The Reflectivity Operator for Curved Interfaces

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Summary

In this paper we present a generalized notion for the reflection operator for a curved interface in an inhomogeneous background configuration. We show how the acquisition-independent properties are obtained. Examples for the reflection operator for a horizontal interface are presented.

Introduction

It is normal practice to decompose the actual wavefield into an incident and a scattered wavefield. The incident wavefield has the same source distribution as the actual, but is measured in a background medium, where the scattering domain is absent. The scattered wavefield finds its origin in a source distribution that is related to the discrepancy in material parameters between the actual and the background medium. The object of medium reconstruction encompasses the process of finding the material parameters of the scattering domain. The determination of the functional behaviour between the scattered and the incident wavefield can be considered as a first step. This behaviour is usually denoted as the reflection operator. In order to be able to represent the intrinsic dynamic properties of the medium, the reflection operator must be independent of the acquisition parameters (Berkhout, 1982). In general the reflection operator is dependent on space and time. In the imaging process the reflection operator is mapped in space, such that at each location in space its time behaviour is frozen at the very time instant of first causal occurrence. In the inversion process the total time behaviour of the reflection operator at each space location is taken into account to find the material parameters. So far, an explicit exact expression for the reflection operator has only been derived for a horizontal interface between two homogeneous half spaces. The aim of this paper is to generalize the notion of the reflectivity operator to a curved interface. To this end we apply the reciprocity relations of the wave field quantities (Fokkema and van den Berg,1993).
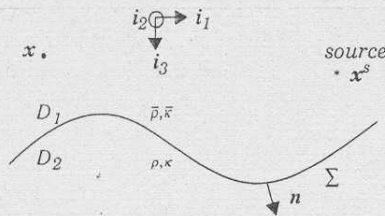

Fig. 1   The scattering inhomogeneous configuration with a curved interface.

The boundary-integral representation of the scattered field

We investigate the direct or forward scattering of acoustic waves by a contrasting domain D_2, present in an embedding or background medium D_1. The media D_1 and D_2 are separated by a curved interface Σ (see Figure 1). The background medium D_1 is characterized by the inhomogeneous mass density $\overline{\rho} = \overline{\rho}(x)$ and the compressibility $\overline{\kappa} = \overline{\kappa}(x)$. The contrasting domain D_2 is characterized by the inhomogeneous mass density $\rho = \rho(x)$ and the compressibility $\kappa = \kappa(x)$.

The total acoustic wavefield in the configuration P is decomposed into the incident wavefield P^{inc} and the scattered wavefield P^{sct}. The incident wavefield is the wavefield that would be present in the entire configuration if the domain D_2 showed no contrast with the embedding.

The total wavefield is generated by a monopole source that is located in D_1 outside the scattering domain. Since the source remains present even if the scattering domain is thought to be absent, it also serves as the source for the incident field.

We start our analysis in the frequency domain and omit the explicit functional dependence on the frequency parameter ω. The scattered wavefield is the difference between the total wavefield and the incident wavefield. Hence,

$$P^{sct} = P - P^{inc}. \tag{1}$$

Through a particular reasoning we want to express that the scattered wavefield originates from the contrast in acoustic properties that the scattering domain shows with respect to the embedding. The total wavefield $P = P(x \mid x^s)$ satisfies the following set of wave equations

$$\partial_k \frac{1}{\rho} \partial_k P + \omega^2 \kappa P = -S\delta(x - x^s), \; x \in D_1, \tag{2}$$

$$\partial_k \frac{1}{\rho} \partial_k P + \omega^2 \kappa P = 0, \qquad x \in D_2, \tag{3}$$

where ∂_k denotes the spatial derivative with respect to x_k and S denotes the source spectrum of the monopole source.

Secondly, the incident wavefield has no sources in D_2, while the material parameters in D_2 have the same values as for the embedding ($\overline{\rho}$ and $\overline{\kappa}$) and can be chosen as a suitable extension of D_1 in D_2.

The governing wave equations for the incident wavefield $P^{inc} = P^{inc}(x \mid x^s)$ are given by

$$\partial_k \frac{1}{\overline{\rho}} \partial_k P^{inc} + \omega^2 \overline{\kappa} P^{inc} = -S\delta(x - x^s), \; x \in D_1, \tag{4}$$

$$\partial_k \frac{1}{\overline{\rho}} \partial_k P^{inc} + \omega^2 \overline{\kappa} P^{inc} = 0, \qquad x \in D_2. \tag{5}$$

Then substracting equations (4) and (5) from equations (2) and (3) and using equation (1), we arrive at the wave equation for the scattered wavefield $P^{sct} = P^{sct}(x \mid x^s)$

$$\partial_k \frac{1}{\rho} \partial_k P^{sct} + \omega^2 \overline{\kappa} P^{sct} = 0, \qquad \boldsymbol{x} \in D_1 \,, \qquad (6)$$

$$\partial_k \frac{1}{\rho} \partial_k P^{sct} + \omega^2 \overline{\kappa} P^{sct} = \partial_k \left(\frac{1}{\overline{\rho}} - \frac{1}{\rho} \right) \partial_k P + \omega^2 (\overline{\kappa} - \kappa) P,$$
$$\boldsymbol{x} \in D_2 \,. \qquad (7)$$

The associated Green's functions of the different wavefields are obtained through the following operation:

$$G(\boldsymbol{x} \mid \boldsymbol{x}^s) = P(\boldsymbol{x} \mid \boldsymbol{x}^s) / S \,, \qquad (8)$$

$$\overline{G}(\boldsymbol{x} \mid \boldsymbol{x}^s) = P^{inc}(\boldsymbol{x} \mid \boldsymbol{x}^s) / S \,, \qquad (9)$$

and

$$G^{sct}(\boldsymbol{x} \mid \boldsymbol{x}^s) = P^{sct}(\boldsymbol{x} \mid \boldsymbol{x}^s) / S \,. \qquad (10)$$

Further all wavefields satisfy the reciprocity relation in which the source and receiver position are interchanged. In particular for the scattered wavefield we have

$$P^{sct}(\boldsymbol{x} \mid \boldsymbol{x}^s) = P^{sct}(\boldsymbol{x}^s \mid \boldsymbol{x}) \,. \qquad (11)$$

The following boundary-integral representation for the scattered wavefield can be derived

$$\chi(\boldsymbol{x}) P^{sct}(\boldsymbol{x} \mid \boldsymbol{x}^s) =$$
$$\int_{\boldsymbol{x}' \in \Sigma} \frac{1}{\overline{\rho}(\boldsymbol{x}')} [\overline{G}(\boldsymbol{x} \mid \boldsymbol{x}') n_k(\boldsymbol{x}') \partial_k' P^{sct}(\boldsymbol{x}' \mid \boldsymbol{x}^s) \qquad (12)$$
$$- \left(n_k(\boldsymbol{x}') \partial_k' \overline{G}(\boldsymbol{x} \mid \boldsymbol{x}') \right) P^{sct}(\boldsymbol{x}' \mid \boldsymbol{x}^s)] dA(\boldsymbol{x}'),$$

where ∂_k' denotes the spatial derivative with respect to x_k' and $\chi(\boldsymbol{x})$ denotes the characteristic function of D_1

$$\chi(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in D_1, \\ \dfrac{1}{2} & \boldsymbol{x} \in \Sigma, \\ 0 & \boldsymbol{x} \in D_2. \end{cases} \qquad (13)$$

In the further analysis we do not use the integral representation in D_2. When $\boldsymbol{x} \in \Sigma$ the Cauchy principal value of the boundary integral is used, denoted by the symbol $\fint$.

The reflection operator in inhomogeneous media

In order to arrive at an expression for the reflection operator we use the reciprocity relation of the type of equation (11) in the scattered wavefield representation of equation (12). Using the definitions of the Green's functions as given in equations (8),(9) and (10), the resulting expression becomes

$$\chi(\boldsymbol{x}) P^{sct}(\boldsymbol{x} \mid \boldsymbol{x}^s) = \int_{\boldsymbol{x}' \in \Sigma} \frac{1}{\overline{\rho}(\boldsymbol{x}')} [n_k'(\boldsymbol{x}') \partial_k' G^{sct}(\boldsymbol{x} \mid \boldsymbol{x}')$$
$$\qquad (14)$$
$$- G^{sct}(\boldsymbol{x} \mid \boldsymbol{x}') n_k(\boldsymbol{x}') \partial_k'] P^{inc}(\boldsymbol{x}' \mid \boldsymbol{x}^s) dA(\boldsymbol{x}').$$

Next we introduce the reflection operator R as

$$R(\boldsymbol{x} \mid \boldsymbol{x}') = \frac{2}{\overline{\rho}(\boldsymbol{x}')} [n_k(\boldsymbol{x}') \partial_k' G^{sct}(\boldsymbol{x} \mid \boldsymbol{x}') -$$
$$\qquad (15)$$
$$G^{sct}(\boldsymbol{x} \mid \boldsymbol{x}') n_k(\boldsymbol{x}') \partial_k'], \ \boldsymbol{x}' \in \Sigma,$$

which allows us to rewrite equation (14) compactly as

$$\chi(\boldsymbol{x}) P^{sct}(\boldsymbol{x} \mid \boldsymbol{x}^s) = \frac{1}{2} \int_{\boldsymbol{x}' \in \Sigma} R(\boldsymbol{x} \mid \boldsymbol{x}') P^{inc}(\boldsymbol{x}' \mid \boldsymbol{x}^s) dA(\boldsymbol{x}'). \qquad (16)$$

From equation (16) it is clear that in operator terminology R is in inhomogeneous media the integral-differential operator that connects the incident wavefield with the scattered wavefield. Thus we are justified to name it the reflection operator. Moreover from equation (15) it follows that it is an intrinsic dynamic property of the contrasting medium, which is independent of the source spectrum and the source and receiver position. It appears that the reflection operator is related to the Green's function of the scattered wavefield, whereby its source point $\boldsymbol{x}'$ is essentially located at the interface. For the plane interface in the homogeneous case, this will be illustrated in the next section. Note that R is not only defined for $\boldsymbol{x}$ on Σ , but also in D_1. Finally we substitute equation (16) into equation (12) and we use equation (9). We thus obtain

$$\chi(\boldsymbol{x}) P^{sct}(\boldsymbol{x} \mid \boldsymbol{x}^S) = \int_{\boldsymbol{x}' \in \Sigma} dA(\boldsymbol{x}') \fint_{\boldsymbol{x}'' \in \Sigma} dA(\boldsymbol{x}'') \frac{1}{\overline{\rho}(\boldsymbol{x}')}$$
$$[\overline{G}(\boldsymbol{x} \mid \boldsymbol{x}') n_k(\boldsymbol{x}') \partial_k' R(\boldsymbol{x}' \mid \boldsymbol{x}'') \overline{G}(\boldsymbol{x}'' \mid \boldsymbol{x}^S) S - \qquad (17)$$
$$\left(n_k(\boldsymbol{x}') \partial_k' \overline{G}(\boldsymbol{x} \mid \boldsymbol{x}') \right) R(\boldsymbol{x}' \mid \boldsymbol{x}'') \overline{G}(\boldsymbol{x}'' \mid \boldsymbol{x}^S) S].$$

Equation (17) represents the "downward" extrapolation, reflection and "upward" extrapolation operator formalism in inhomogeneous media with curved interfaces (generalized WRW model). In the next section we investigate this formalism in homogeneous media.

The reflection operator in homogeneous media

Plane interface

In this section we first consider the scattering problem of two homogeneous media D_1 and D_2, characterized by the material parameters $\{\overline{\rho}, \overline{\kappa}\}$ and $\{\rho, \kappa\}$, respectively, with a plane interface Σ (see Figure 2).
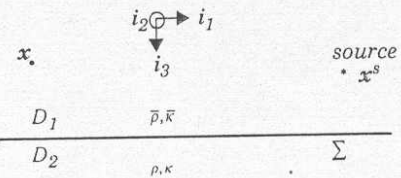

Fig. 2   The scattering homogeneous
configuration with a plane interface.

In this case equation (16) is evaluated at a plane level. Then using a suitable plane-wave representation of $\overline{G}$ and noting that the scattered wavefield is upgoing in D_1, it can be shown that R reduces to the simple kernel function

$$R(\boldsymbol{x} \mid \boldsymbol{x}') = \frac{4}{\rho} H(x_3' - x_3)\partial_3' G^{sct}(\boldsymbol{x} \mid \boldsymbol{x}'), \quad \boldsymbol{x}' \in \Sigma, \quad (18)$$

where H is the Heaviside function.

By the same token equation (17) simplifies in this scattering problem to

$$H(x_3' - x_3)P^{sct}(\boldsymbol{x} \mid \boldsymbol{x}^s) = -\frac{2}{\rho} \int\limits_{\boldsymbol{x}' \in \Sigma} dA(\boldsymbol{x}')$$

$$\oint\limits_{\boldsymbol{x}'' \in \Sigma} dA(\boldsymbol{x}'')\left(\partial_3'\overline{G}(\boldsymbol{x} \mid \boldsymbol{x}')\right)R(\boldsymbol{x}' \mid \boldsymbol{x}'')\overline{G}(\boldsymbol{x}'' \mid \boldsymbol{x}^s)S, \quad (19)$$

which is the well-known WRW model (Berkhout,1982).

As an illustration of $R(\boldsymbol{x} \mid \boldsymbol{x}')$ we modeled the two-dimensional response of a dipole source at $\boldsymbol{x}'$ on a plane interface between two homogeneous half spaces, with propagation velocities $c_1 = (\overline{\rho}\kappa)^{-1/2} = 1500$ m/s and $c_2 = (\overline{\rho}\kappa)^{-1/2} = 3000$ m/s, respectively. A "snapshot" of this response at t=200 ms is shown in Figure 3a; a space-time domain registration by receivers at the interface is shown in Figure 3b. In both Figures the incident wavefield has been removed. Hence, by transforming the registration of Figure 3b from the time domain to the frequency domain we obtain the Green's function $\partial_3 G^{sct}(\boldsymbol{x} \mid \boldsymbol{x}')$ as a function of $\boldsymbol{x}$ for fixed $\boldsymbol{x}'$, both at the interface. Subsequently, the reflection operator $R(\boldsymbol{x} \mid \boldsymbol{x}')$ is obtained by scaling this response by a factor $2/\overline{\rho}$, see equation (18). This example clearly illustrates the non-local character of the reflection operator, due to the waves that propagate along the interface, away from the excitation point $\boldsymbol{x}'$. This non-local behaviour of the reflection operator is essential for obtaining the correct amplitude of P^{sct}, irrespective of the propagation angle of P^{inc} (equation (16)).

We carried out another experiment for a similar configuration with a density contrast only (i.e., no propagation velocity contrast). For this configuration it is known that the reflection properties are independent of the propagation angle of P^{inc}. Indeed we observe in Figure 4b that the response of the dipole at $\boldsymbol{x}'$ is registered only by a receiver at $\boldsymbol{x} = \boldsymbol{x}'$ at the interface. Hence, for the reflection operator we may write for this situation $R(\boldsymbol{x} \mid \boldsymbol{x}') = R_0\delta(x_1' - x_1)$ for $\boldsymbol{x}$ and $\boldsymbol{x}'$ at the interface. Upon substitution in equation (16) we obtain $P^{sct}(\boldsymbol{x} \mid \boldsymbol{x}^s) = R_0 P^{inc}(\boldsymbol{x} \mid \boldsymbol{x}^s)$, which confirms

that the reflection amplitude is independent of the incidence angle. Summarizing, a non-local reflection operator

(Figure 3) corresponds to angle-dependent reflectivity, whereas a local reflection operator (Figure 4) corresponds to angle-independent reflectivity.
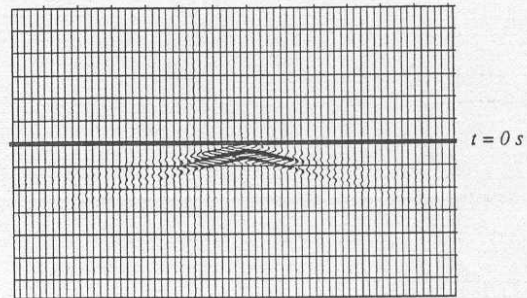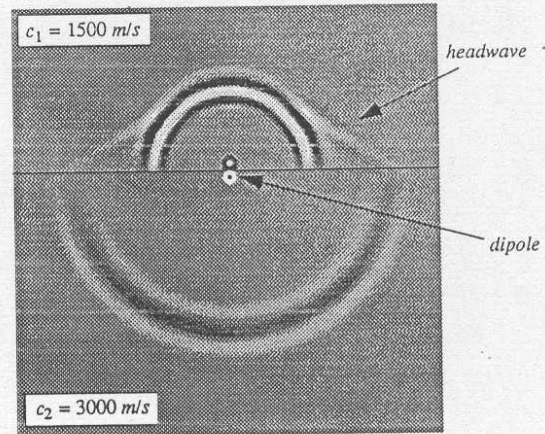

Fig. 3   *The response of a dipole source at an interface between two half-spaces with different propagation velocities*
a. Snapshot at t=200ms

b. Space-time domain registration at the interface

This example illustrates the non-local behaviour of the reflection operator at an interface with angle dependent reflection properties.
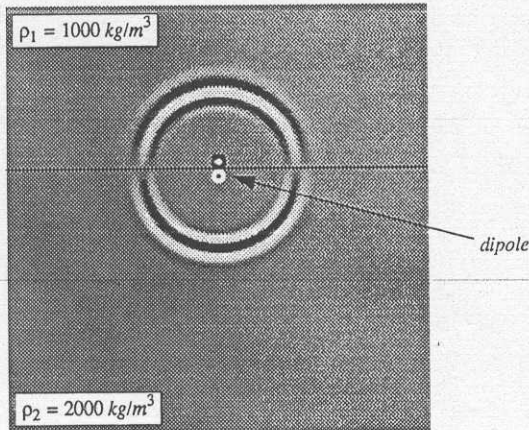
$\rho_1 = 1000 \; kg/m^3$

$\rho_2 = 2000 \; kg/m^3$

dipole

$t = 0 \; s$

Fig. 4 *As in Figure 3, but only with a density contrast at the interface. This example illustrates the local behaviour of the reflection operator at an interface with angle independent reflection properties.*
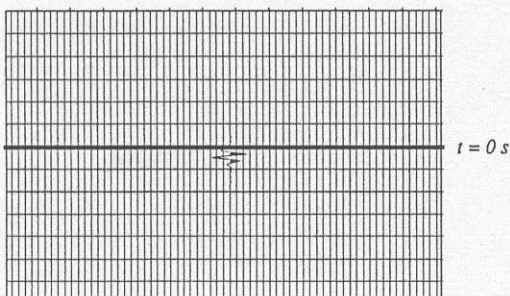
Curved interface

Next we consider the curved interface Σ in the homogeneous situation (see Figure 5).

i_3 $\quad x_3^A$

D_1

$\bar{\rho}, \bar{\kappa}$

D_2

ρ, κ $\quad \Sigma$

n

Fig. 5 *The scattering homogeneous configuration with a curved interface.*

Also in this case the scattered wavefield at level x_3^A is upgoing. However, a reduction of the reflection operator as in equation (18) is not possible, due to the fact that x_3' is functionally dependent on x_1' and x_2' on Σ in the plane-wave representation. As an alternative we could take the level x_3^A as the representation level of the scattered wavefield. Then again we can write

$$R(\boldsymbol{x} \mid \boldsymbol{x}') = \frac{4}{\rho} H(x_3^A - x_3) \partial_3' G^{sct}(\boldsymbol{x} \mid \boldsymbol{x}') \Big|_{x_3 = x_3^A}, \qquad (20)$$

and use equation (19) with $x_3' = x_3'' = x_3^A$ as a representation of the scattered wavefield. Note that for this situation R can be interpreted as the response of the half-space $x_3 \geq x_3^A$. This is equivalent to the data after redatuming from the acquisition level to the depth level x_3^A.

Conclusions

We have shown that starting from the integral representation for the scattered wavefield and using reciprocity, an expression for the reflection operator is obtained. Furthermore it has the required property of being independent of the acquisition parameters. Finally we have shown how the expression of the reflection operator leads to a generalized concept of the WRW model introduced by Berkhout (1982).

References

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