AVA for self-scaling interfaces

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Summary

Angle-dependent reflection functions are generally derived for step-functions of the medium parameters. A step-function is a special case of a self-scaling singularity. In this paper we investigate the AVA reflection behaviour of self-scaling interfaces. It appears that for $f \to 0$ the self-scaling interface acts as a step function, whereas for $f \to \infty$ the AVA reflection behaviour is dominated by the scaling parameters of the interface.

Introduction

Angle-dependent reflection functions are generally derived by solving the boundary conditions at an interface between two homogeneous half-spaces. For two elastic half-spaces this yields the well-known Zoeppritz equations which, in turn, are often used as the basis for AVA-inversion (Ostrander, 1984; Smith and Gidlow, 1987).

By considering two homogeneous half-spaces it is implicitly assumed that the medium parameters as a function of depth are described by step-functions. Obviously a step-function is just one specific example of a more general class of functions that describe the transition from one constant value to another. Recently there has been much interest in the analysis of "self-scaling" singularities (i.e., singular functions that have the same appearance at different scales; Mallat and Hwang, 1992; Herrmann, 1994).

The aim of this paper is to investigate the AVA behaviour of interfaces that are characterized by self-scaling singularities. For convenience we consider the acoustic case. For normal incidence we consider analytical solutions. These serve as a check for the numerical code that is used to model the responses for oblique incidence.

A self-scaling medium

First we consider the rather academic situation of a medium that is self-scaling from $z = -\infty$ to $z = \infty$. Let the compressibility $\kappa(z)$ be defined according to

$$\kappa(z) = \kappa_0 |z|^{\alpha}, \tag{1}$$

where α is a parameter that describes the "strength" of the singularity (with $\alpha > -1$). The mass density is chosen constant, according to $\varrho(z) = \varrho_0$, hence, the propagation velocity reads

$$c(z) = c_0 |z|^{-\frac{\alpha}{2}}, \tag{2}$$

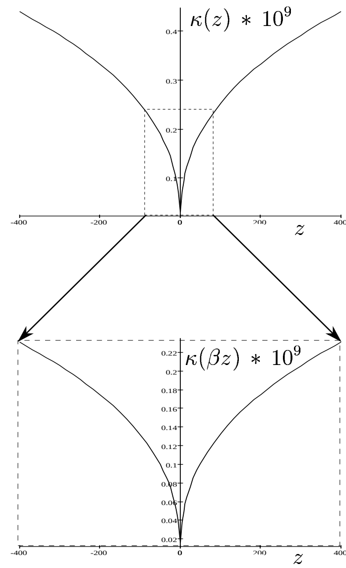

Fig. 1: *The self-scaling function $\kappa(z) = \kappa_0|z|^{\alpha}$, with $\kappa_0 = 4 * 10^{-11}$ and $\alpha = 0.4$. The "zoom-factor" is chosen as $\beta = .2$, hence, the vertical scaling factor is $\beta^{\alpha} = .525$.*

with $c_0 = \{\kappa_0 \varrho_0\}^{-\frac{1}{2}}$. The self-scaling behaviour of $\kappa(z)$ and $c(z)$ can be mathematically expressed as

$$\kappa(\beta z) = \beta^{\alpha} \kappa(z) \quad \text{and} \quad c(\beta z) = \beta^{-\frac{\alpha}{2}} c(z), \tag{3}$$

for $\beta > 0$. This property is visualized in Figure 1, where it becomes clear that the medium "looks the same" at different scales. For vertical wave propagation through the medium defined by equations (1) and (2) the wave equation can be solved analytically (Wapenaar, 1996). Its solution can be expressed in terms of Hankel functions $\mathrm{H}_{\nu}^{(1,2)}$ of fractional order $\nu = (2 + \alpha)^{-1}$, according to

$$P(z, \omega) \sim \sqrt{|z|}\, \mathrm{H}_{\nu}^{(1,2)}(\zeta), \quad \text{for} \quad z \neq 0, \tag{4}$$

etc., where $\zeta = 2\nu\omega|z|^{\frac{1}{2\nu}}/c_0$. Throughout this paper we choose $\omega > 0$. Note that $P(z, \omega)$ obeys the following scaling property

$$P(\beta z, \omega) = \sqrt{\beta}\, P(z, \beta^{\frac{1}{2\nu}} \omega), \tag{5}$$

for $\beta > 0$. Apparently a scaling of the depth implies a scaling of the frequency and vice versa.

Reflection and transmission coefficients

For the half-space $z < 0$ we express the total wave field as the sum of an "incident" and "reflected" wave field, ac-

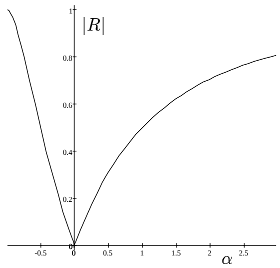

Fig. 2: Modulus of reflection coefficient as a function of α.

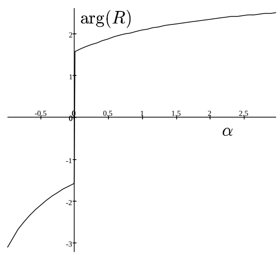

Fig. 3: Phase of reflection coefficient as a function of α.

cording to

$$P(z,\omega) = \sqrt{|z|}\,\mathrm{H}_\nu^{(1)}(\zeta) - R\sqrt{|z|}\,\mathrm{H}_\nu^{(2)}(\zeta) \qquad (6)$$

and for $z > 0$ as a "transmitted" wave field, according to

$$P(z,\omega) = -T\sqrt{|z|}\,\mathrm{H}_\nu^{(2)}(\zeta). \qquad (7)$$

The solutions for $z < 0$ and $z > 0$ are connected by assuming continuity of $P(z,\omega)$ and $\frac{\partial P(z,\omega)}{\partial z}$ at $z = 0$. This yields

$$T = 1 + R \quad \text{with} \quad R = \frac{j\cos(\nu\pi)}{\sin(\nu\pi) - j\cos(\nu\pi)}, \qquad (8)$$

see Figures 2 and 3. Note that these coefficients are independent of the frequency ω ! For small α equation (8) may be approximated by $R \approx \frac{\pi\alpha}{4}j$. This means that in the time domain the reflection response is approximately proportional to the Hilbert transform of the incident field.

Normal incidence example (analytical)

For $\alpha = 0.8$ and $c_0 = 5000$ m/s, the real part of the total pressure (equations 6, 7 and 8) is shown in Figure 4 for frequencies ranging from 5 to 75 Hz. From this figure the scaling behaviour expressed by equation (5) can clearly be

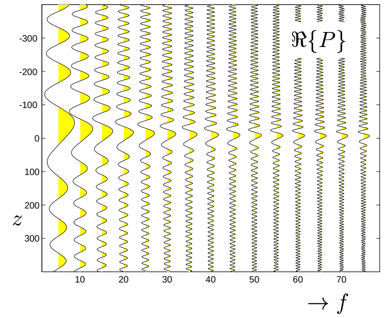

Fig. 4: Real part of total pressure P.

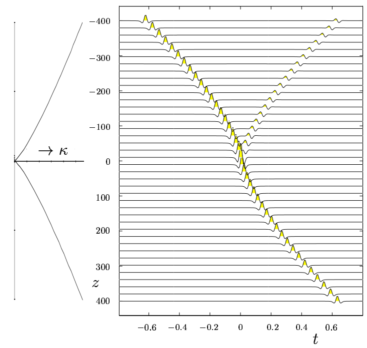

Fig. 5: Time domain solution in a VSP-like presentation.

observed. Time domain solutions were obtained by multiplying these data for each depth level z with the spectrum of a Ricker wavelet (with central frequency f_0 = 20 Hz) and by applying an inverse Fourier transform to the time domain (to this end a much denser frequency sampling was used than is shown in Figure 4). The result is shown in Figure 5. Note that the scattered waves appear to originate from the singularity at $z = 0$ and that the reflection response is indeed approximately proportional to the Hilbert transform of the incident field. This result served as a reference for evaluating the validity of "layer-code" modeling for self-scaling media. For the central frequency of the Ricker wavelet (f_0 = 20 Hz) and the lowest velocity ($c(z = 400) = 450$m/s) the wavelength is 22.5 m. For the numerical modeling we chose a layer thickness of $\Delta z = 1$ m, which is significantly smaller than the wavelength. For normal incidence, the numerical results appeared to be nearly indistinguishable from the analytical results. This confirms that it is allowed to model the response of a self-scaling medium numerically, provided that the discretization interval is sufficiently small in comparison with the seismic wavelength.

Oblique incidence example (numerical)

We performed numerical modeling for a range of rayparameters p (i.e., for a range of propagation angles). The results at $z = -400$ m (the "reflection" response) and at $z = 400$ m (the "transmission" response) are shown in Figure 6 (the traces at $p = 0$ are the same as the first and last trace in Figure 5).

The depth-dependent propagation angle $\theta(z)$ is related to the rayparameter p and the propagation velocity $c(z)$, according to $p = [\sin\theta(z)]/c(z)$. Hence, above the singularity, $\theta(z)$ increases with depth for any fixed $p \neq 0$. Since the velocity at the singularity is infinite, $\theta(z)$ becomes 90 degrees before the singularity is reached. Hence, for any $p \neq 0$ there is a turning point above the singularity. For low p-values this turning point is in the vicinity of the singularity and "tunneling" occurs (that is, waves are transmitted to the lower half-space, despite the turning point, see Figure 6). For larger p-values the turning point is sufficiently far above the singularity (in terms of wavelengths), so that no waves penetrate into the lower half-space and total reflection occurs.

From the reasoning above it follows that the angle-dependent coefficients must be a function of frequency as well. Figure 7 shows the reflection coefficient as a function of the rayparameter p and the frequency f (obtained from the Fourier transformed reflection response in Figure 6). For $p = 0$ this numerically obtained coefficient appears to be frequency-independent (as it should be). Its numerical value reads $|R| = 0.433$ (for comparison, from equation (8) we obtain $|R| = 0.434$). For increasing p the reflection coefficient becomes indeed frequency dependent. For sufficiently high p and f it approaches unity, due to the above mentioned turning point effects.

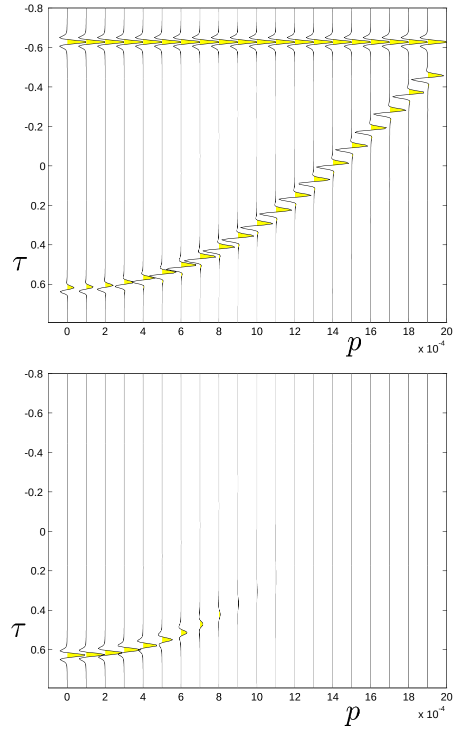

__Fig. 6:__ Angle-dependent "reflection" and "transmission" response obtained with numerical modeling.

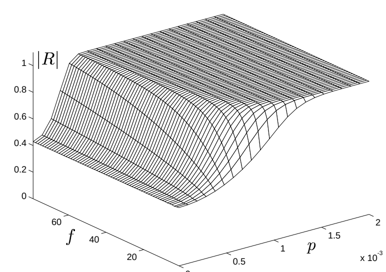

__Fig. 7:__ Angle-dependent $|R|$ for different frequencies.

A self-scaling interface

Next we consider a self-scaling interface, i.e., a self-scaling singularity embedded between two homogeneous half-spaces. Let the velocity function be defined by

$$c(z) = \begin{cases} c_0 & \text{for } z < z_0 \\ c_1|z - z_1|^{-\alpha_1/2} & \text{for } z_0 < z < z_1 \\ c_2|z - z_1|^{-\alpha_2/2} & \text{for } z_1 < z < z_2 \\ c_3 & \text{for } z_2 < z \end{cases} \tag{9}$$

and let this function be continuous at z_0 and z_2. For this situation the normal incidence reflection and transmission coefficients can again be calculated analytically. The resulting expressions are quite involved (Wapenaar, 1996). As can be understood the coefficients appear to be frequency dependent. For $f \to 0$ we obtain $R \to (c_3 - c_0)/(c_3 + c_0)$, hence, the interface acts as a step function. For $f \to \infty$ the reflection and transmission coefficients are dominated by the scaling behaviour between z_0 and z_2; for the special case that $\alpha_1 = \alpha_2$ we obtain

$$R \to \frac{\left(\frac{c_2^{2\nu} - c_1^{2\nu}}{c_2^{2\nu} + c_1^{2\nu}}\right)\sin(\nu\pi) + j\cos(\nu\pi)}{\sin(\nu\pi) - j\cos(\nu\pi)}. \tag{10}$$

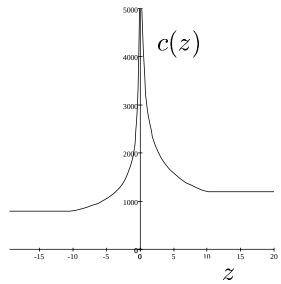

Fig. 8: *Velocity function for a self-scaling interface*

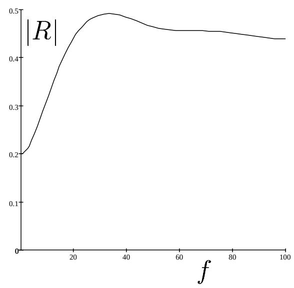

Fig. 9: *Reflection coefficient of a self-scaling interface*

Normal incidence example (analytical)

We consider a self-scaling interface described by eqn (9), with $\{z_0, z_1, z_2\} = \{-10, 0, 10\}$m, $\{c_0, c_1, c_2, c_3\} = \{800, 2000, 3000, 1200\}$m/s and $\{\alpha_1, \alpha_2\} = \{0.8, 0.8\}$, see Figure 8. The analytically obtained frequency-dependent reflection coefficient is shown in Figure 9. Note the limiting behaviour for $f \rightarrow 0$ ($|R| \rightarrow 0.2$) and for $f \rightarrow \infty$ ($|R| \rightarrow 0.453$).

Oblique incidence example (numerical)

For the self-scaling interface defined in the previous subsection we modeled the reflection response numerically for a range of angles and frequencies. The reflection coefficient derived from this response is shown in Figure 10. For $\theta = 0$ and variable frequency we recognize the function of Figure 9. For $f \rightarrow 0$ and variable angle we recognize the well known angle-dependent reflection coefficient of a step function.

Conclusions

We have analyzed the seismic responses of self-scaling singularities. For normal incidence these responses have been modeled analytically as well as numerically. It appeared that the numerically obtained responses match the analytical responses very well, provided that the discretization of the medium is chosen sufficiently fine in relation with the seismic wavelength.

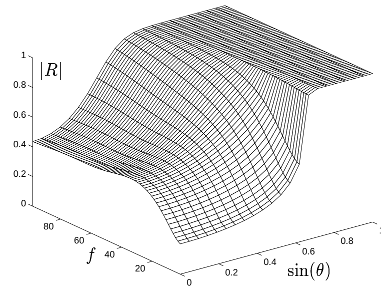

Fig. 10: *Angle-dependent $|R|$ for different frequencies.*

For a medium that is self-scaling from $z = -\infty$ to $z = \infty$, according to equations (1) and (2), it appeared that the normal incidence reflection coefficient is independent of the frequency; the angle-dependent reflection coefficient, however, is different for different frequencies.

For a self-scaling interface (i.e., a self-scaling singularity embedded between homogeneous half-spaces) the reflection coefficient is frequency-dependent for normal incidence as well as for oblique incidence. For $f \rightarrow 0$ the angle-dependent reflection behaviour is fully determined by the parameters of the embedding; for $f \rightarrow \infty$ the angle-dependent effects are dominated by the scaling behaviour of the singularity. It seems worthwhile to investigate the potential of reflection functions of self-scaling interfaces in AVA-inversion.

References

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