Chapter 1
Reflection and Transmission of Waves at an Impermeable Interface between a Fluid and a Porous Medium

A. I. M. Denneman* G. G. Drijkoningen* D. M. J. Smeulders*
C. P. A. Wapenaar*

Abstract

In a previous paper [3] relatively simple closed-form expressions are obtained for the reflection and transmission coefficients belonging to a fluid/porous-medium interface with open-pore boundary conditions. In this paper a similar derivation is given for the case that this permeable interface is replaced by an impermeable one. The resulting expressions for the reflection and transmission coefficients for this sealed-pore case appear to be simpler than the ones belonging to the open-pore case. For both cases the obtained expressions find their application in forward and inverse surface wave analysis.

1 Waves at a fluid/porous-medium interface

In a well-known paper of Biot on waves in porous media [1] it is shown that three different waves may propagate in a porous material: a fast P-wave, a slow P-wave, and a S-wave. Consequently, at an fluid/porous-medium interface an incident P-wave in the fluid is converted simultaneously into (i) a reflected P-wave, (ii) a transmitted fast P-wave, (iii) a transmitted slow P-wave, and (iv) a transmitted S-wave. (see Fig. 1).

The fluid displacements in the x-z plane belonging to the incident and reflected P-wave are in the space-frequency domain given by

$$\mathbf{U}^{\mathrm{I}} = \begin{pmatrix} U_{\mathrm{x}}^{\mathrm{I}}(x,z,\omega) \\ U_{\mathrm{z}}^{\mathrm{I}}(x,z,\omega) \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} A^{\mathrm{I}} \exp\left[-j\omega(px+qz)\right], \tag{1}$$

$$\mathbf{U}^{\mathrm{R}} = \begin{pmatrix} U_{\mathrm{x}}^{\mathrm{R}}(x,z,\omega) \\ U_{\mathrm{z}}^{\mathrm{R}}(x,z,\omega) \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} A^{\mathrm{R}} \exp\left[-j\omega(px-qz)\right], \tag{2}$$

where ω is the angular frequency, p the horizontal slowness, q the vertical slowness with a positive real part and a negative imaginary part, and A^{I} and A^{R} the wave-amplitudes. The slownesses p and q are related to the propagation velocity c as

$$p^2 + q^2 = \frac{1}{c^2} = \frac{\rho}{K}, \tag{3}$$

where ρ is the fluid density and K the fluid bulk modulus. Furthermore, by combining the deformation equation $P = -K\boldsymbol{\nabla}\cdot(\mathbf{U}^{\mathrm{I}}+\mathbf{U}^{\mathrm{R}})$ with Eqs. (1)–(3) one finds that the fluid pressure P is given by

$$P(x,z,\omega) = j\omega\rho\left[A^{\mathrm{I}}\exp\left[-j\omega(px+qz)\right] + A^{\mathrm{R}}\exp\left[-j\omega(px-qz)\right]\right]. \tag{4}$$

*Centre for Technical Geoscience, Delft Univ. of Techn., P.O. Box 5028, 2600 GA Delft, The Netherlands.

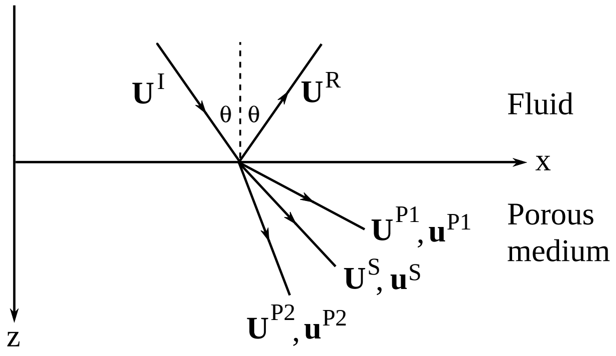


FIG. 1. *Wave conversion at a fluid/porous-medium interface (an incoming P-wave with incident angle θ). The fluid displacements belonging to the incident and reflected P-wave are denoted by $\mathbf{U}^{\mathrm{I}}$ and $\mathbf{U}^{\mathrm{R}}$, respectively. The fluid displacements belonging to the fast P-wave, slow P-wave, and S-wave are denoted by $\mathbf{U}^{\mathrm{P1}}$, $\mathbf{U}^{\mathrm{P2}}$, and $\mathbf{U}^{\mathrm{S}}$, respectively, while the corresponding solid displacements are denoted by $\mathbf{u}^{\mathrm{P1}}$, $\mathbf{u}^{\mathrm{P2}}$, and $\mathbf{u}^{\mathrm{S}}$, respectively.*

The displacements of the solid skeleton in the x-z plane belonging to the fast P-wave, slow P-wave, and S-wave are given by

$$\mathbf{u}^{\mathrm{P1}} = \begin{pmatrix} u_{\mathrm{x}}^{\mathrm{P1}}(x,z,\omega) \\ u_{\mathrm{z}}^{\mathrm{P1}}(x,z,\omega) \end{pmatrix} = \begin{pmatrix} p \\ q_{\mathrm{P1}} \end{pmatrix} A^{\mathrm{P1}} \exp\left[-j\omega(px + q_{\mathrm{P1}}z)\right], \tag{5}$$

$$\mathbf{u}^{\mathrm{P2}} = \begin{pmatrix} u_{\mathrm{x}}^{\mathrm{P2}}(x,z,\omega) \\ u_{\mathrm{z}}^{\mathrm{P2}}(x,z,\omega) \end{pmatrix} = \begin{pmatrix} p \\ q_{\mathrm{P2}} \end{pmatrix} A^{\mathrm{P2}} \exp\left[-j\omega(px + q_{\mathrm{P2}}z)\right], \tag{6}$$

$$\mathbf{u}^{\mathrm{S}} = \begin{pmatrix} u_{\mathrm{x}}^{\mathrm{S}}(x,z,\omega) \\ u_{\mathrm{z}}^{\mathrm{S}}(x,z,\omega) \end{pmatrix} = \begin{pmatrix} -q_{\mathrm{S}} \\ p \end{pmatrix} A^{\mathrm{S}} \exp\left[-j\omega(px + q_{\mathrm{S}}z)\right], \tag{7}$$

where A^{P1}, A^{P2}, and A^{S} are the wave-amplitudes. The vertical slownesses q_{P1}, q_{P2}, and q_{S} (all with a positive real part and a negative imaginary part) are related to the horizontal slowness p and the propagation velocities c_{P1}, c_{P2}, and c_{S} as

$$p^2 + q_{\mathrm{P1}}^2 = \frac{1}{c_{\mathrm{P1}}^2}, \qquad p^2 + q_{\mathrm{P2}}^2 = \frac{1}{c_{\mathrm{P2}}^2}, \qquad p^2 + q_{\mathrm{S}}^2 = \frac{1}{c_{\mathrm{S}}^2}. \tag{8}$$

According to Biot's theory [1, 5] the propagation velocities c_{P1}, c_{P2}, and c_{S} are given by

$$c_{\mathrm{P1}}^2 = \frac{c_1 + \sqrt{c_1^2 - 4c_0c_2}}{2c_0}, \qquad c_{\mathrm{P2}}^2 = \frac{c_1 - \sqrt{c_1^2 - 4c_0c_2}}{2c_0}, \qquad c_{\mathrm{S}}^2 = \frac{G\rho_{22}}{c_0} \tag{9}$$

with

$$c_0 = \rho_{11}\rho_{22} - \rho_{12}^2, \qquad c_1 = R\rho_{11} + (A+2G)\rho_{22} - 2Q\rho_{12}, \qquad c_2 = R(A+2G) - Q^2, \tag{10}$$

where G is the shear modulus of the porous material. The generalized elastic coefficients A, Q, and R are related to measurable quantities by the following expressions [2, 5]

$$A = \frac{(1-\phi)^2 K_{\mathrm{s}}K_{\mathrm{f}} - (1-\phi)K_{\mathrm{b}}K_{\mathrm{f}} + \phi K_{\mathrm{s}}K_{\mathrm{b}}}{K_{\mathrm{f}}(1-\phi-K_{\mathrm{b}}/K_{\mathrm{s}}) + \phi K_{\mathrm{s}}} - \frac{2}{3}G, \tag{11}$$

$$Q = \frac{\phi K_{\mathrm{f}}(K_{\mathrm{s}}(1-\phi) - K_{\mathrm{b}})}{K_{\mathrm{f}}(1-\phi-K_{\mathrm{b}}/K_{\mathrm{s}}) + \phi K_{\mathrm{s}}}, \tag{12}$$

$$R = \frac{\phi^2 K_{\mathrm{f}}K_{\mathrm{s}}}{K_{\mathrm{f}}(1-\phi-K_{\mathrm{b}}/K_{\mathrm{s}}) + \phi K_{\mathrm{s}}}, \tag{13}$$

where K_s is the skeletal grain bulk modulus, K_f the pore fluid bulk modulus, K_b the "jacketed" bulk modulus of the porous material, and ϕ the porosity (pore fluid volume divided by bulk volume).

The density terms ρ_{11}, ρ_{22}, and ρ_{12} in Eqs. (9) and (10) are defined as

$$\rho_{11} = (1-\phi)\rho_\mathrm{s} - \rho_{12}, \qquad \rho_{22} = \phi\rho_\mathrm{f} - \rho_{12}, \qquad \rho_{12} = -(\alpha - 1)\phi\rho_\mathrm{f}, \tag{14}$$

where ρ_s and ρ_f are the densities of the solid skeleton and the pore fluid, respectively. According to Johnson *et al.* [7] the drag coefficient α belonging to a fluid-saturated porous material can be defined as

$$\alpha = \alpha_\infty \left(1 - j\frac{\omega_\mathrm{c}}{\omega}\sqrt{1 + j\frac{M}{2}\frac{\omega}{\omega_\mathrm{c}}}\right) \qquad \text{with} \qquad \omega_\mathrm{c} = \frac{\eta\phi}{k_0\rho_\mathrm{f}\alpha_\infty}, \tag{15}$$

where $M \approx 1$ is the so-called similarity parameter, α_∞ the inertial drag at infinite frequency (or tortuosity), η the pore fluid viscosity, k_0 the permeability of the porous material, and the critical frequency ω_c is the frequency at which the inertial and viscous drag are of comparable magnitude.

According to Biot's theory the pore fluid displacements $\mathbf{U}^\mathrm{P1}$, $\mathbf{U}^\mathrm{P2}$, and $\mathbf{U}^\mathrm{S}$ are related to the solid skeleton displacements $\mathbf{u}^\mathrm{P1}$, $\mathbf{u}^\mathrm{P2}$, and $\mathbf{u}^\mathrm{S}$ as

$$\mathbf{U}^\mathrm{P1} = G_\mathrm{P1}\mathbf{u}^\mathrm{P1} \qquad \text{with} \qquad G_\mathrm{P1} = \frac{Q - c_\mathrm{P1}^2\rho_{12}}{c_\mathrm{P1}^2\rho_{22} - R} = \frac{A + 2G - c_\mathrm{P1}^2\rho_{11}}{c_\mathrm{P1}^2\rho_{12} - Q}, \tag{16}$$

$$\mathbf{U}^\mathrm{P2} = G_\mathrm{P2}\mathbf{u}^\mathrm{P2} \qquad \text{with} \qquad G_\mathrm{P2} = \frac{Q - c_\mathrm{P2}^2\rho_{12}}{c_\mathrm{P2}^2\rho_{22} - R} = \frac{A + 2G - c_\mathrm{P2}^2\rho_{11}}{c_\mathrm{P2}^2\rho_{12} - Q}, \tag{17}$$

$$\mathbf{U}^\mathrm{S} = G_\mathrm{S}\mathbf{u}^\mathrm{S} \qquad \text{with} \qquad G_\mathrm{S} = \frac{-\rho_{12}}{\rho_{22}} = \frac{\alpha - 1}{\alpha}. \tag{18}$$

The pore fluid stress $\boldsymbol{\tau}_\mathrm{f}$ and solid skeleton stress $\boldsymbol{\tau}_\mathrm{s}$ are defined as

$$\boldsymbol{\tau}_\mathrm{f} = -\phi P_\mathrm{f}\boldsymbol{\delta} = Q(\boldsymbol{\nabla}\cdot\mathbf{u})\boldsymbol{\delta} + R(\boldsymbol{\nabla}\cdot\mathbf{U})\boldsymbol{\delta}, \tag{19}$$

$$\boldsymbol{\tau}_\mathrm{s} = -\boldsymbol{\sigma} - (1-\phi)P_\mathrm{f}\boldsymbol{\delta} = G\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T\right] + A(\boldsymbol{\nabla}\cdot\mathbf{u})\boldsymbol{\delta} + Q(\boldsymbol{\nabla}\cdot\mathbf{U})\boldsymbol{\delta} \tag{20}$$

with $\mathbf{u} = \mathbf{u}^\mathrm{P1} + \mathbf{u}^\mathrm{P2} + \mathbf{u}^\mathrm{S}$ and $\mathbf{U} = \mathbf{U}^\mathrm{P1} + \mathbf{U}^\mathrm{P2} + \mathbf{U}^\mathrm{S}$, and where P_f is the pore fluid pressure, $\boldsymbol{\delta}$ a unit tensor, and $\boldsymbol{\sigma}$ the intergranular stress tensor.

2 Reflection and transmission coefficients

The reflection and transmission coefficients R^F, T^P1, T^P2, and T^S are related to the wave-amplitudes A^I, A^R, A^P1, A^P2, and A^S as

$$A^\mathrm{R} = R^\mathrm{F}A^\mathrm{I}, \qquad A^\mathrm{P1} = T^\mathrm{P1}A^\mathrm{I}, \qquad A^\mathrm{P2} = T^\mathrm{P2}A^\mathrm{I}, \qquad A^\mathrm{S} = T^\mathrm{S}A^\mathrm{I}. \tag{21}$$

To solve R^F, T^P1, T^P2, and T^S the boundary conditions associated with an impermeable fluid/porous-medium interface are used, i.e., the sealed-pore boundary conditions [4, 5, 6]. Hence, at the boundary $z = 0$:

$$u_\mathrm{z}^\mathrm{P1} + u_\mathrm{z}^\mathrm{P2} + u_\mathrm{z}^\mathrm{S} = U_\mathrm{z}^\mathrm{I} + U_\mathrm{z}^\mathrm{R} \qquad -\sigma_\mathrm{zz} - P_\mathrm{f} = -P, \tag{22}$$

$$u_\mathrm{z}^\mathrm{P1} + u_\mathrm{z}^\mathrm{P2} + u_\mathrm{z}^\mathrm{S} = U_\mathrm{z}^\mathrm{P1} + U_\mathrm{z}^\mathrm{P2} + U_\mathrm{z}^\mathrm{S} \qquad \sigma_\mathrm{xz} = 0. \tag{23}$$

By combining these four boundary conditions with Eqs. (1), (2), (4)–(8), and (16)–(20) in an appropriate way one obtains the following set of linear equations

$$(24)\quad \begin{bmatrix} q & q_{\mathrm{P1}} & q_{\mathrm{P2}} & p \\ \dfrac{\rho}{2G} & \left(p^2-\dfrac{K_{\mathrm{P1}}}{2Gc_{\mathrm{P1}}^2}\right) & \left(p^2-\dfrac{K_{\mathrm{P2}}}{2Gc_{\mathrm{P2}}^2}\right) & -pq_{\mathrm{S}} \\ 0 & q_{\mathrm{P1}}(G_{\mathrm{P1}}-1) & q_{\mathrm{P2}}(G_{\mathrm{P2}}-1) & p(G_{\mathrm{S}}-1) \\ 0 & pq_{\mathrm{P1}} & pq_{\mathrm{P2}} & \left(p^2-\dfrac{1}{2c_{\mathrm{S}}^2}\right) \end{bmatrix} \begin{bmatrix} R^{\mathrm{F}} \\ T^{\mathrm{P1}} \\ T^{\mathrm{P2}} \\ T^{\mathrm{S}} \end{bmatrix} = \begin{bmatrix} q \\ -\dfrac{\rho}{2G} \\ 0 \\ 0 \end{bmatrix},$$

where the moduli K_{P1} and K_{P2} are defined as

$$(25)\quad K_{\mathrm{P1}} = A + 2G + Q + G_{\mathrm{P1}}\,(Q+R)\,, \qquad K_{\mathrm{P2}} = A + 2G + Q + G_{\mathrm{P2}}\,(Q+R)\,.$$

To obtain relatively simple closed-form expressions for R^{F}, T^{P1}, T^{P2}, and T^{S} it is assumed that both the porous skeleton and the pore fluid are much more compressible than the skeletal solid grains themselves. Consequently, the substitution of $K_{\mathrm{s}} \gg K_{\mathrm{b}}$ and $K_{\mathrm{s}} \gg K_{\mathrm{f}}$ in Eqs. (11)–(13) leads to

$$(26)\quad A = \frac{(1-\phi)^2}{\phi}K_{\mathrm{f}} + K_{\mathrm{b}} - \frac{2}{3}G, \qquad Q = (1-\phi)K_{\mathrm{f}}, \qquad R = \phi K_{\mathrm{f}}.$$

By combining Eq. (26) with Eqs. (9), (10), (14), (16)–(18), and (25) one obtains

$$(27)\quad K_{\mathrm{P1}} = K_{\mathrm{b}} + \frac{4}{3}G + \delta_1\rho_{\mathrm{f}}c_{\mathrm{P1}}^2, \qquad \frac{G_{\mathrm{P1}}-1}{G_{\mathrm{S}}-1} = (1-\delta_1),$$

$$(28)\quad K_{\mathrm{P2}} = K_{\mathrm{b}} + \frac{4}{3}G + \delta_2\rho_{\mathrm{f}}c_{\mathrm{P2}}^2, \qquad \frac{G_{\mathrm{P2}}-1}{G_{\mathrm{S}}-1} = (1-\delta_2),$$

$$(29)\quad c_{\mathrm{P1}}^2c_{\mathrm{P2}}^2 = \frac{(K_{\mathrm{b}}+\frac{4}{3}G)K_{\mathrm{f}}}{G\alpha\rho_{\mathrm{f}}}c_{\mathrm{S}}^2, \qquad \delta_1\delta_2 = -\frac{\alpha G}{\phi\rho_{\mathrm{f}}c_{\mathrm{S}}^2}$$

with the parameters δ_1 and δ_2 given by

$$(30)\quad \delta_1 = \frac{(\alpha-\phi)K_{\mathrm{f}}}{\alpha\phi\rho_{\mathrm{f}}c_{\mathrm{P1}}^2 - \phi K_{\mathrm{f}}}, \qquad \delta_2 = \frac{(\alpha-\phi)K_{\mathrm{f}}}{\alpha\phi\rho_{\mathrm{f}}c_{\mathrm{P2}}^2 - \phi K_{\mathrm{f}}}.$$

By combining Eqs. (24) and (27)–(29) in an appropriate way one finds that the closed-form expressions for R^{F}, T^{P1}, T^{P2}, and T^{S} are given by

$$(31)\quad R^{\mathrm{F}} = \frac{R_1-R_2}{R_1+R_2}, \qquad T^{\mathrm{P1}} = \frac{2q_{\mathrm{P2}}\delta_2}{R_1+R_2}\left(p^2-\frac{K_{\mathrm{P1}}}{2Gc_{\mathrm{P1}}^2}\right),$$

$$(32)\quad T^{\mathrm{S}} = \frac{2q_{\mathrm{P1}}q_{\mathrm{P2}}}{R_1+R_2}(\delta_1-\delta_2), \qquad T^{\mathrm{P2}} = \frac{-2q_{\mathrm{P1}}\delta_1}{R_1+R_2}\left(p^2-\frac{K_{\mathrm{P2}}}{2Gc_{\mathrm{P2}}^2}\right),$$

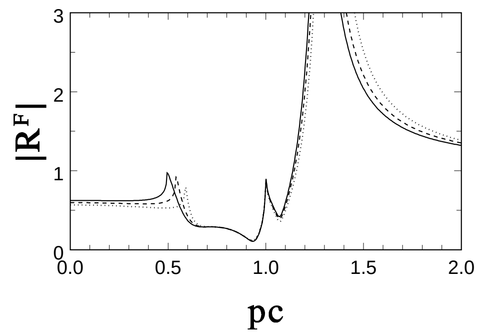
where R_1 and R_2 are defined as

$$(33)\quad R_1 = \frac{2G}{\rho}\left(\delta_1 q_{\mathrm{P1}}\Delta_2 - \delta_2 q_{\mathrm{P2}}\Delta_1\right), \qquad R_2 = \frac{q_{\mathrm{P1}}q_{\mathrm{P2}}}{2qc_{\mathrm{S}}^2}(\delta_1-\delta_2),$$

while Δ_1 and Δ_2 are given by

$$(34)\quad \Delta_1 = p^2q_{\mathrm{S}}q_{\mathrm{P1}} + \left(p^2-\frac{K_{\mathrm{P1}}}{2Gc_{\mathrm{P1}}^2}\right)^2, \qquad \Delta_2 = p^2q_{\mathrm{S}}q_{\mathrm{P2}} + \left(p^2-\frac{K_{\mathrm{P2}}}{2Gc_{\mathrm{P2}}^2}\right)^2.$$

Note that these expressions for R^{F}, T^{P1}, T^{P2}, and T^{S} are much simpler than the ones belonging to a fluid/porous-medium interface with open-pore boundary conditions [3].



$\omega = 15000\,\mathrm{rad\,s^{-1}}$	$K = 2.22\,\mathrm{G\,Pa}$
$\phi = 0.3$	$K_\mathrm{f} = 2.22\,\mathrm{G\,Pa}$
$\rho = 1000\,\mathrm{kg\,m^{-3}}$	$K_\mathrm{b} = 6.67\,\mathrm{G\,Pa}$
$\rho_\mathrm{f} = 1000\,\mathrm{kg\,m^{-3}}$	$G = 4.00\,\mathrm{G\,Pa}$
$\rho_\mathrm{s} = 2640\,\mathrm{kg\,m^{-3}}$	$\alpha_\infty = 2$
$\eta = 0.001\,\mathrm{Pa\,s}$	$M = 1$
$k_0 = 12\,10^{-12}\,\mathrm{m^2}$	

FIG. 2. *The reflection coefficient $|R^\mathrm{F}|$ as a function of the horizontal slowness p times the propagation velocity c of the P-wave in the fluid. The solid lines correspond to the case of rigid solid grains (i.e., $K_\mathrm{s}/K_\mathrm{b} = \infty$); in this case $|R^\mathrm{F}|$ is calculated by using Eqs. (31) and (33). The dashed and dotted lines correspond to the cases $K_\mathrm{s}/K_\mathrm{b} = 5$ and $K_\mathrm{s}/K_\mathrm{b} = 2$, respectively; in these two cases $|R^\mathrm{F}|$ is obtained by solving the set of linear equations given by Eq. (24) and by using the expressions for A, Q, and R given by Eqs. (11)–(13) instead of the ones given in Eq. (26). Note that these results can be easily transformed into a figure showing the reflection coefficient $|R^\mathrm{F}|$ as a function of the incident angle θ by using the relation $\theta = arcsin(pc)$ for the region $|pc| \leq 1$.*

3 Surface waves

The phase velocity of a surface wave traveling along an impermeable fluid/porous-medium interface can be obtained by finding a horizontal slowness p for which the denominator of R^F is minimum (consequently, $|R^\mathrm{F}|$ is very large for this p): the reciprocal of the obtained p is then the surface wave velocity. For example, in Fig. 2 one observes that $|R^\mathrm{F}|$ is very large for $pc \approx 1.3$, so for this specific case the surface wave velocity is 0.8 times the propagation velocity c of the P-wave in the fluid. In general, one might say that the value of the surface wave velocity is only weakly dependent on the actual value of the skeletal grain bulk modulus K_s, which implies that one may use the expression for R^F given by Eqs. (31) and (33) to determine the surface wave velocity. It is clear that the availability of these simple closed-form expressions will facilitate the research in forward and inverse surface wave analysis a lot.

References

[1] M. A. Biot, *Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range and II. Higher frequency range*, J. Acoust. Soc. Am., 28 (1956), pp. 168–191.

[2] M. A. Biot and D. G. Willis, *The elastic coefficients of the theory of consolidation*, J. Appl. Mech., 24 (1957), pp. 594–601.

[3] A. I. M. Denneman, G. G. Drijkoningen, D. M. J. Smeulders, and C. P. A. Wapenaar, *Reflection and transmission of waves at a fluid/porous-medium boundary*, in Proceedings of IUTAM-Symposium on Theoretical and Numerical Methods in Continuum Mechanics of Porous Materials (Stuttgart, September 5–10, 1999), Kluwer Academic, Dordrecht, The Netherlands.

[4] H. Deresiewicz and R. Skalak, *On uniqueness in dynamic poroelasticity*, Bull. Seism. Soc. Am., 53 (1963), pp. 783–788.

[5] S. Feng and D. L. Johnson, *High-frequency acoustic properties of a fluid/porous solid interface. I. New surface mode*, J. Acoust. Soc. Am., 74 (1983), pp. 906–914.

[6] B. Gurevich and M. Schoenberg, *Interface conditions for Biot's equations of poroelasticity*, J. Acoust. Soc. Am., 105 (1999), pp. 2585–2589.

[7] D. L. Johnson, J. Koplik, and R. Dashen, *Theory of dynamic permeability and tortuosity in fluid-saturated porous media*, J. Fluid Mech., 176 (1987), pp. 379–402.