# Initial conditions for elastodynamic Green's function retrieval by the Marchenko method

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## SUMMARY

With the acoustic Marchenko method it is possible to retrieve the Green's response to a virtual source in the subsurface from the single-sided reflection response at the surface. The acoustic Marchenko method relies on the separability of the Green's functions and focusing functions in the time domain. Recently, the first steps have been made towards extending the single-sided Marchenko method to the elastodynamic situation. A complication of the elastodynamic scheme is that the Green's functions and focusing functions partly overlap each other. The iterative Marchenko method does not retrieve the parts of the focusing functions that overlap with the Green's functions. Ideally, the overlapping parts of the focusing functions are defined as the initial condition for the iterative Marchenko method. This leads to a perfect retrieval of the elastodynamic Green's functions. In this paper we investigate the results of the Marchenko method when we use less stringent initial conditions. It appears that using an initial focusing function which contains single events leads to an accurate retrieval of two components of the Green's matrix, provided the converted reflection responses are included in the Marchenko scheme.

## INTRODUCTION

Building on work on 1D acoustic autofocusing of Rose (2001, 2002), Broggini and Snieder (2012) designed a method to retrieve the 1D Green's response to a virtual source in the subsurface from the single-sided reflection response at the surface. We extended this method to more dimensions (Wapenaar et al., 2013) and used it to design a new method of seismic imaging which accounts for internal multiples (Wapenaar et al., 2014; Broggini et al., 2014; Behura et al., 2014; Singh et al., 2015). Because this method is based on iteratively solving the Marchenko equation (Marchenko, 1955; Lamb, 1980; Chadan and Sabatier, 1989), we use the name "Marchenko imaging" next to "autofocusing". Snieder (2015) discusses Marchenko imaging in its broader historical context and van der Neut et al. (2015) explain the method in an intuitive way.

Recently, the first steps have been made towards extending Marchenko imaging to the elastodynamic situation (da Costa et al., 2014; Wapenaar, 2014). The extension from the acoustic to the elastodynamic situation is not trivial. The acoustic Marchenko method relies on the separability of the Green's functions and the so-called focusing functions in the time domain. In the elastodynamic situation the Green's functions and the focusing functions partly overlap each other (Wapenaar and Slob, 2014). Although we do not exclude that it is possible to retrieve these overlapping functions entirely from the reflection response, the methods developed to date do not retrieve the parts of the focusing functions that overlap in time with the Green's functions. These parts of the focusing functions are either ignored or should be estimated separately. This

implies different initial conditions for the iterative Marchenko scheme, leading to different accuracies of the retrieved Green's functions. Evaluating these differences is the aim of this paper.

## GREEN'S FUNCTION REPRESENTATIONS

For the analysis in this paper we restrict ourselves to oblique plane waves in a horizontally layered medium. Consider the following 1D elastodynamic Green's function representations in the rayparameter intercept time $(p, \tau)$ domain (Slob et al., 2014; Wapenaar, 2014)

$$\mathbf{G}^{-,+}(p,z_0,z_i,\tau) + \mathbf{F}_1^-(p,z_0,z_i,\tau) =$$
$$\int_{-\infty}^{\tau} \mathbf{R}(p,z_0,\tau-\tau')\mathbf{F}_1^+(p,z_0,z_i,\tau')\mathrm{d}\tau', \quad (1)$$

$$\mathbf{G}^{-,-}(p,z_0,z_i,\tau) + \mathbf{F}_1^+(-p,z_0,z_i,-\tau) =$$
$$\int_{-\infty}^{\tau} \mathbf{R}(p,z_0,\tau-\tau')\mathbf{F}_1^-(-p,z_0,z_i,-\tau')\mathrm{d}\tau'. \quad (2)$$

Here $\mathbf{R}(p,z_0,\tau)$ is the single-sided elastodynamic reflection response at the acquisition surface $z_0$. This surface is considered to be transparent, which corresponds to the situation after surface-related multiple elimination. Moreover, assuming the reflection response has been decomposed into compressional ($P$) and shear ($S$) waves, it can be written as

$$\mathbf{R}(p,z_0,\tau) = \begin{pmatrix} R_{P,P} & R_{P,S} \\ R_{S,P} & R_{S,S} \end{pmatrix}(p,z_0,\tau). \quad (3)$$

Here $R_{X,Y}$ stands for the reflection response that is obtained with sources for downgoing $Y$-waves at $z_0$, observed by receivers for upgoing $X$-waves at $z_0$. $\mathbf{G}^{-,+}(p,z_0,z_i,\tau)$ is the Green's function with sources for downgoing ($+$) waves at depth level $z_i$ and receivers for upgoing ($-$) waves at the surface $z_0$. It can be seen as a reflection response of which the sources have been redatumed from $z_0$ to $z_i$. $\mathbf{G}^{-,-}(p,z_0,z_i,\tau)$ is defined in a similar way, except that the sources at $z_i$ are now emitting upgoing waves. $\mathbf{F}_1^+(p,z_0,z_i,\tau)$ is a focusing function, containing the downgoing waves at $z_0$ that collapse to $\mathbf{I}\delta(\tau)$ at $z_i$ (if the half-space below $z_i$ would be homogeneous). $\mathbf{F}_1^-(p,z_0,z_i,\tau)$ is the upgoing response to $\mathbf{F}_1^+(p,z_0,z_i,\tau)$, observed at $z_0$ (if the half-space below $z_i$ would be homogeneous). The Green's functions and focusing functions are partitioned in a similar way as the reflection response, hence

$$\mathbf{G}^{-,\pm} = \begin{pmatrix} G_{P,P}^{-,\pm} & G_{P,S}^{-,\pm} \\ G_{S,P}^{-,\pm} & G_{S,S}^{-,\pm} \end{pmatrix}, \mathbf{F}_1^\pm = \begin{pmatrix} f_{P,P}^\pm & f_{P,S}^\pm \\ f_{S,P}^\pm & f_{S,S}^\pm \end{pmatrix}. \quad (4)$$

## ONE-WAY WAVE FIELD EXTRAPOLATION

Before we discuss the Marchenko method, let us briefly review standard one-way wave field extrapolation from $z_0$ to $z_i$.

Figure 1: *Horizontally layered elastic model.*



Figure 3: *Green's functions, obtained by one-way wave field extrapolation.*

Let $\mathbf{T}(p, z_i, z_0, \tau)$ denote the transmission response of the layered medium between $z_0$ and $z_i$. We define $\mathbf{T}_d(p, z_i, z_0, \tau)$ as the direct arrivals of this transmission response, ignoring all multiples and wave conversions. Applying the inverse of this matrix to the reflection response, according to

$$\mathbf{G}^{-,+}(p, z_0, z_i, \tau) \approx \int_{-\infty}^{\tau} \mathbf{R}(p, z_0, \tau - \tau') \mathbf{T}_d^{inv}(p, z_i, z_0, \tau') d\tau', \tag{5}$$

is the standard way of redatuming the sources from $z_0$ to $z_i$. Note that this expression resembles equation 1, with the focusing function $\mathbf{F}_1^+(p, z_0, z_i, \tau)$ replaced by $\mathbf{T}_d^{inv}(p, z_i, z_0, \tau)$, while ignoring $\mathbf{F}_1^-(p, z_0, z_i, \tau)$ on the left-hand side. We illustrate equation 5 for the horizontally layered medium shown in Figure 1. We modelled the elastodynamic reflection response $\mathbf{R}(p, z_0, \tau)$ for $p = 0.0002$ s/m, and convolved it with a Ricker wavelet with a central frequency of 50 Hz (not shown). The focusing depth level is chosen as $z_i = 1800$ m. Figure 2 shows the diagonal elements of $\mathbf{T}_d^{inv}(p, z_i, z_0, \tau)$ (the non-diagonal elements are zero because wave conversions are ignored). Note that both elements contain a single event at a negative time, defined by the $P$ and $S$ velocities of the layered medium. The
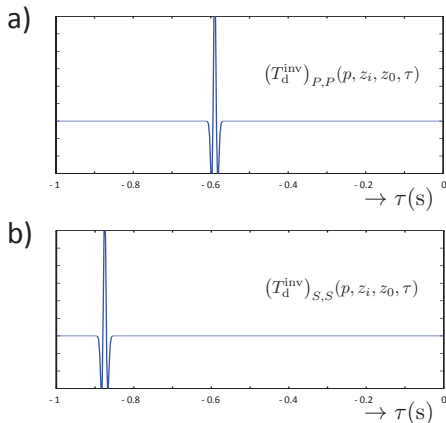


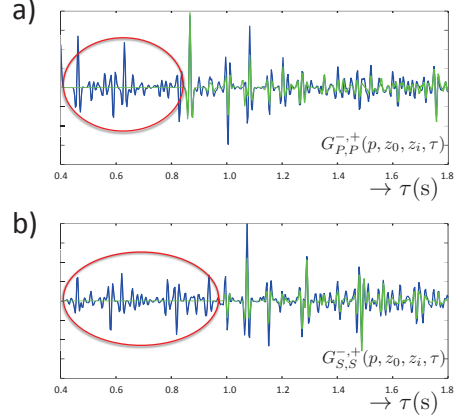Figure 2: *Inverse direct arrivals of transmission response.*

diagonal elements of $\mathbf{G}^{-,+}(p, z_0, z_i, \tau)$, obtained with equation 5, are shown in Figure 3 (in blue), compared with the directly modelled Green's functions (in green). Note that the events indicated by the red ellipses arrive prior to the first arrival of the exact Green's function. These acausal events will cause significant artefacts in imaging. The non-diagonal elements of $\mathbf{G}^{-,+}(p, z_0, z_i, \tau)$ (not shown) exhibit a similar behavior.

### THE ELASTODYNAMIC MARCHENKO EQUATION

Equations (1) and (2) form a set of two equations with four unknowns, namely $\mathbf{G}^{-,+}$, $\mathbf{G}^{-,-}$, $\mathbf{F}_1^+$ and $\mathbf{F}_1^-$. The reflection response $\mathbf{R}$ is assumed to be known. To solve this set of equations, we use the causality of the Green's functions to remove them from the equations. Throughout the following analysis we assume $z_i$ is not "too close" to an interface. We define $\tau_{XY}^d(p)$ as the arrival time of the first arrival of $G_{X,Y}^{-,-}(p, z_0, z_i, \tau)$. Moreover, $\tau_{XY}^{d,\varepsilon}(p) = \tau_{XY}^d(p) - \varepsilon$ defines the onset of the first arrival when the Green's function is convolved with a wavelet. We define a time-window matrix $\mathbf{W}(p, \tau)$, according to

$$\mathbf{W}(p, \tau) = \begin{pmatrix} H(\tau_{PP}^{d,\varepsilon} - \tau) & H(\tau_{PS}^{d,\varepsilon} - \tau) \\ H(\tau_{SP}^{d,\varepsilon} - \tau) & H(\tau_{SS}^{d,\varepsilon} - \tau) \end{pmatrix}, \tag{6}$$

where $H(\tau)$ is the Heaviside step function. Note that $\mathbf{W}(p, \tau) \circ \mathbf{G}^{-,\pm}(p, z_0, z_i, \tau) = \mathbf{O}$, where $\mathbf{O}$ is the null matrix and $\circ$ denotes Hadamard matrix multiplication (i.e., element-wise multiplication). Moreover, with the assumption made above, we have $\mathbf{W}(p, \tau) \circ \mathbf{F}_1^-(p, z_0, z_i, \tau) = \mathbf{F}_1^-(p, z_0, z_i, \tau)$ (Wapenaar and Slob, 2014). Hence, applying $\mathbf{W}(p, \tau)$ to both sides of equations (1) and (2) we obtain

$$\mathbf{F}_1^-(p, z_0, z_i, \tau) = \tag{7}$$
$$\mathbf{W}(p, \tau) \circ \int_{-\infty}^{\tau} \mathbf{R}(p, z_0, \tau - \tau') \mathbf{F}_1^+(p, z_0, z_i, \tau') d\tau',$$

$$\mathbf{W}(p, \tau) \circ \mathbf{F}_1^+(-p, z_0, z_i, -\tau) = \tag{8}$$
$$\mathbf{W}(p, \tau) \circ \int_{-\infty}^{\tau} \mathbf{R}(p, z_0, \tau - \tau') \mathbf{F}_1^-(-p, z_0, z_i, -\tau') d\tau'.$$

We have now a system of two equations for two unknowns. However, the complicating factor is that $\mathbf{F}_1^+$ in the left-hand

a)



$G_{P,P}^{-,+}(p,z_0,z_i,\tau)$

$\tau_{PP}^{d,\varepsilon}$    $\rightarrow \tau(s)$

b)



$G_{S,S}^{-,+}(p,z_0,z_i,\tau)$

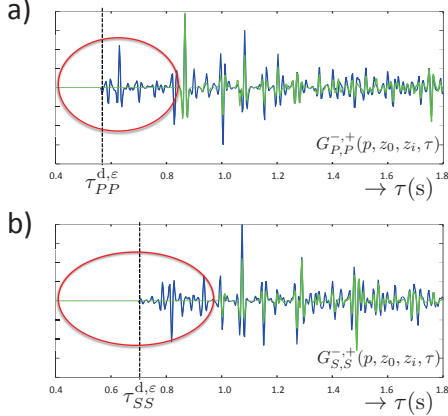$\tau_{SS}^{d,\varepsilon}$    $\rightarrow \tau(s)$

Figure 4: *Green's functions, retrieved with decoupled scalar Marchenko schemes, using initial conditions with single events (Figure 2).*

side of equation 8 is muted by the window function. We define

$$\mathbf{W}(p,\tau) \circ \mathbf{F}_1^+(-p,z_0,z_i,-\tau) = \mathbf{M}^+(-p,z_0,z_i,-\tau) \quad (9)$$

and write the focusing function $\mathbf{F}_1^+(p,z_0,z_i,\tau)$ as

$$\mathbf{F}_1^+(p,z_0,z_i,\tau) = \mathbf{F}_{1,0}^+(p,z_0,z_i,\tau) + \mathbf{M}^+(p,z_0,z_i,\tau). \quad (10)$$

Substituting equations 9 and 10 into equations 7 and 8 yields a set of two equations for the unknowns $\mathbf{M}^+$ and $\mathbf{F}_1^-$. We call this the elastodynamic Marchenko equations. Assuming $\mathbf{R}$ and $\mathbf{F}_{1,0}^+$ are known, the elastodynamic Marchenko equations can be solved iteratively, taking $\mathbf{F}_{1,0}^+$ as initial estimate of $\mathbf{F}_1^+$. After convergence, the Green's functions $\mathbf{G}^{-,+}$ and $\mathbf{G}^{-,-}$ are obtained from equations 1, 2, 9 and 10. In the following sections we investigate the effect of the choice of initial condition $\mathbf{F}_{1,0}^+$.

**APPROXIMATE INITIAL CONDITION**

In the acoustic case, $f_{1,0}^+$ consists of a single event, at the arrival time of the time-reversed direct arrival of the Green's function $G^{-,-}$. This underlies the relative simplicity of the acoustic Marchenko scheme. In this section, we let also the elements of the elastodynamic initial condition $\mathbf{F}_{1,0}^+$ consist of single events. To this end, we define $\mathbf{F}_{1,0}^+(p,z_0,z_i,\tau) = \mathbf{T}_d^{inv}(p,z_i,z_0,\tau)$, see Figure 2.

In a first experiment we replace $\mathbf{R}(p,z_0,\tau)$ by its diagonal (i.e., we set $R_{S,P}$ and $R_{P,S}$ in equation 3 to zero). Because the initial condition $\mathbf{F}_{1,0}^+(p,z_0,z_i,\tau)$ is diagonal as well, the elastodynamic Marchenko equations decouple in this case into two independent scalar equations. As a matter of fact, the scheme for the upper-left element reduces to the acoustic scheme, applied to the elastic response $R_{P,P}$. The results of the two independent schemes are shown in Figure 4. Note that there is no significant improvement compared with the standard one-way extrapolation results in Figure 3. Acausal artefacts, indicated by the red ellipses, still exist, except that prior to $\tau_{XX}^{d,\varepsilon}(p)$ they

have been suppressed by the window functions. The remaining acausal events will cause artefacts in imaging.

In a second experiment we apply the elastodynamic Marchenko scheme to the full response matrix $\mathbf{R}(p,z_0,\tau)$, but still use the initial condition with single events (Figure 2). The result is shown in Figure 5. Note that the retrieved Green's functions $G_{P,P}^{-,+}$ and $G_{S,P}^{-,+}$ in Figures 5a,c match the directly modeled Green's functions (green) remarkably well. The acausal events in the red ellipses have almost entirely disappeared. The other two results, $G_{P,S}^{-,+}$ and $G_{S,S}^{-,+}$ in Figures 5b,d, contain again acausal events. Experiments with other configurations indicate that this overall behavior is persistent, which is promising for imaging applications. By correlating $G_{P,P}^{-,+}(p,z_0,z_i,\tau)$ with the direct downgoing $P$-wave (Wapenaar et al., 1987), an estimate of $R_{P,P}(p,z_i,\tau)$ is obtained of which the first event corresponds to the first reflector below $z_i$. This forms a basis for imaging the primary $PP$ reflectivity, without artefacts related to internal multiples and conversions.

**EXACT INITIAL CONDITION**

In a previous analysis we found that the initial condition $\mathbf{F}_{1,0}^+$ is ideally defined as the inverse of the "forward-scattering" transmission response matrix $\mathbf{T}_{fs}(p,z_i,z_0,\tau)$, i.e., the part of the transmission response that includes direct and forward converted waves, but no internal multiples (Wapenaar and Slob, 2014). Hence, we define $\mathbf{F}_{1,0}^+(p,z_0,z_i,\tau) = \mathbf{T}_{fs}^{inv}(p,z_i,z_0,\tau)$, see Figure 6. Applying the elastodynamic Marchenko scheme to the full response matrix $\mathbf{R}(p,z_0,\tau)$, using the exact initial condition, yields a near exact estimate of the Green's functions $\mathbf{G}^{-,+}(p,z_0,z_i,\tau)$ (Figure 7) and $\mathbf{G}^{-,-}(p,z_0,z_i,\tau)$ (not shown). In Figure 7 the retrieved Green's functions (blue) are almost exactly overlaid by the directly modelled Green's functions (green). Also the acausal artefacts have almost entirely disappeared (red ellipses). These Green's functions form the perfect starting point for elastodynamic Marchenko imaging by deconvolution (see the supporting information in Wapenaar and Slob (2014)). Note that defining the forward-scattering transmission response matrix $\mathbf{T}_{fs}(p,z_i,z_0,\tau)$ requires more information about the medium than defining the direct transmission response matrix $\mathbf{T}_d(p,z_i,z_0,\tau)$.

**CONCLUSIONS**

We have compared four ways to retrieve the elastodynamic Green's matrix $\mathbf{G}^{-,+}(p,z_0,z_i,\tau)$ from the reflection response matrix $\mathbf{R}(p,z_0,\tau)$ at the surface.

(1) Standard one-way wave field extrapolation involves application of the inverse of the direct transmission response matrix $\mathbf{T}_d(p,z_i,z_0,\tau)$ (Figure 2) to the reflection response matrix $\mathbf{R}(p,z_0,\tau)$ (equation 5). The result (Figure 3) contains acausal events prior to the first arrival, which will cause significant artefacts in imaging. This standard one-way method can be seen as the first step of the Marchenko method (compare equation 5 with equation 1).
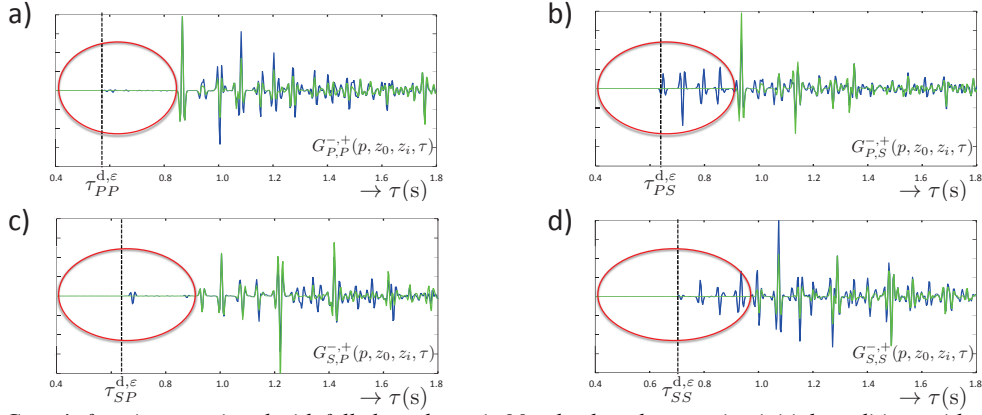
Figure 5: *Green's functions, retrieved with full elastodynamic Marchenko scheme, using initial conditions with single events.*
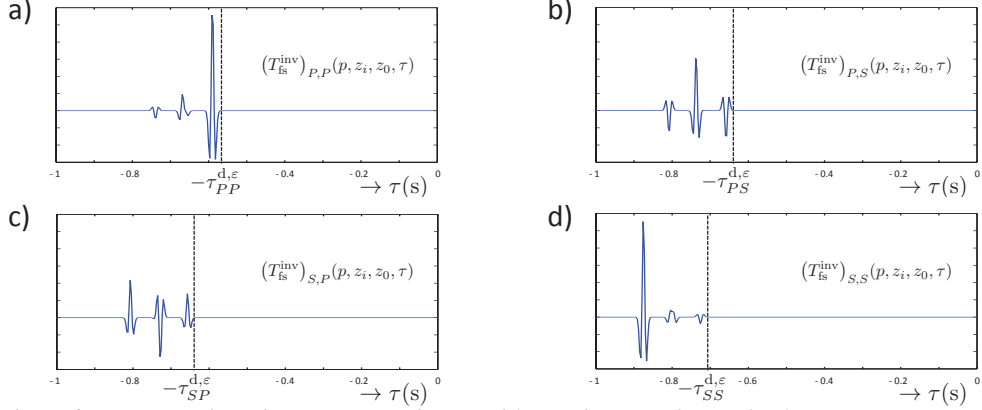


Figure 6: *Exact initial conditions: inverse direct and forward scattered arrivals of transmission response.*
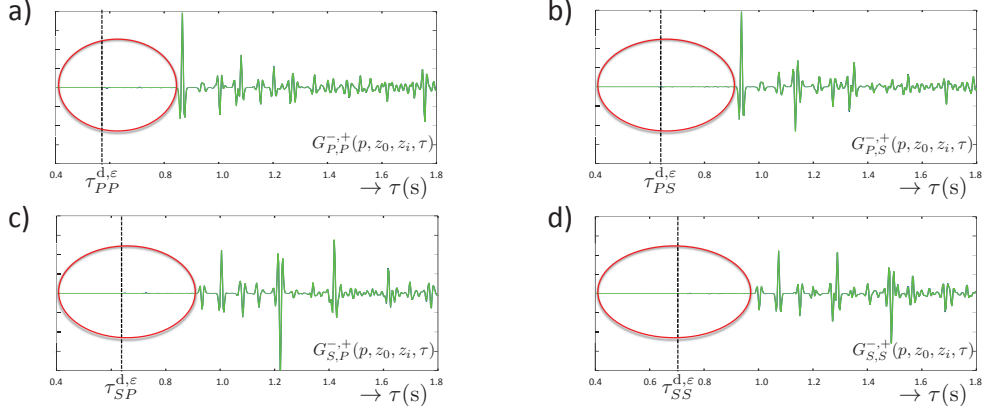


Figure 7: *Green's functions, retrieved with full elastodynamic Marchenko scheme; exact initial conditions (Figure 6).*

(2) Applying the Marchenko method with initial condition $\mathbf{F}_{1,0}^+(p,z_0,z_i,\tau) = \mathbf{T}_{\mathrm{d}}^{\mathrm{inv}}(p,z_i,z_0,\tau)$ (Figure 2) to the diagonal of the reflection response matrix results in two simple scalar schemes, which do not give a significant improvement (Figure 4).

(3) Applying the Marchenko method with the same initial condition to the full reflection response matrix (including the conversions) gives a significant improvement for two components of the retrieved Green's matrix (Figures 5a,c).

(4) Applying the Marchenko method with initial condition $\mathbf{F}_{1,0}^+(p,z_0,z_i,\tau) = \mathbf{T}_{\mathrm{fs}}^{\mathrm{inv}}(p,z_i,z_0,\tau)$ (the inverse of the forward scattered transmission matrix, Figure 6) gives a near perfect result (Figure 7).

Method (4) gives the best results but requires more knowledge about the medium than the other methods. For practical applications method (3) seems to be a good compromise. Note that da Costa et al. (2014) discuss similar approximations as discussed here under (2) and (3), but they do not apply wave field decomposition prior to applying their Marchenko method. It remains to be investigated whether our observations for methods (2) and (3) also hold for their approach.

**Initial conditions for elastodynamic Marchenko method**

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