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Reciprocity Theorems for Electromagnetic One-Way Wave Fields in 2-D Inhomogeneous Media

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One-way wave equations play a fundamental role in the formulation of imaging schemes for reflection measurements, as in seismic exploration and ground-penetrating radar. In this paper we review the theory for electromagnetic one-way wave fields, including dissipation, and we derive reciprocity theorems for these one-way wave fields. First we consider the special situation of a 2-D medium in which the medium parameters vary only in the lateral direction. Next we generalize the theory for 2-D inhomogeneous media. For the latter situation we derive one-way reciprocity theorems of the convolution type and of the correlation type. These reciprocity theorems find their applications in forward and inverse problems, respectively.

Key Words. Reciprocity, one-way wave fields, GPR, imaging.

1. Introduction

One-way wave equations play a fundamental role in the formulation of imaging schemes for reflection measurements. For seismic applications, pioneering work in this field has been done by Claerbout [1], Gazdag [2], Berkhout [3] and others. For ground-penetrating radar, similar methods have been used by Mast and Johansson [4] and Kagalenko and Weedon [5]. The basic idea behind "one-way wave equation imaging" is that a reflection experiment can be described in terms of downward and upward propagating wave fields, which are coupled by the reflection properties of the subsurface.

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By eliminating the downward and upward propagation effects from the reflection data, followed by applying an imaging principle, the reflection properties of the subsurface are obtained. In this paper we will not discuss imaging schemes, but we will consider theoretical aspects of the one-way wave equations underlying these schemes. In particular we will derive reciprocity theorems for electromagnetic one-way wave equations. For the greater part, the derivation will be along the same lines as that of the reciprocity theorems for the acoustic one-way wave equations [6]. The main difference, however, stems from the fact that dissipation plays a more prominent role in ground-penetrating radar data than in seismic data. The reciprocity theorems that we derive in this paper apply to electromagnetic one-way wave fields in dissipative media. We restrict ourselves to the two-dimensional situation, for which the electromagnetic vectorial wave equation reduces to a scalar wave equation for either transverse electric waves (TE) or transverse magnetic waves (TM). To show the principle of one-way reciprocity first, we start by considering a medium which varies in the lateral direction but which is constant in the depth direction. This is often a useful approximation for recursive wave field extrapolation through thin slabs. Later we will generalize the electromagnetic one-way reciprocity theorems to the situation of arbitrary 2-D inhomogeneous media.

2. One-Way Reciprocity in a Laterally Variant Medium

We derive a reciprocity relation for electromagnetic one-way wave fields in a dissipative medium which varies in the lateral (x_1) direction but which is constant in the depth (x_3) direction (the x_3-axis is pointing downward).

2.1. The Forward and Inverse Fourier Transform

We define the Fourier transform with respect to time (t) of a real-valued function as

$$U(\omega) = \int_{-\infty}^{\infty} \exp(-j\omega t) u(t)\, dt \tag{1}$$

and its inverse as

$$u(t) = \frac{1}{\pi} \mathrm{Re} \int_{-\infty}^{\infty} \exp(j\omega t)\, \chi(\omega)\, U(\omega)\, d\omega, \tag{2}$$

where ω denotes the angular frequency, j is the imaginary unit, Re denotes that the real part is taken, and $\chi(\omega)$ is the characteristic function, defined as

$$\chi(\omega) = \begin{cases} 0 & \text{for } \omega < 0 \\ \frac{1}{2} & \text{for } \omega = 0 \\ 1 & \text{for } \omega > 0. \end{cases} \tag{3}$$

In this paper, the angular frequency is chosen non-negative and real-valued, with the remark that its vanishing imaginary part is chosen negative when $u(t)$ is a causal function of time and positive when $u(t)$ is anti-causal (i.e., ($\text{Im}(\omega) \uparrow 0$ when $u(t) = 0$ for $t < 0$ and $\text{Im}(\omega) \downarrow 0$ when $u(t) = 0$ for $t > 0$).

2.2. One-Way Wave Equations

Our starting point for the discussion of the one-way wave equations is the two-dimensional Helmholtz equation for a laterally varying medium, given by

$$\partial_1^2 P + \partial_3^2 P + k^2 P = 0. \tag{4}$$

The wave field $P = P(\mathbf{x}, \omega)$ (with $\mathbf{x} = (x_1, x_3)$) and the wavenumber $k = k(x_1, \omega)$ are specified in Table 1 for the TE and TM case. Note that lateral medium variations occur either in ε and σ (for the TE case) or in μ and Γ (for the TM case). More general situations will be considered later.

Beyond some finite $|x_1|$-value the medium will be considered homogeneous and lossless and the wave field $P(\mathbf{x}, \omega)$ is assumed to have "sufficient decay" for $x_1 \to \pm\infty$. We rewrite Eq. (4) as

Table 1. Overview of Electromagnetic Wave Fields and Wavenumbers in the Helmholtz Equation (4) for Transverse Electric (TE) and Transverse Magnetic (TM) Waves in a Laterally Varying Medium

	$P(\mathbf{x}, \omega)$	$k^2(x_1, \omega)$
TE	$E_2(\mathbf{x}, \omega)$	$\left(\varepsilon(x_1) + \frac{\sigma(x_1)}{j\omega}\right)\left(\mu + \frac{\Gamma}{j\omega}\right)\omega^2$
TM	$H_2(\mathbf{x}, \omega)$	$\left(\mu(x_1) + \frac{\Gamma(x_1)}{j\omega}\right)\left(\varepsilon + \frac{\sigma}{j\omega}\right)\omega^2$

Note: Wave field quantities: E_2 (transverse electric field strength) and H_2 (transverse magnetic field strength); medium parameters: ε (permittivity), μ (permeability), σ (conductivity) and Γ (magnetic hysteresis loss term).

$$\partial_3^2 P = -\hat{\mathscr{H}}_2 P, \tag{5}$$

where $\hat{\mathscr{H}}_2$ is the one-dimensional Helmholtz operator, defined as

$$\hat{\mathscr{H}}_2 = k^2(x_1, \omega) + \partial_1^2. \tag{6}$$

The circumflex denotes an operator containing the lateral differentiation operator ∂_1. Note that $\hat{\mathscr{H}}_2$ is symmetric, in the sense that

$$\int_{-\infty}^{\infty} \{\hat{\mathscr{H}}_2 f\}(x_1) g(x_1)\, dx_1 = \int_{-\infty}^{\infty} f(x_1) \{\hat{\mathscr{H}}_2 g\}(x_1)\, dx_1 \tag{7}$$

(assuming the scalar functions $f(x_1)$ and $g(x_1)$ have sufficient decay for $x_1 \to \pm\infty$).

Equation (5) can be decoupled into two independent one-way wave equations for downgoing (+) and upgoing (−) wave fields, according to

$$\partial_3 P^+ = -j\hat{\mathscr{H}}_1 P^+, \tag{8}$$

$$\partial_3 P^- = +j\hat{\mathscr{H}}_1 P^-, \tag{9}$$

where $\hat{\mathscr{H}}_1$ is the square-root operator, defined as

$$\hat{\mathscr{H}}_1 = \hat{\mathscr{H}}_2^{1/2} \tag{10}$$

([1],[6–10]). Unlike $\hat{\mathscr{H}}_2$, the square-root operator $\hat{\mathscr{H}}_1$ cannot be written as a polynomial in ∂_1. Therefore $\hat{\mathscr{H}}_1$ is a so-called pseudo-differential operator. The square-root of an operator is not uniquely defined. In the following we assume that the square-root is taken such that the imaginary part of the eigenvalue spectrum of $\hat{\mathscr{H}}_1$ is negative for all eigenvalues. This implies

$$\lim_{x_3 \to \infty} P^+ = 0 \quad \text{and} \quad \lim_{x_3 \to -\infty} P^- = 0. \tag{11}$$

It can be shown that the square-root operator is symmetric as well [11]. Hence

$$\int_{-\infty}^{\infty} \{\hat{\mathscr{H}}_1 f\}(x_1) g(x_1)\, dx_1 = \int_{-\infty}^{\infty} f(x_1) \{\hat{\mathscr{H}}_1 g\}(x_1)\, dx_1. \tag{12}$$

2.3. One-Way Reciprocity Relations for Green's Wave Fields

We introduce Green's wave fields associated to the one-way wave equations (8) and (9). The downgoing Green's wave field $G^+(\mathbf{x}, \mathbf{x}_A)$, due to a source at $\mathbf{x}_A = (x_{1,A}, x_{3,A})$, obeys the one-way wave equation

$$\partial_3 G^+ = -j\hat{\mathscr{H}}_1 G^+ + \delta(\mathbf{x} - \mathbf{x}_A), \tag{13}$$

with initial condition $G^{+}(\mathbf{x},\mathbf{x}_A)=0$ for $x_3<x_{3,A}$. Similarly, the upgoing Green's wave field $G^{-}(\mathbf{x},\mathbf{x}_B)$, due to a source at $\mathbf{x}_B=(x_{1,B},x_{3,B})$, obeys the one-way wave equation

$$\partial_3 G^{-}=+j\hat{\mathscr{H}}_1 G^{-}+\delta(\mathbf{x}-\mathbf{x}_B), \tag{14}$$

with initial condition $G^{-}(\mathbf{x},\mathbf{x}_B)=0$ for $x_3>x_{3,B}$. We define an "interaction quantity" $\partial_3(G^{+}G^{-})$. Applying the product rule for differentiation and substituting one-way wave equations (13) and (14) into the result yields

$$\begin{aligned}\partial_3(G^{+}G^{-})&=(\partial_3 G^{+})G^{-}+G^{+}(\partial_3 G^{-})\\&=[-j\hat{\mathscr{H}}_1 G^{+}+\delta(\mathbf{x}-\mathbf{x}_A)]G^{-}+G^{+}[j\hat{\mathscr{H}}_1 G^{-}+\delta(\mathbf{x}-\mathbf{x}_B)].\end{aligned} \tag{15}$$

Next, we consider a domain $\mathscr{D}$, bounded by $\partial\mathscr{D}=\partial\mathscr{D}^{-}\cup\partial\mathscr{D}^{+}$, where $\partial\mathscr{D}^{-}$ denotes the depth level $x_3=x_3^{-}$ and $\partial\mathscr{D}^{+}$ the depth level $x_3=x_3^{+}$, see Figure 1. We choose the Green's source points $\mathbf{x}_A$ and $\mathbf{x}_B$ both in $\mathscr{D}$, with $x_3^{-}<x_{3,A}<x_{3,B}<x_3^{+}$. Integrating both sides of Eq. (15) over the domain $\mathscr{D}$ yields

$$\begin{aligned}\int_{\partial\mathscr{D}^{+}}(G^{+}G^{-})\,dx_1-\int_{\partial\mathscr{D}^{-}}(G^{+}G^{-})\,dx_1&=-j\int_{\mathscr{D}}\{(\hat{\mathscr{H}}_1 G^{+})G^{-}-G^{+}(\hat{\mathscr{H}}_1 G^{-})\}\,d^2\mathbf{x}\\&\quad+G^{-}(\mathbf{x}_A,\mathbf{x}_B)+G^{+}(\mathbf{x}_B,\mathbf{x}_A).\end{aligned} \tag{16}$$

According to the initial conditions for the Green's wave fields, we have $G^{+}=0$ at $\partial\mathscr{D}^{-}$ and $G^{-}=0$ at $\partial\mathscr{D}^{+}$, hence, the boundary integrals on the left-hand side of Eq. (16) vanish. The domain integral on the right-hand side of Eq. (16) can be written as $\int_{\mathscr{D}}\{\cdot\}\,d^2\mathbf{x}=\int_{x_3^{-}}^{x_3^{+}}dx_3\int_{-\infty}^{\infty}\{\cdot\}\,dx_1$. Hence, according to the symmetry relation for the square-root operator $\hat{\mathscr{H}}_1$ (Eq. (12)), this domain integral vanishes as well. Hence, we are left with

$$G^{-}(\mathbf{x}_A,\mathbf{x}_B)=-G^{+}(\mathbf{x}_B,\mathbf{x}_A). \tag{17}$$

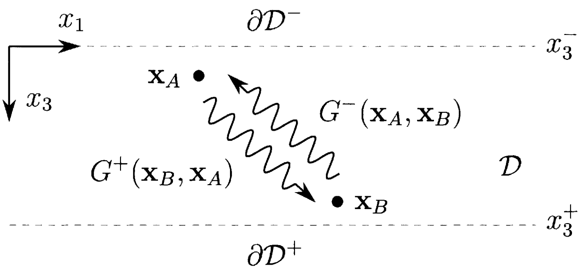

Figure 1. Configuration in which the reciprocity theorems are derived.

This equation states that the upgoing Green's wave field at observation point $\mathbf{x}_A$ due to a source for upgoing waves at $\mathbf{x}_B$, is identical to minus the downgoing Green's wave field at observation point $\mathbf{x}_B$ due to a source for downgoing waves at $\mathbf{x}_A$, see Figure 1. This result is less trivial than it seems, because it applies to solutions of the one-way wave equations (13) and (14) rather than the full wave equation (4). Central in the derivation is the symmetry property of the square-root operator $\hat{\mathscr{H}}_1$ (Eq. (12)), which makes that the domain integral on the right-hand side of Eq. (16) vanishes.

Equation (17) is just a special case of a more general reciprocity theorem for one-way wave fields, which will be discussed in the following sections.

3. Electromagnetic Wave Equation in a 2-D Inhomogeneous Medium

In this section we discuss a general equation for electromagnetic wave fields in dissipative 2-D inhomogeneous media. First we give the basic scalar equations. Next we recast these equations into a matrix-vector form that will be useful as a starting point for the decomposition into one-way wave equations.

3.1. Basic Scalar Equations

We consider again a 2-D configuration, but this time all medium parameters may vary in the x_1 and x_3 directions. Using the definitions in Table 2, the general form of the 2-D basic equations in the space-frequency $(\mathbf{x}, \omega)$ domain for transverse electric (TE) and transverse magnetic (TM) waves is given by

$$j\omega\alpha P + \partial_i Q_i = B, \tag{18}$$

$$j\omega\beta Q_i + \partial_i P = C_i, \tag{19}$$

Table 2. Overview of Electromagnetic Wave Field Quantities, Medium Parameters and Source Functions in Eqs. (18) and (19) for Transverse Electric (TE) and Transverse Magnetic (TM) Waves in a 2-D Inhomogeneous Medium

	P	Q_1	Q_3	α	β	B	C_1	C_3
TE	E_2	H_3	$-H_1$	$\varepsilon+\sigma/j\omega$	$\mu+\Gamma/j\omega$	$-J_2^e$	$-J_3^m$	J_1^m
TM	H_2	$-E_3$	E_1	$\mu+\Gamma/j\omega$	$\varepsilon+\sigma/j\omega$	$-J_2^m$	J_3^e	$-J_1^e$

Note: Wave field quantities: $E_i(\mathbf{x}, \omega)$ and $H_i(\mathbf{x}, \omega)$; medium parameters: $\varepsilon(\mathbf{x}, \omega)$, $\mu(\mathbf{x}, \omega)$, $\sigma(\mathbf{x}, \omega)$ and $\Gamma(\mathbf{x}, \omega)$; source functions: $J_i^e(\mathbf{x}, \omega)$ (electric current density) and $J_i^m(\mathbf{x}, \omega)$ (magnetic current density).

where i only takes on the values 1 and 3 and Einstein's summation convention applies to repeated subscripts. $P(\mathbf{x}, \omega)$, $Q_1(\mathbf{x}, \omega)$, and $Q_3(\mathbf{x}, \omega)$ represent the wave fields, $B(\mathbf{x}, \omega)$, $C_1(\mathbf{x}, \omega)$, and $C_3(\mathbf{x}, \omega)$ are the source distributions and $\alpha(\mathbf{x}, \omega)$ and $\beta(\mathbf{x}, \omega)$ denote the medium parameters. In dissipative media the parameters α and β are complex-valued frequency-dependent functions, with a negative imaginary part. Note that ε, μ, σ, and Γ in itself may be complex-valued and frequency-dependent as well. For the derivation of the reciprocity theorems later in this paper, it is useful to distinguish between dissipative and effectual media [12]. A wave propagating through an effectual medium gains energy. For effectual media the imaginary parts of α and β are positive.

Also for later use, we introduce adjoint medium parameters α' and β', according to

$$\alpha'(\mathbf{x}, \omega) = \alpha^*(\mathbf{x}, \omega), \tag{20}$$

$$\beta'(\mathbf{x}, \omega) = \beta^*(\mathbf{x}, \omega), \tag{21}$$

where $*$ denotes complex conjugation. Note that when a medium is dissipative, its adjoint medium is effectual and vice versa. In the light of the discussion below Eq. (3) it should be noted that the primes not only involve a change of sign of the imaginary parts of α and β, but also of the vanishing imaginary part of ω.

3.2. Wave Equation in Matrix–Vector Form

From here onward we assume that the preferred propagation direction is along the depth (x_3) axis. In the lateral (x_1) direction the medium is unbounded; beyond some finite $|x_1|$-value it will be assumed that the medium is laterally homogeneous, lossless and source-free. Moreover, the wave fields are again assumed to have sufficient decay for $x_1 \to \pm\infty$.

We reorganize the general wave equation into a form that acknowledges the direction of preference. By eliminating Q_1 from the system of Eqs. (18) and (19) we obtain

$$\partial_3 \mathbf{Q} = \hat{\underset{\sim}{\mathbf{A}}} \mathbf{Q} + \mathbf{D}, \tag{22}$$

where the wave field vector $\mathbf{Q}$ and the source vector $\mathbf{D}$ are given by

$$\mathbf{Q} = \begin{pmatrix} P \\ Q_3 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} C_3 \\ B - \dfrac{1}{j\omega}\partial_1\left(\dfrac{1}{\beta}C_1\right) \end{pmatrix} \tag{23}$$

and the operator matrix $\hat{\underset{\sim}{\mathbf{A}}}$ by

$$\hat{\underset{\sim}{\mathbf{A}}} = \begin{pmatrix} 0 & -j\omega\beta \\ -j\omega\hat{A} & 0 \end{pmatrix}, \tag{24}$$

with

$$\hat{A} = \alpha + \frac{1}{\omega^2}\partial_1\left(\frac{1}{\beta}\partial_1 \cdot\right). \tag{25}$$

In order to simplify the differential operator in the right-hand side of this equation, we redefine the Helmholtz operator $\hat{\mathscr{H}}_2$ via

$$\hat{\mathscr{H}}_2 = \omega^2\beta^{1/2}(\hat{A}\beta^{1/2}\cdot), \tag{26}$$

or, using Eq. (25),

$$\hat{\mathscr{H}}_2 = k^2 + \partial_1^2, \tag{27}$$

where

$$k^2 = \alpha\beta\omega^2 - \frac{3(\partial_1\beta)^2}{4\beta^2} + \frac{\partial_1^2\beta}{2\beta}. \tag{28}$$

Using Eq. (26) we may reformulate the operator matrix $\hat{\underset{\sim}{\mathbf{A}}}$, as defined in Eq. (24), as follows

$$\hat{\underset{\sim}{\mathbf{A}}} = \begin{pmatrix} 0 & -j\omega\beta \\ \frac{1}{j\omega\beta^{1/2}}(\hat{\mathscr{H}}_2\beta^{-1/2}\cdot) & 0 \end{pmatrix}. \tag{29}$$

We redefine $\hat{\mathscr{H}}_1$ as the square-root of the modified Helmholtz operator $\hat{\mathscr{H}}_2$, according to

$$\hat{\mathscr{H}}_1 = \hat{\mathscr{H}}_2^{1/2}. \tag{30}$$

In the following we assume that the square-root is taken such that the imaginary part of the eigenvalue spectrum of $\hat{\mathscr{H}}_1$ is negative for dissipative media and positive for effectual media. The redefined operators $\hat{\mathscr{H}}_2$ and $\hat{\mathscr{H}}_1$ again obey the symmetry relations (7) and (12). Given the Helmholtz and square-root operators in a specific medium (either dissipative or effectual), we obtain for the operators in the adjoint medium (defined by the parameters α' and β')

$$\hat{\mathscr{H}}_2' = \hat{\mathscr{H}}_2^* \quad \text{and} \quad \hat{\mathscr{H}}_1' = \hat{\mathscr{H}}_1^*. \tag{31}$$

Note that an "operator for the adjoint medium" is not by definition the same as the "adjoint operator for the original medium."

4. One-Way Wave Equation in a 2-D Inhomogeneous Medium

In this section we derive the matrix–vector form of the coupled system of equations for electromagnetic one-way wave fields in dissipative 2-D inhomogeneous media (which we will call for short the "one-way wave equation").

4.1. Diagonalization of the Operator Matrix

Using the square-root operator introduced in Eq. (30), we may write for the operator matrix $\hat{\underset{\sim}{\mathbf{A}}}$

$$\hat{\underset{\sim}{\mathbf{A}}} = \hat{\underset{\sim}{\mathbf{L}}}\hat{\underset{\sim}{\mathbf{H}}}\hat{\underset{\sim}{\mathbf{L}}}^{-1}, \tag{32}$$

where $\hat{\underset{\sim}{\mathbf{H}}}$ is a diagonal operator matrix, containing the square-root operator on its diagonal, according to

$$\hat{\underset{\sim}{\mathbf{H}}} = \begin{pmatrix} -j\hat{\mathscr{H}}_1 & 0 \\ 0 & j\hat{\mathscr{H}}_1 \end{pmatrix} \tag{33}$$

and where

$$\hat{\underset{\sim}{\mathbf{L}}} = \begin{pmatrix} \hat{L}_1 & \hat{L}_1 \\ \hat{L}_2 & -\hat{L}_2 \end{pmatrix}, \qquad \hat{\underset{\sim}{\mathbf{L}}}^{-1} = \tfrac{1}{2}\begin{pmatrix} \hat{L}_1^{-1} & \hat{L}_2^{-1} \\ \hat{L}_1^{-1} & -\hat{L}_2^{-1} \end{pmatrix}, \tag{34}$$

with

$$\hat{L}_1 = \left(\frac{\omega\beta}{2}\right)^{1/2} \hat{\mathscr{H}}_1^{-1/2}, \qquad \tfrac{1}{2}\hat{L}_1^{-1} = (\hat{\mathscr{H}}_1^{1/2}(2\omega\beta)^{-1/2}\,\cdot), \tag{35}$$

$$\hat{L}_2 = (2\omega\beta)^{-1/2}\hat{\mathscr{H}}_1^{1/2}, \qquad \tfrac{1}{2}\hat{L}_2^{-1} = \left(\hat{\mathscr{H}}_1^{-1/2}\left(\frac{\omega\beta}{2}\right)^{1/2}\cdot\right). \tag{36}$$

Analogous to Eq. (31), we have the following relations for the operators $\hat{L}_1'$ and $\hat{L}_2'$ in the adjoint medium

$$\hat{L}_1' = \hat{L}_1^* \quad \text{and} \quad \hat{L}_2' = \hat{L}_2^*. \tag{37}$$

4.2. One-Way Wave Equation in Matrix–Vector Form

We introduce a "one-way wave field vector" $\mathbf{P}$ and a "one-way source vector" $\mathbf{S}$, according to

$$\mathbf{P} = \begin{pmatrix} P^+ \\ P^- \end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} S^+ \\ S^- \end{pmatrix}, \tag{38}$$

where P^+ and P^- are the downward and upward propagating one-way wave fields, respectively, and S^+ and S^- are the sources for these one-way wave fields. We relate the one-way wave field and source vectors to the full wave field and source vectors, according to

$$\mathbf{Q} = \hat{\underset{\sim}{\mathbf{L}}}\mathbf{P}, \qquad \mathbf{D} = \hat{\underset{\sim}{\mathbf{L}}}\mathbf{S}. \tag{39}$$

Substituting these expressions, together with Eq. (32) into Eq. (22) yields the one-way wave equation in matrix–vector form

$$\partial_3\mathbf{P} = \hat{\underset{\sim}{\mathbf{B}}}\mathbf{P} + \mathbf{S}, \tag{40}$$

where the one-way operator matrix $\hat{\underset{\sim}{\mathbf{B}}}$ is defined as

$$\hat{\underset{\sim}{\mathbf{B}}} = \hat{\underset{\sim}{\mathbf{H}}} - \hat{\underset{\sim}{\mathbf{L}}}^{-1}\partial_3\hat{\underset{\sim}{\mathbf{L}}}. \tag{41}$$

Using Eqs. (33) and (34) we may write for $\hat{\underset{\sim}{\mathbf{B}}}$, as defined in Eq. (41),

$$\hat{\underset{\sim}{\mathbf{B}}} = \begin{pmatrix} -j\hat{\mathscr{H}}_1 & 0 \\ 0 & j\hat{\mathscr{H}}_1 \end{pmatrix} + \begin{pmatrix} \hat{T} & -\hat{R} \\ -\hat{R} & \hat{T} \end{pmatrix}, \tag{42}$$

where $\hat{R}$ and $\hat{T}$ are reflection and transmission operators, respectively, given by

$$\hat{R} = (\hat{L}_1^{-1}\partial_3\hat{L}_1 - \hat{L}_2^{-1}\partial_3\hat{L}_2)/2, \tag{43}$$

$$\hat{T} = -(\hat{L}_1^{-1}\partial_3\hat{L}_1 + \hat{L}_2^{-1}\partial_3\hat{L}_2)/2. \tag{44}$$

On account of Eq. (37) we have

$$\hat{R}' = \hat{R}^* \quad \text{and} \quad \hat{T}' = \hat{T}^*. \tag{45}$$

Hence, given the one-way operator matrix $\hat{\underset{\sim}{\mathbf{B}}}$ in a specific medium (either dissipative or effectual), we obtain for the one-way operator matrix in the adjoint medium, using Eqs. (31) and (45),

$$\hat{\underset{\sim}{\mathbf{B}}}' = \begin{pmatrix} -j\hat{\mathscr{H}}_1^* & 0 \\ 0 & j\hat{\mathscr{H}}_1^* \end{pmatrix} + \begin{pmatrix} \hat{T}^* & -\hat{R}^* \\ -\hat{R}^* & \hat{T}^* \end{pmatrix}. \tag{46}$$

5. One-Way Reciprocity in a 2-D Inhomogeneous Medium

We derive reciprocity theorems that interrelate the one-way wave vectors, operator matrices and source vectors in two different states. These states will be distinguished by the subscripts A and B, see Table 3.

The reciprocity theorems will be derived for the domain $\mathscr{D}$, introduced below Eq. (15), see Figure 1. The wave field conditions that have been discussed above Eq. (22) are assumed to apply to $\mathbf{P}_A$ as well as to $\mathbf{P}_B$.

Table 3. States in the One-Way Reciprocity Theorems

	State A	State B
Wave field	$\mathbf{P}_A$	$\mathbf{P}_B$
Operator	$\hat{\underset{\sim}{\mathbf{B}}}_A$	$\hat{\underset{\sim}{\mathbf{B}}}_B$
Source	$\mathbf{S}_A$	$\mathbf{S}_B$

5.1. Convolution Type Reciprocity Theorem for One-Way Wave Fields

We define a convolution type interaction quantity between oppositely propagating waves in both states, according to

$$\partial_3\{P_A^+ P_B^- - P_A^- P_B^+\}, \tag{47}$$

or, using a more compact notation,

$$\partial_3\{\mathbf{P}_A^t \underset{\sim}{\mathbf{N}} \mathbf{P}_B\}, \tag{48}$$

where ^{t} denotes transposition and

$$\underset{\sim}{\mathbf{N}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{49}$$

Applying the product rule for differentiation, substituting the one-way wave equation (40) for states A and B, integrating the result over domain $\mathscr{D}$ with boundary $\partial\mathscr{D}$ and applying the theorem of Gauss, yields

$$\int_{\partial\mathscr{D}} \mathbf{P}_A^t \underset{\sim}{\mathbf{N}} \mathbf{P}_B n_3 \, dx_1 = \int_{\mathscr{D}} \{\mathbf{P}_A^t \underset{\sim}{\mathbf{N}} \hat{\underset{\sim}{\mathbf{B}}}_B \mathbf{P}_B + (\hat{\underset{\sim}{\mathbf{B}}}_A \mathbf{P}_A)^t \underset{\sim}{\mathbf{N}} \mathbf{P}_B\} \, d^2\mathbf{x}$$

$$+ \int_{\mathscr{D}} \{\mathbf{P}_A^t \underset{\sim}{\mathbf{N}} \mathbf{S}_B + \mathbf{S}_A^t \underset{\sim}{\mathbf{N}} \mathbf{P}_B\} \, d^2\mathbf{x}, \tag{50}$$

where the component n_3 of the outward pointing normal vector on $\partial\mathscr{D}$ is defined as $n_3 = -1$ on $\partial\mathscr{D}^-$ and $n_3 = +1$ on $\partial\mathscr{D}^+$. Hence, $\int_{\partial\mathscr{D}} \{\cdot\} n_3 \, dx_1$ stands for $\int_{\partial\mathscr{D}^+} \{\cdot\} \, dx_1 - \int_{\partial\mathscr{D}^-} \{\cdot\} \, dx_1$. On account of the following symmetry property [11]

$$\int_{-\infty}^{\infty} \{\hat{\underset{\sim}{\mathbf{B}}} \mathbf{f}\}^t \underset{\sim}{\mathbf{N}} \mathbf{g} \, dx_1 = -\int_{-\infty}^{\infty} \mathbf{f}^t \underset{\sim}{\mathbf{N}} \hat{\underset{\sim}{\mathbf{B}}} \mathbf{g} \, dx_1, \tag{51}$$

we may rewrite Eq. (50) as

$$\int_{\partial\mathscr{D}} \mathbf{P}_A^t \underset{\sim}{\mathbf{N}} \mathbf{P}_B n_3 \, dx_1 = \int_{\mathscr{D}} \mathbf{P}_A^t \underset{\sim}{\mathbf{N}} (\hat{\underset{\sim}{\mathbf{B}}}_B - \hat{\underset{\sim}{\mathbf{B}}}_A) \mathbf{P}_B \, d^2\mathbf{x}$$

$$+ \int_{\mathscr{D}} \{\mathbf{P}_A^t \underset{\sim}{\mathbf{N}} \mathbf{S}_B + \mathbf{S}_A^t \underset{\sim}{\mathbf{N}} \mathbf{P}_B\} \, d^2\mathbf{x}. \tag{52}$$

Equation (52) formulates a reciprocity theorem of the convolution type for one-way wave fields. It relates the one-way wave field vectors at the boundary $\partial\mathscr{D}$ to the one-way sources and the contrast between the one-way operators in both states in domain $\mathscr{D}$. Equation (52) applies to lossless as well as dissipative and effectual media. Note that the contrast term in the right-hand side of Eq. (52) vanishes when the medium parameters in both states are identical. In its general form Eq. (52) provides a basis for, amongst others, representations of scattered one-way wave fields in terms of generalized Bremmer series expansions or in terms of generalized primaries ([10], [13–15]).

5.2. One-Way Reciprocity Relations for Green's Wave Fields

We introduce Green's wave fields associated to the one-way wave equation (40). The Green's wave field matrix $\underset{\sim}{\mathbf{G}}(\mathbf{x}, \mathbf{x}_A)$, related to sources for downgoing and upgoing waves at $\mathbf{x}_A = (x_{1,A}, x_{3,A})$, obeys the one-way wave equation

$$\partial_3 \underset{\sim}{\mathbf{G}}(\mathbf{x}, \mathbf{x}_A) = \underset{\sim}{\hat{\mathbf{B}}}\underset{\sim}{\mathbf{G}}(\mathbf{x}, \mathbf{x}_A) + \underset{\sim}{\mathbf{I}}\delta(\mathbf{x} - \mathbf{x}_A). \tag{53}$$

$\underset{\sim}{\mathbf{I}}$ is the 2×2 identity matrix and $\underset{\sim}{\mathbf{G}}(\mathbf{x}, \mathbf{x}_A)$ is a 2×2 matrix which has the following form

$$\underset{\sim}{\mathbf{G}}(\mathbf{x}, \mathbf{x}_A) = \begin{pmatrix} G^{+,+}(\mathbf{x}, \mathbf{x}_A) & G^{+,-}(\mathbf{x}, \mathbf{x}_A) \\ G^{-,+}(\mathbf{x}, \mathbf{x}_A) & G^{-,-}(\mathbf{x}, \mathbf{x}_A) \end{pmatrix}, \tag{54}$$

where the superscripts refer to the propagation direction at $\mathbf{x}$ and $\mathbf{x}_A$, respectively. A similar Green's wave field matrix $\underset{\sim}{\mathbf{G}}(\mathbf{x}, \mathbf{x}_B)$, related to sources at $\mathbf{x}_B = (x_{1,B}, x_{3,B})$, obeys the one-way wave equation

$$\partial_3 \underset{\sim}{\mathbf{G}}(\mathbf{x}, \mathbf{x}_B) = \underset{\sim}{\hat{\mathbf{B}}}\underset{\sim}{\mathbf{G}}(\mathbf{x}, \mathbf{x}_B) + \underset{\sim}{\mathbf{I}}\delta(\mathbf{x} - \mathbf{x}_B). \tag{55}$$

We let these Green's matrices play the role of states A and B in the one-way reciprocity theorem (52), according to

$$\int_{\partial\mathscr{D}} \underset{\sim}{\mathbf{G}}^t(\mathbf{x}, \mathbf{x}_A)\underset{\sim}{\mathbf{N}}\underset{\sim}{\mathbf{G}}(\mathbf{x}, \mathbf{x}_B) n_3 \, dx_1$$
$$= \int_{\mathscr{D}} \{\underset{\sim}{\mathbf{G}}^t(\mathbf{x}, \mathbf{x}_A)\underset{\sim}{\mathbf{N}}\delta(\mathbf{x} - \mathbf{x}_B) + \delta(\mathbf{x} - \mathbf{x}_A)\underset{\sim}{\mathbf{N}}\underset{\sim}{\mathbf{G}}(\mathbf{x}, \mathbf{x}_B)\} \, d^2\mathbf{x}. \tag{56}$$

When we choose the Green's source points $\mathbf{x}_A$ and $\mathbf{x}_B$ both in domain $\mathscr{D}$ (see Fig. 1) and the medium outside $\partial\mathscr{D}$ is homogeneous, then there are only outgoing contributions at $\partial\mathscr{D}$, so the boundary integral on the left-hand side of Eq. (56) vanishes, yielding

$$\underset{\sim}{\mathbf{O}} = \underset{\sim}{\mathbf{G}}^t(\mathbf{x}_B, \mathbf{x}_A)\underset{\sim}{\mathbf{N}} + \underset{\sim}{\mathbf{N}}\underset{\sim}{\mathbf{G}}(\mathbf{x}_A, \mathbf{x}_B), \tag{57}$$

($\underset{\sim}{\mathbf{O}}$ is the 2×2 null-matrix) or, using the notation of Eq. (54),

$$\begin{pmatrix} G^{+,+}(\mathbf{x}_A, \mathbf{x}_B) & G^{+,-}(\mathbf{x}_A, \mathbf{x}_B) \\ G^{-,+}(\mathbf{x}_A, \mathbf{x}_B) & G^{-,-}(\mathbf{x}_A, \mathbf{x}_B) \end{pmatrix} = \begin{pmatrix} -G^{-,-}(\mathbf{x}_B, \mathbf{x}_A) & G^{+,-}(\mathbf{x}_B, \mathbf{x}_A) \\ G^{-,+}(\mathbf{x}_B, \mathbf{x}_A) & -G^{+,+}(\mathbf{x}_B, \mathbf{x}_A) \end{pmatrix}. \tag{58}$$

Note that for the lower-right element we have $G^{-,-}(\mathbf{x}_A, \mathbf{x}_B) = -G^{+,+}(\mathbf{x}_B, \mathbf{x}_A)$, which is a generalization of Eq. (17) for arbitrary 2-D inhomogeneous media.

5.3. Correlation Type Reciprocity Theorem for One-Way Wave Fields

We define a correlation type interaction quantity, according to

$$\partial_3\{(P_A^+)^* P_B^+ - (P_A^-)^* P_B^-\}, \tag{59}$$

or, using a more compact notation,

$$\partial_3\{\mathbf{P}_A^\dagger \underset{\sim}{\mathbf{J}} \mathbf{P}_B\}, \tag{60}$$

where † denotes complex conjugation and transposition and

$$\underset{\sim}{\mathbf{J}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{61}$$

Applying the product rule for differentiation, substituting the one-way wave equation (40) for states A and B, integrating the result over domain $\mathscr{D}$ with boundary $\partial\mathscr{D}$ and applying the theorem of Gauss, yields

$$\int_{\partial\mathscr{D}} \mathbf{P}_A^\dagger \underset{\sim}{\mathbf{J}} \mathbf{P}_B n_3 \, dx_1$$
$$= \int_{\mathscr{D}} \{\mathbf{P}_A^\dagger \underset{\sim}{\mathbf{J}} \underset{\sim}{\hat{\mathbf{B}}}_B \mathbf{P}_B + (\underset{\sim}{\hat{\mathbf{B}}}_A \mathbf{P}_A)^\dagger \underset{\sim}{\mathbf{J}} \mathbf{P}_B\} \, d^2\mathbf{x} + \int_{\mathscr{D}} \{\mathbf{P}_A^\dagger \underset{\sim}{\mathbf{J}} \mathbf{S}_B + \mathbf{S}_A^\dagger \underset{\sim}{\mathbf{J}} \mathbf{P}_B\} \, d^2\mathbf{x}. \tag{62}$$

On account of the following symmetry property [11]

$$\int_{-\infty}^{\infty} \{\underset{\sim}{\hat{\mathbf{B}}}\mathbf{f}\}^\dagger \underset{\sim}{\mathbf{J}} \mathbf{g} \, dx_1 = -\int_{-\infty}^{\infty} \mathbf{f}^\dagger \underset{\sim}{\mathbf{J}} \underset{\sim}{\hat{\mathbf{B}}}' \mathbf{g} \, dx_1, \tag{63}$$

we may rewrite Eq. (62) as

$$\int_{\partial\mathscr{D}} \mathbf{P}_A^{\dagger}\,\underset{\sim}{\mathbf{J}}\mathbf{P}_B n_3\,dx_1 = \int_{\mathscr{D}} \mathbf{P}_A^{\dagger}\underset{\sim}{\mathbf{J}}(\hat{\underset{\sim}{\mathbf{B}}}_B - \hat{\underset{\sim}{\mathbf{B}}}'_A)\mathbf{P}_B\,d^2\mathbf{x} + \int_{\mathscr{D}} \{\mathbf{P}_A^{\dagger}\underset{\sim}{\mathbf{J}}\mathbf{S}_B + \mathbf{S}_A^{\dagger}\underset{\sim}{\mathbf{J}}\mathbf{P}_B\}\,d^2\mathbf{x}. \quad (64)$$

Equation (64) formulates a reciprocity theorem of the correlation type for one-way wave fields. It applies to lossless as well as dissipative and effectual media. Note that the contrast term in the right-hand side of Eq. (64) vanishes when the medium parameters in one state are the adjoint versions of the medium parameters in the other state. In its general form Eq. (64) provides a basis for reflection imaging based on inverse one-way wave field propagators [14].

6. Conclusions

We have derived reciprocity theorems for electromagnetic one-way wave fields in dissipative 2-D media. We started with the situation in which the medium parameters vary only in the lateral direction but are constant in depth. Later we generalized the theory for 2-D inhomogeneous media. For the latter situation we obtained one-way reciprocity theorems of the convolution type (Eq. 52) and of the correlation type (Eq. 64). They are particularly suited for wave propagation problems in which the "preferred direction of propagation" is along the x_3-axis, like in ground-penetrating radar measurements. Both theorems relate the one-way wave field vectors at the boundary $\partial\mathscr{D} = \partial\mathscr{D}^- \cup \partial\mathscr{D}^+$ of some domain $\mathscr{D}$ (Fig. 1) to the one-way sources and the contrast between the one-way operators in both states in the domain $\mathscr{D}$. The contrast term in the reciprocity theorem of the convolution type vanishes when the medium parameters in both states are identical; the contrast term in the reciprocity theorem of the correlation type vanishes when the medium parameters in one state are the adjoint versions of the medium parameters in the other state.

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