c$ that provide the travel time in the layer. An elastic version of this concept can potentially be used to determine the resonances in the low-velocity layer in the shallow subsurface from the autocorrelation of recorded noise.

8. Connection with the generalized optical theorem.

When the surface ∂V in (1) is a large sphere with constant impedance ρc and acoustic waves propagate out off that sphere, (1) for acoustic waves can be written as [24]

$$\frac{1}{2i}\left(G(\mathbf{r}_A,\mathbf{r}_B) - G^*(\mathbf{r}_B,\mathbf{r}_A)\right) = \frac{\omega}{\rho c}\oint_{\partial V} G(\mathbf{r},\mathbf{r}_A)G^*(\mathbf{r},\mathbf{r}_B)dS. \tag{30}$$

The Green's function extraction for the Schrödinger equation is similar [24]. In scattering problems the Green's function can be written as the sum of the unperturbed field G_0 and the scattered waves G_S:

$$G = G_0 + G_S. \tag{31}$$

There is an advantage to considering the Green's function extraction for the scattered waves. To see this we write expression (30) in shorthand notation as

$$\mathrm{Im}\,(G) = \oint GG^*. \tag{32}$$

The same expression holds for the unperturbed field

$$\mathrm{Im}\,(G_0) = \oint G_0 G_0^*. \tag{33}$$

Subtracting expression (33) from (32) and using the decomposition (31) gives

$$\mathrm{Im}\,(G_S) = \oint G_0 G_S^* + \oint G_S G_0^* + \oint G_S G_S^*. \tag{34}$$

Note that this equation differs from expressions (32) and (33). The reason for this difference is that G_S alone does not satisfy a field equation of the same form as the ones satisfied by G_0 or G. In fact, the Green's function extraction for the wave field perturbation differs from that of the full Green's function [48].

The significance of using expression (34) rather than the original equation (30) is that the scattered field G_S usually is much smaller than G_0. Unraveling G_S as a small perturbation to G_0 can be an inaccurate process, while expression (34) directly yields G_S. Note that because of the employed subtraction the term $\oint G_0 G_0^*$ is absent from the right-hand side of (34). In practice, one often uses the scattering amplitude A_k rather than G_S; hence we consider the scattering amplitude in the following. There exists a connection between the equivalent of expression (34) for the scattering amplitude and the generalized optical theorem. This theorem, which concerns the scattering amplitude $A_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}')$ of scattered waves with wave number k and $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ unit vectors representing the directions of the outgoing and incoming waves, respectively, states that

$$\frac{1}{2i}\left(A_k(\hat{\mathbf{n}}_A, \hat{\mathbf{n}}_B) - A_k^*(\hat{\mathbf{n}}_B, \hat{\mathbf{n}}_A)\right) = \frac{k}{4\pi} \oint A_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}_A) A_k^*(\hat{\mathbf{n}}, \hat{\mathbf{n}}_B) d^2\hat{\mathbf{n}}. \tag{35}$$

This theorem has been derived in quantum mechanics [49, 50, 51] and in acoustics [52].

Note that the generalized optical theorem (35) and equation (30) for the extraction of the Green's function in acoustics or quantum mechanics have the same functional form. Equation (30) holds, however, in the space domain, while the generalized optical theorem (35) is formulated in the wave number domain. The optical theorem follows from expression (35) by setting $\hat{\mathbf{n}}_A = \hat{\mathbf{n}}_B = \hat{\mathbf{n}}_0$ [53, 54]:

$$\sigma_e = \oint |A_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}_0)|^2 d^2\hat{\mathbf{n}} \; = \frac{4\pi}{k} \mathrm{Im}\,(A_k(\hat{\mathbf{n}}_0, \hat{\mathbf{n}}_0))\,, \tag{36}$$

where σ_e is the scattering cross-section and $A_k(\hat{\mathbf{n}}_0, \hat{\mathbf{n}}_0)$ is the scattering amplitude in the forward direction. The scattering cross-section measures the efficacy of the object in scattering waves [51]. It has the physical dimension of area and can be interpreted as a measure of the size of the scatterer, as seen by the waves. The optical theorem, which relates the radiation loss by scattering to the properties of the forward-scattered wave, is equivalent to the general expression (9) that relates the imaginary component of the Green's function at the source to the loss of generalized energy.

We show in Appendix A that for uncorrelated noise sources with power spectrum $|S(\omega)|^2$ on a large sphere surrounding the scattering object [31], (35) can be written, analogously to (2), as

$$\frac{1}{2i}\left(A_k(\hat{\mathbf{n}}_A, \hat{\mathbf{n}}_B) - A_k^*(\hat{\mathbf{n}}_B, \hat{\mathbf{n}}_A)\right) = \frac{k}{4\pi} \frac{\langle \Psi^s(-\hat{\mathbf{n}}_A) \Psi^{s*}(-\hat{\mathbf{n}}_B) \rangle}{|S(\omega)|^2}, \tag{37}$$

where $\Psi^s(\hat{\mathbf{n}})$ is the scattered field in the direction $\hat{\mathbf{n}}$ excited by the noise. By setting $\hat{\mathbf{n}}_A = \hat{\mathbf{n}}_B = \hat{\mathbf{n}}_0$, the scattering cross-section of (36) is given by

$$\sigma_e = \frac{4\pi}{k}\mathrm{Im}\,(A_k(\hat{\mathbf{n}}_0, \hat{\mathbf{n}}_0)) = \frac{\langle|\Psi^s(-\hat{\mathbf{n}}_0)|^2\rangle}{|S(\omega)|^2}. \tag{38}$$

The scattering amplitude and scattering cross-section thus can be inferred from the autocorrelation of field fluctuations. Note the resemblance between expressions (24) and (38).

9. Discussion. We have shown the formal relation between the autocorrelation of field fluctuations, the imaginary part of the Green's function at the source, the flow and dissipation of generalized energy, and the generalized optical theorem. The generalized optical theorem and the equation for the extraction of the Green's function have the same functional form. Both equations express the same physical principle but in different spaces. The generalized optical theorem is given in wave number space while the imaginary part of the Green's function is expressed in position space. There is an even deeper connection with scattering theory, because these expressions are equivalent to relations used in the Marchenko equation for inverse scattering (see, e.g., [55]).

The optical theorem and the general expression for the imaginary part of the Green's function at the source account for generalized energy loss at the source. We use the term *generalized energy*, because the imaginary component of the Green's function is related to integrals that are quadratic in the field considered; these integrals represent energy for acoustic waves and probability in quantum mechanics.

In general, one cannot measure the Green's function at the source because it is singular. The imaginary part of the Green's function, however, is regular. As shown in expression (27), this quantity can be inferred from the autocorrelation of measured field fluctuations. The autocorrelation of observed field fluctuations has been used for monitoring a volcano and a fault zone during an earthquake [11] and to study the coherent backscattering effect for ultrasound [56]. The example of the acoustic layer in section 7 illustrates that the formal relationship, presented here, between the autocorrelation of the field fluctuations and the imaginary part of the Green's function can be used to infer the backscattered waves from field fluctuations recorded by a single sensor. These backscattered waves can be used to image the medium.

Appendix A. Derivation of expression (37). Consider a scatter potential $V(\mathbf{r})$ with compact support around the origin. Assuming an incident plane wave with unit amplitude, the asymptotic expression for the wave field is expressed as

$$\psi(\mathbf{k}, \mathbf{r}) = \psi^i(\mathbf{k}, \mathbf{r}) + \psi^s(\mathbf{k}, \mathbf{r}), \tag{39}$$

where the incident field $\psi^i(\mathbf{k}, \mathbf{r})$ is given by

$$\psi^i(\mathbf{k}, \mathbf{r}) = \exp(i\mathbf{k}\cdot\mathbf{r}), \tag{40}$$

and the scattered field $\psi^s(\mathbf{k}, \mathbf{r})$ by [57]

$$\psi^s(\mathbf{k}, \mathbf{r}) = A_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}_k)\exp(ikr)/r, \tag{41}$$

with $r = |\mathbf{r}| \to \infty$, $\hat{\mathbf{n}}_k = \mathbf{k}/k$, and $\hat{\mathbf{n}} = \mathbf{r}/r$. Equation (41) accounts for primary and multiple scattering in the finite domain where $V(\mathbf{r}) \neq 0$.

Using reciprocity, i.e., $A_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}_k) = A_k(-\hat{\mathbf{n}}_k, -\hat{\mathbf{n}})$, the scattered field $\psi^s(\mathbf{k}, \mathbf{r})$ can also be interpreted as the field from a source at $\mathbf{r}$, observed in the direction $-\hat{\mathbf{n}}_k$. Next we consider a distribution of noise sources on a spherical surface ∂V. We define the frequency spectrum of the noise signal at $\mathbf{r}$ on ∂V as $N(\mathbf{r}, \omega)$. When all noise sources act simultaneously, we may write for the observed scattered fields in the directions $-\hat{\mathbf{n}}_A$ and $-\hat{\mathbf{n}}_B$, respectively,

$$\Psi^s(-\hat{\mathbf{n}}_A) = \oint_{\partial V} \psi^s(\mathbf{k}_A, \mathbf{r}) N(\mathbf{r}\ , \omega) d^2\mathbf{r}, \tag{42}$$

$$\Psi^s(-\hat{\mathbf{n}}_B) = \oint_{\partial V} \psi^s(\mathbf{k}_B, \mathbf{r}') N(\mathbf{r}'\ , \omega) d^2\mathbf{r}'. \tag{43}$$

We assume that two noise sources $N(\mathbf{r}, \omega)$ and $N(\mathbf{r}', \omega)$ are mutually uncorrelated for any $\mathbf{r} \neq \mathbf{r}'$ at ∂V and that their power spectrum is the same for all $\mathbf{r}$. Hence, we assume that these noise sources obey the relation

$$\langle N(\mathbf{r}, \omega) N^*(\mathbf{r}', \omega)\rangle = \delta(\mathbf{r} - \mathbf{r}')|S(\omega)|^2, \tag{44}$$

where $\langle\cdot\rangle$ denotes a spatial ensemble average and $|S(\omega)|^2$ denotes the power spectrum of the noise. The cross-correlation of $\Psi^s(-\hat{\mathbf{n}}_A)$ and $\Psi^s(-\hat{\mathbf{n}}_B)$ gives, using equations (42)–(44),

$$\langle \Psi^s(-\hat{\mathbf{n}}_A) \Psi^{s*}(-\hat{\mathbf{n}}_B)\rangle = \oint_{\partial V} \psi^s(\mathbf{k}_A, \mathbf{r}) \psi^{s*}(\mathbf{k}_B, \mathbf{r}) |S(\omega)|^2 d^2\mathbf{r} \tag{45}$$

or, using (41),

$$\langle \Psi^s(-\hat{\mathbf{n}}_A) \Psi^{s*}(-\hat{\mathbf{n}}_B)\rangle = \oint_{\partial V} A_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}_A) A_k^*(\hat{\mathbf{n}}, \hat{\mathbf{n}}_B) |S(\omega)|^2 \frac{d^2\mathbf{r}}{r^2} \tag{46}$$

or, using $\hat{\mathbf{n}} = \mathbf{r}/r$,

$$\frac{\langle \Psi^s(-\hat{\mathbf{n}}_A) \Psi^{s*}(-\hat{\mathbf{n}}_B)\rangle}{|S(\omega)|^2} = \oint A_k(\hat{\mathbf{n}}, \hat{\mathbf{n}}_A) A_k^*(\hat{\mathbf{n}}, \hat{\mathbf{n}}_B) d^2\hat{\mathbf{n}}. \tag{47}$$

Comparing this with (35) gives expression (37).

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