Scale-dependency of Thomsen parameters for layers with intermediate thickness

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Summary

The Thomsen anisotropy parameters (Thomsen, 1986) describe the most important aspects of anisotropy on velocity. In the case of anisotropy caused by fine layering effects, the parameters are in fact defined in the long wavelength limit, meaning that it is implicitly assumed that all the fine layering is very fine compared to the dominant seismic wavelength. When this assumption is not met (which is often the case to some degree in practical cases), the Thomsen anisotropy parameters can still be calculated using a moving averaging technique. It will be shown with an example that the anisotropy parameters derived in this way may become strongly dependent on the scale (i.e. related to the size of the averaging window) they are calculated on.

Introduction: long wavelength limit and assumptions

In the case of anisotropy caused by fine layering, the Thomsen anisotropy parameters (Thomsen, 1986) are derived in the long wavelength limit, implicitly assuming that all the fine layering is much smaller than the seismic wavelength. A second assumption is that the interval used to calculate the anisotropy parameters, i.e. the thickness of the model layer, should be large compared to the average thickness of the fine layering within. A third implicit assumption is that there should be no change in the characteristics of the fine layering within such a model layer/interval, i.e. the statistics of the fine layering should be constant in depth.

In reality, these assumptions are often not met; it is often observed that the thickness of the "layering" extends beyond what is allowed for the long wavelength assumption (roughly situated at about $\frac{\lambda_{\text{dom}}}{d} \approx 5$ - 10, with λ_{dom} the dominant wavelength and d the layering thickness (Sams and Williamson, 1994; Rio et al., 1996)). The second assumption is often violated because of the trend towards more detailed models, so the model layer thicknesses used nowadays often become small compared to the thickness of some of the fine layering within. In addition, the fine layering is often changing within a given model layer, thus violating assumption three. This may be caused for instance by changing sedimentological conditions over the often large time-interval of deposition.

When these assumptions are not all honored, the Thomsen parameters should not be calculated in the conventional way, i.e. using a long wavelength limit approach. An alternative method is then to calculate the Thomsen parameters using a moving averaging approach.

In the next section, the conventional calculation of the Thomsen parameters will first be reviewed, this means under the assump-

tion of a long wavelength limit. The method will subsequently be extended to cases where the long wavelength assumption cannot be made in the following section. Finally the proposed method will be applied to a well-log and the results will be discussed.

Thomsen parameters in the long wavelength limit

Hooke's law and VTI-media
In general, a linear elastic material is decribed using Hooke's law. Using the so-called condensed Voigt notation, it can be written in the following way:

$$\vec{\sigma} = \mathbf{c}\ \vec{\epsilon}, \tag{1}$$

with $\vec{\epsilon}$ the vector with strain components and $\vec{\sigma}$ the stress components. The elastic modulus matrix $\mathbf{c}$ is symmetric and characterized by 21 independent elastic constants in the most general case (Aki and Richards, 1980; Dahlen and Tromp, 1998).

An important special case is when the z-axis is a symmetry axis, i.e. the material is a so-called Vertical Transversely Isotropic (VTI) material. The elastic matrix $\mathbf{c}$ for a VTI-material is described by:

$$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}, \tag{2}$$

with $c_{12} = c_{11} - 2c_{66}$, resulting in five independent elastic coefficients c_{ij}, for this specific type of anisotropic material.

Fine layering: anisotropic equivalent medium
Three basic classes can be recognized in the causes for anisotropy (Crampin et al., 1984), one of which is fine layering. Since long, it is known that fine layering expresses itself on a larger scale as anisotropy (Postma, 1955; Backus, 1962); even when individual layers are isotropic, the effect of fine layering on seismic wave propagation at low frequencies is the same as when the layered medium is replaced by a homogeneous, transversely isotropic material. When the layering is perpendicular to the z-axis, this replacement medium becomes a VTI-medium. The elastic coefficients for the equivalent medium can be derived from the properties of the individual layers by requiring that the elastic energy density for the equivalent medium needs to be the same as the sum of the elastic energy density in the individual layers, and this for every possible stress and strain condition (Aki and Richards, 1980). This leads to the following four relationships between the elastic coefficients of the individual layers and the ones for the equivalent medium (denoted by

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the subscript "e"):

$$c_{11e} - \frac{c_{13e}^2}{c_{33e}} = \left\langle c_{11} - \frac{c_{13}^2}{c_{33}} \right\rangle, \qquad \frac{1}{c_{33e}} = \left\langle \frac{1}{c_{33}} \right\rangle,$$

$$\frac{1}{c_{44e}} = \left\langle \frac{1}{c_{44}} \right\rangle, \qquad c_{66e} = \langle c_{66} \rangle, \qquad (3)$$

$$\text{with } \langle x \rangle = \frac{\sum_i h_i x_i}{\sum_i h_i},$$

the weighted average of property x, using the thickness of each layer, h_i, as weight. These four relationships are however not sufficient to obtain the five independent elastic coefficients. An additional relationship was suggested by Bruggeman (1937), i.e.:

$$\frac{c_{13e}}{c_{33e}} = \left\langle \frac{c_{13}}{c_{33}} \right\rangle, \qquad (4)$$

but also alternative relationships are possible (see e.g. Backus (1962)). Using the additional relationship, the five elastic coefficients for the equivalent (anisotropic) medium can be calculated. This system of equations can also be replaced by any five independent linear combinations. In the special case of isotropic individual layers, the so-called Backus parameters (Backus, 1962) are a particular useful set; since the Backus parameters can be formulated in quantities directly observed in well-logs (Folstad and Schoenberg, 1992):

$$S = \left\langle \rho \frac{c_S^4}{c_P^2} \right\rangle, \qquad T = \left\langle \frac{c_S^2}{c_P^2} \right\rangle,$$

$$R = \left\langle \frac{1}{\rho c_P^2} \right\rangle, \qquad \frac{1}{L} = \left\langle \frac{1}{\rho c_S^2} \right\rangle, \qquad (5)$$

$$M = \left\langle \rho c_S^2 \right\rangle.$$

The elastic parameters for the equivalent medium can in turn be calculated from these Backus parameters as (Backus, 1962):

$$c_{11e} = \frac{(1 - 2T)^2}{R} + 4(M - S),$$

$$c_{13e} = \frac{1 - 2T}{R}, \qquad c_{33e} = \frac{1}{R}, \qquad (6)$$

$$c_{44e} = L, \qquad c_{66e} = M.$$

Thomsen anisotropic parameters
Note, that it is hard to appreciate the strength of anisotropy and its effects on seismic wavefields from the elastic coefficients as derived in the previous subsection. The P-wave phase velocity c_ϕ in a VTI-medium can be written as a function of the elastic coefficients c_{ij} in the following way[1] (Tsvankin, 1996):

$$\frac{c_\phi^2(\theta)}{\alpha^2} = 1 + \epsilon \sin^2 \theta - \frac{f}{2}$$
$$+ \frac{f}{2} \sqrt{\left(1 + \frac{2\epsilon \sin^2 \theta}{f}\right)^2 - \frac{2(\epsilon - \delta) \sin^2 2\theta}{f}}, \qquad (7)$$

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[1] Note that the elastic coefficients c_{ij} do not have a subscript "e" because these formulas are valid for any type of VTI-medium, not only for equivalent media caused by fine layering.

where $f = 1 - \frac{\beta^2}{\alpha^2} = 1 - \frac{c_{44}}{c_{33}}$,

and with θ the phase angle measured from the symmetry axis, and

$$\alpha = \sqrt{\frac{c_{33}}{\rho}}, \qquad \beta = \sqrt{\frac{c_{44}}{\rho}},$$

$$\gamma = \frac{c_{66} - c_{44}}{2c_{44}}, \qquad \epsilon = \frac{c_{11} - c_{33}}{2c_{33}}, \qquad (8)$$

$$\delta = \frac{(c_{13} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})},$$

the so-called Thomsen parameters (Thomsen, 1986). Assuming weak anisotropy ($|\delta| \ll 1$ and $|\epsilon| \ll 1$), Equation (7) leads to:

$$c_\phi(\theta) = \alpha(1 + \delta \sin^2 \theta \cos^2 \theta + \epsilon \sin^4 \theta), \qquad (9)$$

the weak-anisotropy approximation as derived by Thomsen (1986). From this equation, it is easy to understand the physical meaning of the two anisotropy parameters δ and ϵ. It is clear from Equation (9) that the parameter δ is more important than ϵ in describing the near-vertical effects of anisotropy. On the other hand, ϵ is the important factor describing the deviation from α towards the large angles.

For the derivations in this section, it was assumed that all the layering is fine enough compared to the wavelength of the seismic wave. The anisotropy parameters are often calculated for relative long intervals, assuming that the statistics of the fine layering remain constant. The choice of the intervals is also an important factor in the result. This is because the calculation of the anisotropy parameters is not a linear operation. This means that the (mean) anisotropy parameters for a long interval will not be the same as the mean of the anisotropy parameters for sub-intervals.

Thomsen parameters at intermediate wavelengths

Moving averaging procedure
When a part of the layering is not fine enough to be treated in the long wavelenght limit, another approach needs to be taken. This was already noticed by Backus (1962), as he clearly states: "... *the averaging process converts the finely layered, highly variable medium to a smoothed, transversely isotropic, long-wave equivalent (STILWE) medium* ...". Backus (1962) states as well that only the fine layering which is sufficiently fine compared to the seismic wavelength will be averaged out, a qualitative statement not further quantified.

Backus did not quantify the upper limit for the layer thicknesses that should be averaged out. Sams and Williamson (1994) and Rio et al. (1996) estimated the limit for normal incidence propagation from physical and numerical experiments at $\frac{\lambda_{\text{dom}}}{d} \approx 5$ - 10 (with λ_{dom} the dominant wavelength and d the layering thickness). The limit is dependent on the contrast between the materials, and for most practical cases (with relative modest contrasts) it can be taken as 5. For oblique incidence, it is known that the resolution of a plane wave decreases with increasing incidence angle (Wapenaar et al., 1999). The relationship becomes

then:

$$\frac{\lambda_{\text{dom}}}{d \, \cos \theta} \approx 5, \tag{10}$$

with θ the angle between the normal on the plane wave and the vertical. The fine layering with a thickness $d < \frac{\lambda_{\text{dom}}}{5 \, \cos \theta}$ should be averaged using equivalent medium techniques, while the intermediate and coarse layering with $d > \frac{\lambda_{\text{dom}}}{5 \, \cos \theta}$ should be preserved in the smoothed medium. A way to get this effect was proposed (for normal incidence) by Sams and Williamson (1994) and Rio et al. (1996) by means of a moving averaging procedure. This method is adapted here using a Gaussian window function, which is a smooth window. This makes the procedure related to the continuous Gaussian wavelet transform (Mallat, 1999) and the related analysis of scaling behaviour (Herrmann, 1997, 1998).

The equations for the Backus parameters (assuming that the individual layers are isotropic) can be rewritten using a moving averaging procedure. Equation (5) thus becomes:

$$\breve{S}(\sigma_z, z) = \frac{1}{\sigma_z} \int_{-\infty}^{+\infty} \rho(z') \frac{c_S^4(z')}{c_P^2(z')} \, e^{-\pi \left(\frac{z'-z}{\sigma_z}\right)^2} \, \mathrm{d}z',$$

$$\breve{T}(\sigma_z, z) = \frac{1}{\sigma_z} \int_{-\infty}^{+\infty} \frac{c_S^2(z')}{c_P^2(z')} \, e^{-\pi \left(\frac{z'-z}{\sigma_z}\right)^2} \, \mathrm{d}z',$$

$$\breve{R}(\sigma_z, z) = \frac{1}{\sigma_z} \int_{-\infty}^{+\infty} \frac{1}{\rho(z') c_P^2(z')} \, e^{-\pi \left(\frac{z'-z}{\sigma_z}\right)^2} \, \mathrm{d}z', \quad (11)$$

$$\breve{L}^{-1}(\sigma_z, z) = \frac{1}{\sigma_z} \int_{-\infty}^{+\infty} \frac{1}{\rho(z') c_S^2(z')} \, e^{-\pi \left(\frac{z'-z}{\sigma_z}\right)^2} \, \mathrm{d}z',$$

$$\breve{M}(\sigma_z, z) = \frac{1}{\sigma_z} \int_{-\infty}^{+\infty} \rho(z') c_S^2(z') \, e^{-\pi \left(\frac{z'-z}{\sigma_z}\right)^2} \, \mathrm{d}z',$$

which are the Backus parameters regularized towards the scale σ_z. The prime on the symbols for the Backus parameters denotes that the property is a result from a moving averaging procedure. Note that the Backus parameters are now scale-dependent. The scale-dependent elastic parameters can in turn be calculated using equations similar to Equation (6), using the scale-dependent Backus parameters. Subsequently the scale-dependent Thomsen parameters can be derived using equations similar to Equation (8).

Application: scale-dependent Thomsen parameters

Figure 1 shows the results for well A of the Mobil AVO data set (Keys and Foster, 1998). We note that the Thomsen parameter $\breve{\delta}$ has a strong scale-dependency, whereas $\breve{\varepsilon}$ shows almost no scale-dependency. There is a relatively large interval, $z = [2150$ m, 2600 m], where $\breve{\delta}$ is almost zero. This is caused by a high correlation between c_P and c_S in this interval. This means that in this interval the medium has an almost constant Poisson's ratio ν, leading to $\breve{\delta} \approx 0$ (Berryman et al., 1997). Some more examples will be discussed during the presentation.

Using Equations (9) and (10) the appropriate angle-dependent velocity can be calculated. This velocity may deviate from the equivalent medium velocity, calculated in the long wavelenght limit. This deviation becomes often more apparent for increasing angles/ray parameter. When using this velocity as background velocity in inversion, the resulting contrast parameters will also be different due to the non-linear dependency between contrast parameters and background velocity (van Wijngaarden, 1998). Commonly, the deviation will be minor for the acoustic impedance contrast, $\frac{\Delta Z}{Z}$, but relatively strong for the contrasts in P-wave velocity, $\frac{\Delta \bar{c}_P}{\bar{c}_P}$, and in shear modulus $\frac{\Delta \mu}{\bar{\mu}}$.

The results were also verified using numerical modeling, which compare favourable.

Conclusions

A method is proposed for the calculation of the Thomsen parameters in the case not all the fine layering is fine enough to be treated in the long wavelength limit. The method uses a moving averaging procedure, with the width of the window as a scale parameter. The Thomsen parameters derived in this way become scale-dependent. Using an example, it was shown that the scale-dependency can become large at some locations.

A complete discussion can be found in Verhelst (2000).

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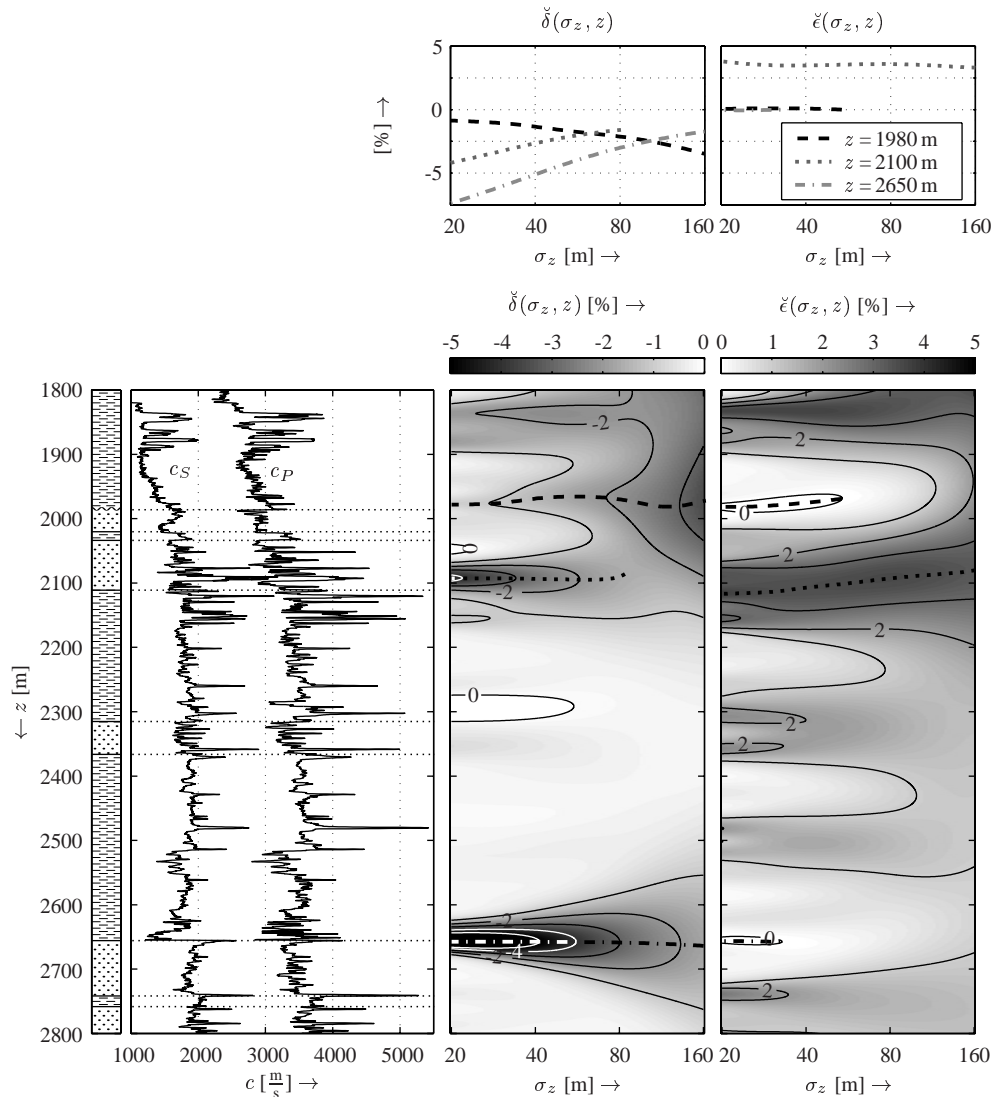
Fig. 1: The Thomsen parameters $\breve{\delta}$ and $\breve{\epsilon}$ for well A of the Mobil AVO data set (see Keys and Foster (1998) for a brief discussion). This well has oil and gas bearing sandstones. All well logs were measured (c_P, c_S and ρ). For a relatively large interval, $z = [2150\text{ m}, 2600\text{ m}]$, $\breve{\delta}$ is almost zero. In this interval, c_P and c_S have a high correlation coefficient of 0.94. There are two locations where $|\breve{\delta}|$ shows higher values: around $z = 2100$ m, and at about $z = 2650$ m. At these locations, the correlation between c_P and c_S is resp. -0.05 and 0.11. At these two locations (and at the Base Cretaceous Unconformity (BCU); $z = 1980$ m) the scale-dependency of two Thomsen parameters is plotted (the displays at the top). The Thomsen parameter $\breve{\delta}$ shows a clear scale dependency, whereas $\breve{\epsilon}$ almost shows no dependency at all.

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