Seismic applications of one-way acoustic reciprocity

Kees Wapenaar, Jacob Fokkema and Menno Dillen
Centre for Technical Geoscience
Delft University of Technology
The Netherlands

December 8, 2000

Introduction

One-way wave equations have played a prominent role in seismic processing since the pioneering work of Claerbout (1971, Geophysics), Berkhout (1979, Geoph. Prosp.) and others. The reason for this is that a seismic experiment can be explained in terms of 'downgoing' waves traveling from the source at the Earth's surface to a target in the subsurface and 'upgoing' waves traveling from the target to the receivers at the surface. One-way wave equations naturally honour this distinction between downgoing and upgoing waves.

This paper starts with a review of reciprocity theorems for one-way wave fields. These reciprocity theorems formulate general relations between the one-way wave fields in two different 'states'. One of these states is an actual seismic experiment, while the other state can either be a computational state (e.g. a wave field propagator), a desired state (e.g. multiple-free data) or another seismic measurement (characterizing time-lapse differences in the target). The title of this paper refers to the book 'Seismic applications of acoustic reciprocity' by Fokkema and van den Berg (1993, Elsevier). These authors derive seismic processing techniques from Rayleigh's reciprocity theorem for total acoustic wave fields. In this paper the one-way reciprocity theorems form the starting point. These theorems provide a theoretical frame-work for current seismic processing techniques based on the one-way wave equations. Some applications will be indicated.

One-way reciprocity theorems in media with losses

The one-way wave equation and its symmetry properties

We review the acoustic one-way wave equation for downgoing and upgoing waves in an inhomogeneous medium with losses. We introduce a one-way wave vector $\mathbf{P}$ and a one-way source vector $\mathbf{S}$, according to

$$\mathbf{P} = \begin{pmatrix} P^+ \\ P^- \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} S^+ \\ S^- \end{pmatrix}. \tag{1}$$

The superscripts $+$ and $-$ stand for 'downgoing' and 'upgoing', respectively. In the space-frequency $(\mathbf{x}, \omega)$ domain, the one-way wave equation reads

$$\partial_3 \mathbf{P} = \hat{\underset{\sim}{\mathbf{B}}}\mathbf{P} + \mathbf{S}, \tag{2}$$

where the one-way operator matrix $\hat{\underset{\sim}{\mathbf{B}}}$ is defined as

$$\hat{\underset{\sim}{\mathbf{B}}} = \begin{pmatrix} -j\hat{\mathcal{H}}_1 & O \\ O & j\hat{\mathcal{H}}_1 \end{pmatrix} + \begin{pmatrix} \hat{T} & -\hat{R} \\ -\hat{R} & \hat{T} \end{pmatrix} \tag{3}$$

(the circumflex denotes a pseudo-differential containing ∂_1 and ∂_2). $\hat{\mathcal{H}}_1$ is the well-known square-root operator and $\hat{R}$ and $\hat{T}$ are the reflection and transmission operators.

We define the bilinear form $\langle \cdot, \cdot \rangle_b$ and the sesquilinear form $\langle \cdot, \cdot \rangle_s$ according to

$$\langle \mathbf{f}, \mathbf{g} \rangle_b = \int \mathbf{f}^t(\mathbf{x}_H)\mathbf{g}(\mathbf{x}_H)\mathrm{d}^2\mathbf{x}_H \quad \text{and} \quad \langle \mathbf{f}, \mathbf{g} \rangle_s = \int \mathbf{f}^\dagger(\mathbf{x}_H)\mathbf{g}(\mathbf{x}_H)\mathrm{d}^2\mathbf{x}_H, \tag{4}$$

where ^{t} denotes transposition, † denotes adjoint (here: transposition and complex conjugation) and $\mathbf{x}_H = (x_1, x_2)$. We introduce the transposed operator $\hat{\underset{\sim}{\mathbf{B}}}^t$ and the adjoint operator $\hat{\underset{\sim}{\mathbf{B}}}^\dagger$ via

$$\langle \hat{\underset{\sim}{\mathbf{B}}}\mathbf{f}, \mathbf{g}\rangle_b = \langle \mathbf{f}, \hat{\underset{\sim}{\mathbf{B}}}^t \mathbf{g}\rangle_b \quad \text{and} \quad \langle \hat{\underset{\sim}{\mathbf{B}}}\mathbf{f}, \mathbf{g}\rangle_s = \langle \mathbf{f}, \hat{\underset{\sim}{\mathbf{B}}}^\dagger \mathbf{g}\rangle_s. \tag{5}$$

Using these definitions, the transposed and adjoint one-way operator matrices obey

$$\hat{\underset{\sim}{\mathbf{B}}}^t \underset{\sim}{\mathbf{N}} = -\underset{\sim}{\mathbf{N}}\hat{\underset{\sim}{\mathbf{B}}} \quad \text{and} \quad \hat{\underset{\sim}{\mathbf{B}}}^\dagger \underset{\sim}{\mathbf{J}} = -\underset{\sim}{\mathbf{J}}\hat{\underset{\sim}{\mathbf{B}}}', \tag{6}$$

with $\underset{\sim}{\mathbf{N}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\underset{\sim}{\mathbf{J}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The first symmetry relation in equation (6) was derived in Wapenaar and Grimbergen (1996, Geoph. J. Int.) for lossless media. Using an approach modified after Dillen (2000, Ph.D. thesis, Delft), it appears to hold in media with losses as well. The prime $'$ in the second symmetry relation in equation (6) denotes that this operator is defined in the adjoint (=complex conjugate) medium. When a medium is passive, its adjoint medium is active and vice versa.

Reciprocity theorem of the convolution type for one-way wave fields

We introduce two different states that will be distinguished by the subscripts A and B. For these two states we consider the interaction quantity $\partial_3\{\mathbf{P}^t_A \underset{\sim}{\mathbf{N}} \mathbf{P}_B\}$, or, written alternatively, $\partial_3\{P^+_A P^-_B - P^-_A P^+_B\}$. [For comparison, Fokkema and van den Berg consider the interaction quantity $\partial_k\{P_A V_{k,B} - V_{k,A} P_B\}$]. Apparently, we consider the interaction between oppositely propagating waves. Applying the product rule for differentiation, substituting the one-way wave equation (2) for states A and B, integrating the result over a cylindrical volume $\mathcal{V}$ with boundary $\partial\mathcal{V}_0 \cup \partial\mathcal{V}_1$ (see Figure 1), applying the theorem of Gauss and using the first symmetry relation in equation (6) yields the following one-way reciprocity theorem

$$\int_{\mathbf{x}\in\partial\mathcal{V}_0} \mathbf{P}^t_A \underset{\sim}{\mathbf{N}} \mathbf{P}_B n_3 \mathrm{d}^2\mathbf{x}_H = \int_{\mathbf{x}\in\mathcal{V}} \mathbf{P}^t_A \underset{\sim}{\mathbf{N}} (\hat{\underset{\sim}{\mathbf{B}}}_B - \hat{\underset{\sim}{\mathbf{B}}}_A) \mathbf{P}_B \mathrm{d}^3\mathbf{x} + \int_{\mathbf{x}\in\mathcal{V}} \{\mathbf{P}^t_A \underset{\sim}{\mathbf{N}} \mathbf{S}_B + \mathbf{S}^t_A \underset{\sim}{\mathbf{N}} \mathbf{P}_B\} \mathrm{d}^3\mathbf{x}, \tag{7}$$

with $n_3 = -1$ at the upper surface of $\partial\mathcal{V}_0$ and $n_3 = +1$ at the lower surface of $\partial\mathcal{V}_0$.

Reciprocity theorem of the correlation type for one-way wave fields

We consider the interaction quantity $\partial_3\{\mathbf{P}^\dagger_A \underset{\sim}{\mathbf{J}} \mathbf{P}_B\}$, or, written alternatively, $\partial_3\{(P^+_A)^* P^+_B - (P^-_A)^* P^-_B\}$, where * denotes complex conjugation. Following the same procedure as above, using the second symmetry relation in equation (6), yields the following one-way reciprocity theorem

$$\int_{\mathbf{x}\in\partial\mathcal{V}_0} \mathbf{P}^\dagger_A \underset{\sim}{\mathbf{J}} \mathbf{P}_B n_3 \mathrm{d}^2\mathbf{x}_H = \int_{\mathbf{x}\in\mathcal{V}} \mathbf{P}^\dagger_A \underset{\sim}{\mathbf{J}} (\hat{\underset{\sim}{\mathbf{B}}}_B - \hat{\underset{\sim}{\mathbf{B}}}'_A) \mathbf{P}_B \mathrm{d}^3\mathbf{x} + \int_{\mathbf{x}\in\mathcal{V}} \{\mathbf{P}^\dagger_A \underset{\sim}{\mathbf{J}} \mathbf{S}_B + \mathbf{S}^\dagger_A \underset{\sim}{\mathbf{J}} \mathbf{P}_B\} \mathrm{d}^3\mathbf{x}. \tag{8}$$

Representation theorem for one-way wave fields

We introduce a one-way Green's matrix $\underset{\sim}{\mathbf{G}}$ which satisfies the following one-way wave equation

$$\partial_3 \underset{\sim}{\mathbf{G}}(\mathbf{x}, \mathbf{x}') = \bar{\hat{\underset{\sim}{\mathbf{B}}}}(\mathbf{x}) \underset{\sim}{\mathbf{G}}(\mathbf{x}, \mathbf{x}') + \underset{\sim}{\mathbf{I}} \delta(\mathbf{x} - \mathbf{x}'), \tag{9}$$

with $\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3)$ and $\underset{\sim}{\mathbf{I}}$ being the 2×2 identity matrix. $\bar{\hat{\underset{\sim}{\mathbf{B}}}}$ is some reference one-way operator.

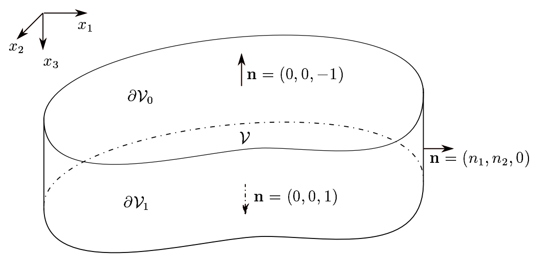


Figure 1: *Configuration for the one-way reciprocity theorems. The combination of the two planar surfaces is denoted by $\partial\mathcal{V}_0$; the cylindrical surface is denoted by $\partial\mathcal{V}_1$.*

The two columns of $\underset{\sim}{\mathbf{G}}(\mathbf{x},\mathbf{x}')$ represent two independent Green's one-way wave vectors at observation point $\mathbf{x}$, related to two independent one-way sources at source point $\mathbf{x}'$. Using reciprocity theorem (7) we can derive $\underset{\sim}{\mathbf{G}}(\mathbf{x}',\mathbf{x}'') = -\underset{\sim}{\mathbf{N}}^{-1}\underset{\sim}{\mathbf{G}}^t(\mathbf{x}'',\mathbf{x}')\underset{\sim}{\mathbf{N}}$, or

$$\underset{\sim}{\mathbf{G}}(\mathbf{x}',\mathbf{x}'') = \begin{pmatrix} G^{+,+}(\mathbf{x}',\mathbf{x}'') & G^{+,-}(\mathbf{x}',\mathbf{x}'') \\ G^{-,+}(\mathbf{x}',\mathbf{x}'') & G^{-,-}(\mathbf{x}',\mathbf{x}'') \end{pmatrix} = \begin{pmatrix} -G^{-,-}(\mathbf{x}'',\mathbf{x}') & G^{+,-}(\mathbf{x}'',\mathbf{x}') \\ G^{-,+}(\mathbf{x}'',\mathbf{x}') & -G^{+,+}(\mathbf{x}'',\mathbf{x}') \end{pmatrix}. \tag{10}$$

Next we derive a representation for the one-way wave field vector $\mathbf{P}$, obeying equation (2). This wave field vector will be used as state B in the reciprocity theorem (7). The Green's matrix $\underset{\sim}{\mathbf{G}}$, obeying equation (9), will play the role of state A. We thus obtain

$$\chi(\mathbf{x}')\mathbf{P}(\mathbf{x}') = \int_{\mathcal{V}} \underset{\sim}{\mathbf{G}}(\mathbf{x}',\mathbf{x})\mathbf{S}(\mathbf{x})\mathrm{d}^3\mathbf{x} - \int_{\partial\mathcal{V}} \underset{\sim}{\mathbf{G}}(\mathbf{x}',\mathbf{x})\mathbf{P}(\mathbf{x})n_3\mathrm{d}^2\mathbf{x}_H + \int_{\mathcal{V}} \underset{\sim}{\mathbf{G}}(\mathbf{x}',\mathbf{x})\hat{\underset{\sim}{\boldsymbol{\Delta}}}(\mathbf{x})\mathbf{P}(\mathbf{x})\mathrm{d}^3\mathbf{x}, \tag{11}$$

where the characteristic function χ and the contrast operator $\hat{\underset{\sim}{\boldsymbol{\Delta}}}$ are defined as

$$\chi(\mathbf{x}') = \begin{cases} 1 & \text{for } \mathbf{x}' \in \mathcal{V} \\ 1/2 & \text{for } \mathbf{x}' \in \partial\mathcal{V} \\ 0 & \text{for } \mathbf{x}' \notin \mathcal{V}\cup\partial\mathcal{V} \end{cases} \quad \text{and} \quad \hat{\underset{\sim}{\boldsymbol{\Delta}}}(\mathbf{x}) = \hat{\underset{\sim}{\mathbf{B}}}(\mathbf{x}) - \hat{\bar{\underset{\sim}{\mathbf{B}}}}(\mathbf{x}). \tag{12}$$

Note that the right-hand side of equation (11) contains, respectively, a 'direct wave' contribution, a boundary integral over the 'interaction quantity' $\underset{\sim}{\mathbf{G}}\mathbf{P}$ and a volume integral over the 'contrast operator' $\hat{\underset{\sim}{\boldsymbol{\Delta}}}$.

Applications

The one-way reciprocity theorems (7) and (8) as well as the one-way representation theorem (11) have many applications. We mention forward and inverse extrapolation in finely layered media, multiple elimination, and time-lapse seismics. As an example, here we discuss multiple elimination, using a similar approach as Fokkema and van den Berg (1993) and van Borselen et al. (1996, Geophysics). We start by deriving an integral equation for multiples, related to a reflecting boundary. This boundary may represent the Earth's free surface, the ocean bottom, or any reflector in the subsurface. We denote this reflecting boundary by Σ. We define the volume $\mathcal{V}$ entirely below Σ, in such a way that its upper boundary approaches Σ (as a limiting process) from below and its lower boundary lies below all inhomogeneities in the subsurface. We denote the upper boundary by Σ^+ and the lower by Σ_m, respectively, see Figure 2b. We employ representation (11) to this configuration, where $\mathbf{P}$ denotes the one-way wave vector related to the actual situation (hence, P^+ and P^- include the multiples related to the reflector Σ). We assume that the source is situated at or above Σ, so the first volume integral on the right-hand side of equation (11) vanishes. Moreover, the boundary integral in equation (11) reduces to an integral over Σ^+ only, with $n_3 = -1$. Throughout $\mathcal{V}$ we choose the reference medium equal to the actual medium, so the last volume integral on the right-hand side of equation (11) vanishes as well. Hence, for $\mathbf{x}' \in \Sigma^+$ we obtain

$$\frac{1}{2}\mathbf{P}(\mathbf{x}') = \int_{\Sigma^+} \underset{\sim}{\mathbf{G}}(\mathbf{x}',\mathbf{x})\mathbf{P}(\mathbf{x})\mathrm{d}^2\mathbf{x}_H. \tag{13}$$

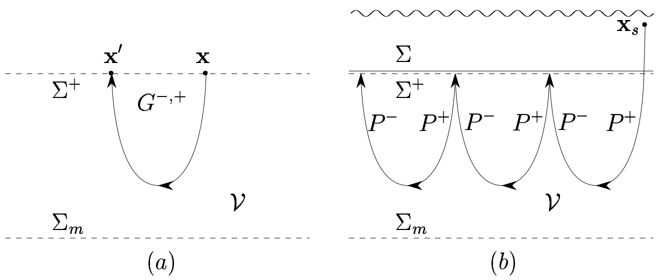
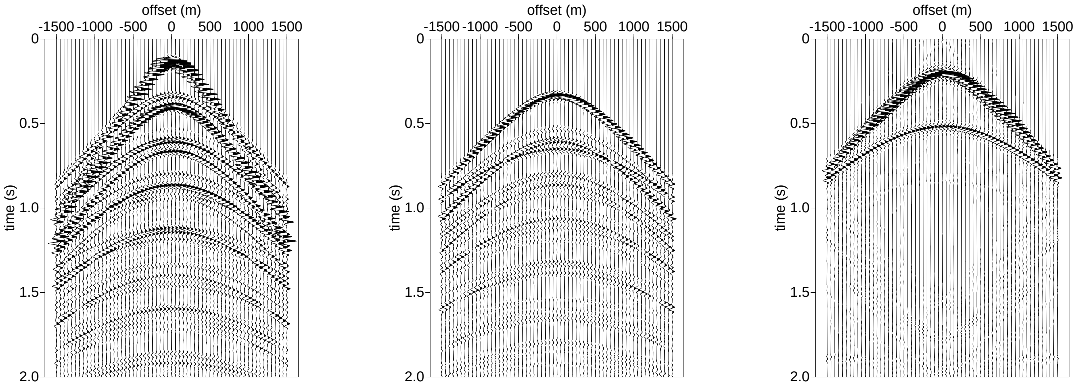


Figure 2: *(a) State A: the multiple-free impulse response* $G^{-,+}(\mathbf{x}',\mathbf{x})$. *(b) State B: the actual one-way responses* $P^+(\mathbf{x})$ *and* $P^-(\mathbf{x}')$*, including ocean-bottom related multiples.*

We have not yet specified the reference medium for the Green's matrix outside $\mathcal{V}$. In the following we choose a non-scattering reference medium in the half-space above Σ^+ (Figure 2a). The lower-left element



(a) P^+ at Σ^+ *(b) P^- at Σ^+* *(c) $G^{-,+}$ at Σ^+*

Figure 3: *Downgoing (a) and upgoing (b) wave fields at Σ^+, obtained by decomposing the pressure and velocity at the ocean bottom Σ. (c) The multiple-free response $G^{-,+}$, obtained by inverting equation (15).*

of the Green's matrix, i.e., $G^{-,+}(\mathbf{x}',\mathbf{x})$ represents the reflection impulse response of the medium in $\mathcal{V}$ *without* multiples related to the reflector Σ; this is the response that we are after. The upper-right element is the reflection response of the medium above Σ^+, hence it equals zero. The diagonal elements of $\underset{\sim}{\mathbf{G}}$ are the direct downgoing and upgoing waves at Σ^+. From equation (9) it follows that they are given by $\pm\frac{1}{2}\delta(\mathbf{x}'_H - \mathbf{x}_H) = \pm\frac{1}{2}\delta(x'_1 - x_1)\delta(x'_2 - x_2)$. The factor $\frac{1}{2}$ is due to the fact that $\mathbf{x}'$ and $\mathbf{x}$ are both on Σ^+; the plus and minus signs follow from the boundary conditions (i.e., outgoing waves for $x_3 \to -\infty$ and $x_3 \to \infty$). Hence, we may rewrite equation (13) for $\mathbf{x}' \in \Sigma^+$ as

$$\frac{1}{2}\begin{pmatrix} P^+(\mathbf{x}') \\ P^-(\mathbf{x}') \end{pmatrix} = \int_{\Sigma^+} \begin{pmatrix} \frac{1}{2}\delta(\mathbf{x}'_H - \mathbf{x}_H) & 0 \\ G^{-,+}(\mathbf{x}',\mathbf{x}) & -\frac{1}{2}\delta(\mathbf{x}'_H - \mathbf{x}_H) \end{pmatrix} \begin{pmatrix} P^+(\mathbf{x}) \\ P^-(\mathbf{x}) \end{pmatrix} \mathrm{d}^2\mathbf{x}_H, \tag{14}$$

from which we obtain

$$P^-(\mathbf{x}') = \int_{\Sigma^+} G^{-,+}(\mathbf{x}',\mathbf{x}) P^+(\mathbf{x}) \mathrm{d}^2\mathbf{x}_H, \tag{15}$$

for $\mathbf{x}' \in \Sigma^+$. Note that we have obtained an integral equation for the reflection impulse response $G^{-,+}(\mathbf{x}',\mathbf{x})$ which does not contain multiples related to the reflector Σ (Figure 2a). The one-way wave fields $P^+(\mathbf{x})$ and $P^-(\mathbf{x}')$ represent the actual situation including the multiples related to Σ. Since $G^{-,+}$ depends on $\mathbf{x}'$ as well as on $\mathbf{x}$, it can not be resolved from a single seismic experiment (except when there are no lateral variations in which case $G^{-,+}(\mathbf{x}',\mathbf{x})$ is invariant with respect to lateral shifts). Hence, in general equation (15) can only be resolved when many independent seismic experiments (related to different source positions) are available. In principle it involves a multi-dimensional deconvolution of the upgoing wave fields by the downgoing wave fields. The idea of resolving the multiple-free response by 'dividing' the upgoing waves by the downgoing waves has been discussed before by amongst others Kennett (1979, Geoph. Prosp.), Berkhout (1982, Elsevier), Wapenaar and Verschuur (1996, Delphi Acquisition project, Delft), Ziolkowski et al. (1998, SEG) and Amundsen (1999, SEG). The latter author also uses representation theory, analogous to Fokkema and van den Berg, to arrive at an integral equation for surface multiples, similar to equation (15). The present result is valid for surface multiples, ocean bottom multiples or internal multiples. Unfortunately, direct inversion of equation (15) is unstable, because the spectrum of the wave field $P^+(\mathbf{x})$ contains notches due to the multiples. In Wapenaar and Verschuur (1996, Delphi Acquisition project, Delft) we discussed a procedure to stabilize this inversion. An example is presented in Figure 3.

Conclusions

We have derived reciprocity and representation theorems for one-way wave fields in media with anelastic losses. Applications are found in forward and inverse extrapolation in finely layered media, multiple elimination (free-surface, ocean bottom or internal) and in time-lapse seismics.