Chapter 1
Reciprocity theorem for one-way electromagnetic and acoustic wave fields in inhomogeneous media with losses

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Abstract

A system of coupled one-way wave equations for oppositely propagating electromagnetic or acoustic waves in inhomogeneous lossy media is derived. It is shown that the square-root operator appearing in these equations is symmetric. Based on this symmetry a general reciprocity theorem for one-way wave fields is derived.

1 Introduction

In many wave propagation problems one can define a "preferred direction of propagation". Electromagnetic and acoustic waveguides are obvious examples, but also in laterally unbounded media it is often advantageous to define a preferred propagation direction. In all those situations it is useful to decompose the wave equation into a system of coupled "one-way wave equations" for oppositely propagating waves [1],[3],[6] (or "bidirectional beams" [4],[5]). In this paper we derive a reciprocity theorem for one-way electromagnetic and acoustic wave fields in inhomogeneous lossy media.

2 Wave equation in matrix-vector form

For simplicity we consider a 2-D configuration, that is, we assume that the wave fields, material parameters and source distributions are functions of two coordinates ($\mathbf{x} = (x_1, x_3)$) only. For transverse electric waves (TE), transverse magnetic waves (TM), compressional waves in fluids (P) and horizontally polarized shear waves in solids (SH), the general form of the coupled system of basic equations reads in the space-frequency $(\mathbf{x}, \omega)$ domain

$$j\omega\alpha P + \partial_1 Q_1 + \partial_3 Q_3 = B, \quad j\omega\beta Q_1 + \partial_1 P = C_1, \quad j\omega\beta Q_3 + \partial_3 P = C_3, \tag{1}$$

where $P(\mathbf{x}, \omega)$, $Q_1(\mathbf{x}, \omega)$ and $Q_3(\mathbf{x}, \omega)$ represent the wave fields, $\alpha(\mathbf{x}, \omega)$ and $\beta(\mathbf{x}, \omega)$ denote the material parameters (with negative imaginary parts for lossy media) and $B(\mathbf{x}, \omega)$, $C_1(\mathbf{x}, \omega)$ and $C_3(\mathbf{x}, \omega)$ are the source distributions. These functions are further specified in Table 1 for the four wave types discussed above. From here onward we assume that the preferred direction of propagation is along the x_3-axis, see Figure 1. Hence, x_1 and x_3 will be referred to as the lateral and axial coordinates, respectively. In the lateral direction the medium may be bounded or unbounded. In the former case homogeneous Dirichlet or Neumann boundary conditions will be imposed at $x_1 = x_{1,a}$ and $x_1 = x_{1,b}$; in the latter case the wave fields are assumed to have "sufficient decay" for $x_{1,a} \to -\infty$ and $x_{1,b} \to +\infty$.

We reorganize the general wave equation into a form that acknowledges the direction of

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TABLE 1

Overview of electromagnetic and acoustic field quantities, material parameters and source functions. **Electromagnetic waves:** *Field quantities: E_i (electric field strength) and H_i (magnetic field strength); material parameters: ϵ (permittivity), μ (permeability) and σ (conductivity); source functions: J_i^e (electric current density) and J_i^m (magnetic current density).* **Acoustic waves:** *Field quantities: V_i (particle velocity), P (acoustic pressure) and T_{ij} (stress); material parameters: κ (compressibility), μ (shear modulus) and ϱ (mass density); source functions: F_i (force density) and D_{ij} (deformation rate density).*

	P	Q_1	Q_3	α	β	B	C_1	C_3
TE	E_2	H_3	$-H_1$	$\epsilon - j\sigma/\omega$	μ	$-J_2^e$	$-J_3^m$	J_1^m
TM	H_2	$-E_3$	E_1	μ	$\epsilon - j\sigma/\omega$	$-J_2^m$	J_3^e	$-J_1^e$
P (fluid)	P	V_1	V_3	κ	ϱ	$D_{11}+D_{33}$	F_1	F_3
SH (solid)	V_2	$-T_{21}$	$-T_{23}$	ϱ	$1/\mu$	F_2	$D_{12}+D_{21}$	$D_{23}+D_{32}$

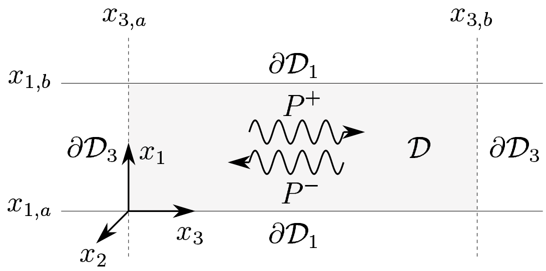

FIG. 1. *Configuration for which we will derive a reciprocity theorem for one-way wave fields. The direction of preference is chosen along the x_3-axis. The coupled one-way wave fields P^+ and P^- will be defined in a later section.*

preference. By eliminating Q_1 from the coupled system of equations (1) we obtain

$$\partial_3 \mathbf{Q} = \hat{\mathbf{A}}\mathbf{Q} + \mathbf{D}, \tag{2}$$

where

$$\mathbf{Q} = \begin{pmatrix} P \\ Q_3 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} C_3 \\ B - \frac{1}{j\omega}\partial_1(\frac{1}{\beta}C_1) \end{pmatrix}, \quad \hat{\mathbf{A}} = \begin{pmatrix} 0 & -j\omega\beta \\ -j\omega\hat{A} & 0 \end{pmatrix}, \tag{3}$$

with

$$\hat{A} = \alpha + \frac{1}{\omega^2}\partial_1\Big(\frac{1}{\beta}\partial_1\,\cdot\Big). \tag{4}$$

The circumflex denotes an operator containing the lateral differentiation operator ∂_1. We introduce an operator $\hat{\mathcal{H}}_2$ via

$$\hat{\mathcal{H}}_2 = \omega^2\beta^{\frac{1}{2}}(\hat{A}\beta^{\frac{1}{2}}\,\cdot), \tag{5}$$

or, using equation (4),

$$\hat{\mathcal{H}}_2 = \Big(\frac{\omega}{c}\Big)^2 + \partial_1^2, \quad \text{where} \quad \Big(\frac{\omega}{c}\Big)^2 = \alpha\beta\omega^2 - \frac{3(\partial_1\beta)^2}{4\beta^2} + \frac{\partial_1^2\beta}{2\beta}. \tag{6}$$

Note that $\hat{\mathcal{H}}_2$ represents the Helmholtz operator, with $c(\mathbf{x},\omega)$ being the (complex valued) propagation velocity. We introduce the transposed Helmholtz operator $\hat{\mathcal{H}}_2^t$ via

$$\langle f, \hat{\mathcal{H}}_2^t g\rangle_b = \langle \hat{\mathcal{H}}_2 f, g\rangle_b, \tag{7}$$

where the bilinear form $\langle \cdot, \cdot \rangle_b$ is defined according to

$$\langle f, g \rangle_b = \int_{x_{1,a}}^{x_{1,b}} f(x_1) g(x_1) \mathrm{d}x_1. \tag{8}$$

Using integration by parts and employing the above mentioned boundary conditions at $x_{1,a}$ and $x_{1,b}$ we find that $\hat{\mathcal{H}}_2$ is a *symmetric* operator, according to

$$\hat{\mathcal{H}}_2^t = \hat{\mathcal{H}}_2. \tag{9}$$

3 Diagonalization of the operator matrix

We diagonalize the operator matrix $\hat{\mathbf{A}}$, according to

$$\hat{\mathbf{H}} = \hat{\mathbf{L}}^{-1} \hat{\mathbf{A}} \hat{\mathbf{L}}, \tag{10}$$

with

$$\hat{\mathbf{H}} = \begin{pmatrix} -j\hat{\mathcal{H}}_1 & 0 \\ 0 & j\hat{\mathcal{H}}_1 \end{pmatrix}, \tag{11}$$

where the *pseudo-differential* operator $\hat{\mathcal{H}}_1$ is the square-root of the Helmholtz operator, such that $\hat{\mathcal{H}}_1 \hat{\mathcal{H}}_1 = \hat{\mathcal{H}}_2$, or $\hat{\mathcal{H}}_1 = \hat{\mathcal{H}}_2^{\frac{1}{2}}$. Furthermore,

$$\hat{\mathbf{L}} = \begin{pmatrix} \hat{L}_1 & \hat{L}_1 \\ \hat{L}_2 & -\hat{L}_2 \end{pmatrix}, \quad \hat{\mathbf{L}}^{-1} = \frac{1}{2} \begin{pmatrix} \hat{L}_1^{-1} & \hat{L}_2^{-1} \\ \hat{L}_1^{-1} & -\hat{L}_2^{-1} \end{pmatrix}, \tag{12}$$

with

$$\hat{L}_1 = \left(\frac{\omega\beta}{2}\right)^{\frac{1}{2}} \hat{\mathcal{H}}_1^{-\frac{1}{2}}, \quad \frac{1}{2}\hat{L}_1^{-1} = \left(\hat{\mathcal{H}}_1^{\frac{1}{2}} (2\omega\beta)^{-\frac{1}{2}} \cdot\right), \tag{13}$$

$$\hat{L}_2 = (2\omega\beta)^{-\frac{1}{2}} \hat{\mathcal{H}}_1^{\frac{1}{2}}, \quad \frac{1}{2}\hat{L}_2^{-1} = \left(\hat{\mathcal{H}}_1^{-\frac{1}{2}} (\frac{\omega\beta}{2})^{\frac{1}{2}} \cdot\right). \tag{14}$$

4 Symmetry of square-root operators

We demonstrate the symmetry of square-root operators, using a generic notation. The approach we follow is modified after [2]. Let $\hat{\mathcal{U}} = \hat{\mathcal{U}}(x_1, \partial_1)$ be an arbitrary symmetric operator (in the sense of equations (7) to (9)) and let $\hat{\mathcal{V}} = \hat{\mathcal{V}}(x_1, \partial_1)$ be its square-root: $\hat{\mathcal{V}} = \hat{\mathcal{U}}^{\frac{1}{2}}$. In order to derive the symmetry properties of $\hat{\mathcal{V}}$ we construct the following pseudo-differential equation

$$\partial_z P = -j\hat{\mathcal{V}} P, \tag{15}$$

with $P = P(x_1, z)$ (note that z is a new variable, which bears no relation with x_3). The square-root of an operator is not unique. We assume that the square-root has been taken such that

$$\lim_{z \to \infty} P = 0, \tag{16}$$

for any P obeying equation (15). This is equivalent with stating that the imaginary part of the spectrum of $\hat{\mathcal{V}}$ is chosen to be negative. Note that, since $\hat{\mathcal{V}}$ is not a function of z, equation (15) implies

$$\partial_z^2 P = -\hat{\mathcal{U}} P. \tag{17}$$

Let P_A and P_B be two linearly independent solutions of equation (15). We introduce an interaction quantity $\mathcal{I}$, according to

$$\mathcal{I} = \langle P_A, \partial_z P_B \rangle_b - \langle \partial_z P_A, P_B \rangle_b. \tag{18}$$

We evaluate the z-derivative of $\mathcal{I}$, which yields

$$\partial_z \mathcal{I} = \langle P_A, \partial_z^2 P_B \rangle_b - \langle \partial_z^2 P_A, P_B \rangle_b = \langle \hat{\mathcal{U}} P_A, P_B \rangle_b - \langle P_A, \hat{\mathcal{U}} P_B \rangle_b = 0, \tag{19}$$

where we used the fact that $\hat{\mathcal{U}}$ is symmetric. From equations (16), (18) and (19) we now obtain $\mathcal{I} = \langle P_A, \partial_z P_B \rangle_b - \langle \partial_z P_A, P_B \rangle_b = 0$. This implies, together with equation (15),

$$\langle P_A, \hat{\mathcal{V}} P_B \rangle_b = \langle \hat{\mathcal{V}} P_A, P_B \rangle_b \quad \text{or} \quad \hat{\mathcal{V}}^t = \hat{\mathcal{V}}. \tag{20}$$

Hence, under the assumptions made above, the square-root of a symmetric operator is symmetric. Using induction, it follows that the operator $\hat{\mathcal{U}}^{\frac{1}{2^n}}$ is symmetric for any $n \geq 0$. Hence, for any fixed value of x_3 we find

$$\hat{\mathcal{H}}_1^t = \hat{\mathcal{H}}_1, \quad (\hat{\mathcal{H}}_1^{\frac{1}{2}})^t = \hat{\mathcal{H}}_1^{\frac{1}{2}}, \quad \text{etc.}, \tag{21}$$

and, using the fact that the inverse of a symmetric operator is symmetic as well, $(\hat{\mathcal{H}}_1^{-\frac{1}{2}})^t = \hat{\mathcal{H}}_1^{-\frac{1}{2}}$. Consequently, for the transposed operators $\hat{L}_1^t$ and $\hat{L}_2^t$ we obtain

$$\hat{L}_1^t = \frac{1}{2}\hat{L}_2^{-1} \quad \text{and} \quad \hat{L}_2^t = \frac{1}{2}\hat{L}_1^{-1}. \tag{22}$$

5 One-way wave equation in matrix-vector form

We define a "one-way wave field vector" $\mathbf{P}$ and a "one-way source vector" $\mathbf{S}$, according to

$$\mathbf{P} = \begin{pmatrix} P^+ \\ P^- \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} S^+ \\ S^- \end{pmatrix}. \tag{23}$$

We relate these vectors to the wave field and source vectors in equation (2), according to

$$\mathbf{P} = \hat{\mathbf{L}}^{-1}\mathbf{Q} \quad \text{and} \quad \mathbf{S} = \hat{\mathbf{L}}^{-1}\mathbf{D}. \tag{24}$$

Using equations (2) and (10) we obtain after some straightforward manipulations the following equation for $\mathbf{P}$

$$\partial_3 \mathbf{P} = \hat{\mathbf{B}}\mathbf{P} + \mathbf{S}, \tag{25}$$

where the one-way operator matrix $\hat{\mathbf{B}}$ is defined as

$$\hat{\mathbf{B}} = \hat{\mathbf{H}} + (\partial_3 \hat{\mathbf{L}}^{-1})\hat{\mathbf{L}}. \tag{26}$$

Equation (25) represents a system of coupled equations for the one-way wave fields P^+ and P^-, which propagate in the positive and negative x_3-direction, respectively, see Figure 1. In equation (26) $\hat{\mathbf{H}}$ accounts for propagation and $(\partial_3 \hat{\mathbf{L}}^{-1})\hat{\mathbf{L}}$ for scattering. In the following we refer to equation (25) as the "one-way wave equation". Analogous to equation (7) we introduce the transposed one-way operator matrix $\hat{\mathbf{B}}^t$ via

$$\langle \mathbf{f}, \hat{\mathbf{B}}^t \mathbf{g} \rangle_b = \langle \hat{\mathbf{B}}\mathbf{f}, \mathbf{g} \rangle_b, \tag{27}$$

where, analogous to equation (8), the bilinear form for vector functions is defined as

$$\langle \mathbf{f}, \mathbf{g} \rangle_b = \int_{x_{1,a}}^{x_{1,b}} \mathbf{f}^t(x_1)\mathbf{g}(x_1)\mathrm{d}x_1. \tag{28}$$

From equations (11), (12), (21), (22) and (26) we thus find for any fixed x_3

$$\hat{\mathbf{B}}^t\mathbf{N} = -\mathbf{N}\hat{\mathbf{B}}, \quad \text{with} \quad \mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{29}$$

6 Reciprocity theorem for one-way wave fields

We derive a reciprocity theorem that interrelates the one-way wave vectors, operator matrices and source vectors in two different states. These states will be distinguished by the subscripts A and B. The domain $\mathcal{D}$ for which we derive the reciprocity theorem is defined as $\mathcal{D} = \{\mathbf{x} \,|\, x_{1,a} \leq x_1 \leq x_{1,b} \wedge x_{3,a} \leq x_3 \leq x_{3,b}\}$, with boundary $\partial\mathcal{D} = \partial\mathcal{D}_1 \cup \partial\mathcal{D}_3$, see Figure 1. We define an interaction quantity between oppositely propagating waves in both states, according to

$$(30) \qquad \partial_3\{P_A^+ P_B^- - P_A^- P_B^+\} = \partial_3\{\mathbf{P}_A^t \mathbf{N} \mathbf{P}_B\}.$$

Applying the product rule for differentiation, substituting the one-way wave equation (25) for states A and B, integrating the result over domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$, applying the theorem of Gauss and using equations (27), (28) and (29) yields the following reciprocity theorem for one-way wave fields

$$(31) \qquad \int_{\partial\mathcal{D}_3} \mathbf{P}_A^t \mathbf{N} \mathbf{P}_B n_3 \mathrm{d}x_1 = \int_{\mathcal{D}} \mathbf{P}_A^t \mathbf{N} \hat{\mathbf{\Delta}} \mathbf{P}_B \mathrm{d}^2\mathbf{x} + \int_{\mathcal{D}} \{\mathbf{P}_A^t \mathbf{N} \mathbf{S}_B + \mathbf{S}_A^t \mathbf{N} \mathbf{P}_B\} \mathrm{d}^2\mathbf{x},$$

where the contrast operator $\hat{\mathbf{\Delta}}$ is defined as

$$(32) \qquad \hat{\mathbf{\Delta}} = \hat{\mathbf{B}}_B - \hat{\mathbf{B}}_A$$

and where the component n_3 of the outward pointing normal vector on $\partial\mathcal{D}_3$ is defined as $n_3 = -1$ for $x_3 = x_{3,a}$ and $n_3 = +1$ for $x_3 = x_{3,b}$.

7 Conclusions

We have derived a reciprocity theorem for electromagnetic or acoustic one-way wave fields (or bidirectional beams) in inhomogeneous lossy media. The same result was derived before for lossless acoustic media [6]. Reciprocity theorem (31) honors the natural separation between propagation and scattering in the one-way wave equations (see equations (25) and (26)). It forms a suitable starting point for the study of forward and inverse scattering problems in situations with a preferred direction of propagation.

References

[1] J. F. Claerbout, *Toward a unified theory of reflector mapping*, Geophysics, 36 (1971), pp. 467–481.

[2] M. W. P. Dillen, *Time-lapse seismic monitoring of subsurface stress dynamics*, PhD thesis, Delft University of Technology, Delft, 2000.

[3] L. Fishman and J. J. McCoy, *Derivation and application of extended parabolic wave theories. I. The factorized Helmholtz equation*, J. Math. Phys., 25 (1984), pp. 285–296.

[4] H. J. W. M. Hoekstra, *On beam propagation methods for modelling in integrated optics*, Optical and Quantum Electronics, 29 (1997), pp. 157–171.

[5] M. J. N. van Stralen, *Directional decomposition of electromagnetic and acoustic wave-fields*, PhD thesis, Delft University of Technology, Delft, 1997.

[6] C. P. A. Wapenaar and J. L. T. Grimbergen, *Reciprocity theorems for one-way wave fields*, Geoph. J. Int., 127 (1996), pp. 169–177.