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Causality aspects of the elastodynamic Marchenko method

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Summary
With the acoustic single-sided Marchenko method it is possible to retrieve the Green's function of a virtual source in the subsurface from the single-sided reflection response of the medium and an estimate of the first arrival of the Green's function. Important ingredients of the Marchenko method are the so-called focusing functions. One of the underlying ideas of the acoustic Marchenko method is that the Green's functions and focusing functions reside in different time intervals in the time domain. We call this the causality condition. The only overlap of the Green's functions and focusing functions occurs at one time instant, namely at the time of the direct arrival of the Green's function.

In this paper we analyze the causality condition for the elastodynamic extension of the single-sided Marchenko method. It appears that the overlap of the elastodynamic Green's functions and focusing functions occurs in an extended time interval. The parts of the focusing functions that overlap with the Green's functions cannot be retrieved with the Marchenko method, and must therefore be specified separately. It appears that these overlapping parts are defined as the inverse of the forward-scattered part of the transmission response of the medium.

Introduction

For a brief historic overview and an explanation of single-sided Marchenko Imaging, please refer to the paper by Roel Snieder at this workshop (Snieder, 2015). Current research on the Marchenko method concerns the extension to the elastodynamic situation (da Costa et al., 2014a,b; Wapenaar and Slob, 2014; Wapenaar, 2014). The extension of the Marchenko method from the acoustic to the elastodynamic situation is hampered by the fact that simple causality arguments, underlying the acoustic method, do not hold for the elastodynamic method. The aim of this paper is to emphasize the role of causality in the theory of the Marchenko method and to discuss how this plays a role in the elastodynamic extension.

Review of the causality aspects of the 1D acoustic Marchenko scheme

The causality aspects are best illustrated at the hand of the 1D version of the acoustic Marchenko scheme. Consider the following 1D Green's function representations (Wapenaar et al., 2014; Slob et al., 2014)

$$G^{-,+}(z_0,z_i,t)+f_1^-(z_0,z_i,t) = \int_{-\infty}^{t} R(z_0,t-t')f_1^+(z_0,z_i,t')\mathrm{d}t', \quad (1)$$

$$G^{-,-}(z_0,z_i,t)+f_1^+(z_0,z_i,-t) = \int_{-\infty}^{t} R(z_0,t-t')f_1^-(z_0,z_i,-t')\mathrm{d}t', \quad (2)$$

respectively. Here $R(z_0,t)$ is the 1D single-sided reflection response of a horizontally layered medium below a homogeneous half-space, observed at the surface $z=z_0$ (in practice this is the response obtained after surface-related multiple elimination and deconvolution for the source wavelet). $G^{-,+}(z_0,z_i,t)$ and $G^{-,-}(z_0,z_i,t)$ are the decomposed Green's functions for a receiver at z_0 and a source at z_i. The superscripts refer, in the same order, to the propagation direction at these depth levels. The focusing functions $f_1^+(z_0,z_i,t)$ and $f_1^-(z_0,z_i,t)$ are defined in a reference medium, which is identical to the actual medium above z_i and homogeneous below this depth level. The focusing function $f_1^+(z_0,z_i,t)$ is a downgoing wave field at z_0, shaped such that it focuses at z_i, i.e., $f_1^+(z_i,z_i,t)=\delta(t)$. The focusing function $f_1^-(z_0,z_i,t)$ is the reflection response of $f_1^+(z_0,z_i,t)$ in the reference medium, observed at the surface z_0. Note that in these representations, z_i denotes the focal point of the focusing functions and the (virtual) source point of the Green's functions. A second set of focusing functions exists, $f_2^+(z_i,z_0,t)$ and $f_2^-(z_i,z_0,t)$, where $f_2^-(z_i,z_0,t)$ is an upgoing wave field at z_i, shaped such that it focuses at z_0, and $f_2^+(z_i,z_0,t)$ is its reflection response, observed at z_i. The mutual relations between the focusing functions are

$$f_1^+(z_0,z_i,t) = f_2^-(z_i,z_0,t), \quad (3)$$

$$-f_1^-(z_0,z_i,-t) = f_2^+(z_i,z_0,t). \quad (4)$$

Equations (1) and (2) form a set of two equations with four unknowns, namely $G^{-,+}$, $G^{-,-}$, f_1^+ and f_1^-. To solve this set of equations, we consider causality arguments to separate the Green's functions $G^{-,+}$ and $G^{-,-}$ from the focusing functions f_1^+ and f_1^-. Let the travel time of the direct arrival between z_0 and z_i be defined as t^{d}. Then, assuming z_i does not correspond with an interface, we have

$$G^{-,-}(z_0,z_i,t)=0, \quad \text{for} \quad t<t^{\mathrm{d}}, \quad (5)$$

$$G^{-,+}(z_0,z_i,t)=0, \quad \text{for} \quad t\leq t^{\mathrm{d}}. \quad (6)$$

Note the subtle difference of the intervals where these functions are zero. For $G^{-,-}$ this excludes the point $t=t^{\mathrm{d}}$ (because $G^{-,-}$ contains the direct arrival), whereas $G^{-,+}$ equals zero also at $t=t^{\mathrm{d}}$ (assuming the source at z_i does not lie on an interface). The focusing function $f_1^+(z_0,z_i,t)$, which focuses at z_i, is by definition the inverse of the transmission response $T(z_i,z_0,t)$ of the medium between z_0 and z_i. For the transmission response we have $T(z_i,z_0,t)=0$ for $t<t^{\mathrm{d}}$. The coda of the transmission response is causal and minimum-phase (Anstey and O'Doherty, 1971). The inverse of a minimum-phase signal is causal and minimum-phase as well (Robinson, 1954). Hence, for $f_1^+(z_0,z_i,t)$, which is the inverse of $T(z_i,z_0,t)$, the time-reversed direct arrival at $-t^{\mathrm{d}}$ is followed by a causal coda, hence $f_1^+(z_0,z_i,t)=0$ for $t<-t^{\mathrm{d}}$. The left-hand side of equation (2) contains the time-reversal of f_1^+, for which we thus have

$$f_1^+(z_0,z_i,-t)=0, \quad \text{for} \quad t>t^{\mathrm{d}}. \quad (7)$$

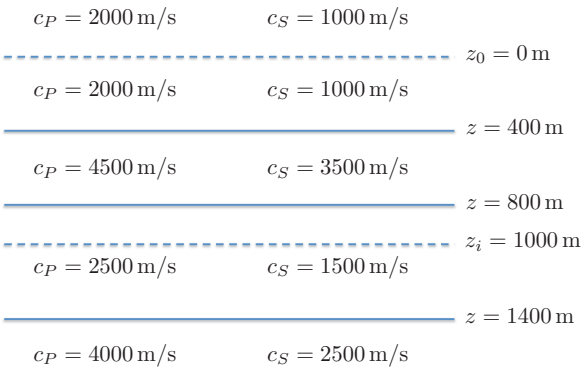

Figure 1 *Horizontally layered medium. The indicated shear-wave velocities only apply to the elastodynamic example. The mass density is 2000 kg/m^3 in all layers.*

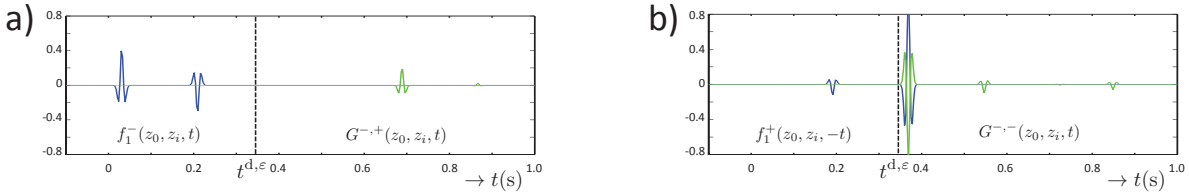

Figure 2 *Green's functions (green) and focusing functions (blue) for the 1D acoustic case.*

Since f_1^- is the reflection response of f_1^+, we have $f_1^-(z_0,z_i,t) = 0$ for $t \leq -t^{\mathrm{d}}$. Similarly, since f_2^+ is the reflection response of f_2^-, we have $f_2^+(z_i,z_0,t) = 0$ for $t \leq -t^{\mathrm{d}}$. Combining this with equation (4) we find

$$f_1^-(z_0,z_i,t) = 0, \quad \text{for} \quad t \geq t^{\mathrm{d}}. \tag{8}$$

Equations (6) and (8) show that the functions at the left-hand side of equation (1) are completely separated in time. This is illustrated in Figure 2a for the horizontally layered medium of Figure 1. Equations (5) and (7) show that the functions at the left-hand side of equation (2) overlap only at $t = t^{\mathrm{d}}$, which is illustrated in Figure 2b. Note that in these figures the Green's functions and focusing functions have been convolved with a wavelet, and $t^{\mathrm{d},\varepsilon} = t^{\mathrm{d}} - \varepsilon$ is defined as the onset of the direct arrival. Using these properties, we deduce from equations (1) and (2)

$$f_1^-(z_0,z_i,t) = \int_{-\infty}^{t} R(z_0,t-t')f_1^+(z_0,z_i,t')\mathrm{d}t', \quad \text{for} \quad t < t^{\mathrm{d}}, \tag{9}$$

$$f_1^+(z_0,z_i,-t) = \int_{-\infty}^{t} R(z_0,t-t')f_1^-(z_0,z_i,-t')\mathrm{d}t', \quad \text{for} \quad t < t^{\mathrm{d}}. \tag{10}$$

Note that equation (9) covers the entire function $f_1^-(z_0,z_i,t)$, whereas equation (10) covers the function $f_1^+(z_0,z_i,-t)$, except its arrival at $t = t^{\mathrm{d}}$. Equations (9) and (10) form a set of two equations with two unknowns, f_1^+ and f_1^-. Assuming the reflection response $R(z_0,t)$ and the arrival at $t = -t^{\mathrm{d}}$ of $f_1^+(z_0,z_i,t)$ are known, these equations can be solved via an iterative Marchenko scheme (Slob et al., 2014).

Note that this causality discussion holds perfectly for the 1D acoustic situation. For the 3D acoustic situation it holds only in an approximate sense (Wapenaar et al., 2014). For the elastodynamic situation it needs to be reconsidered, which we will do in the next section for a 1D medium.

Causality aspects of the 1D elastodynamic Marchenko scheme

We consider oblique plane waves in a horizontally layered medium. The propagation angle is represented via the ray parameter p. Consider the following 1D Green's function representations in the

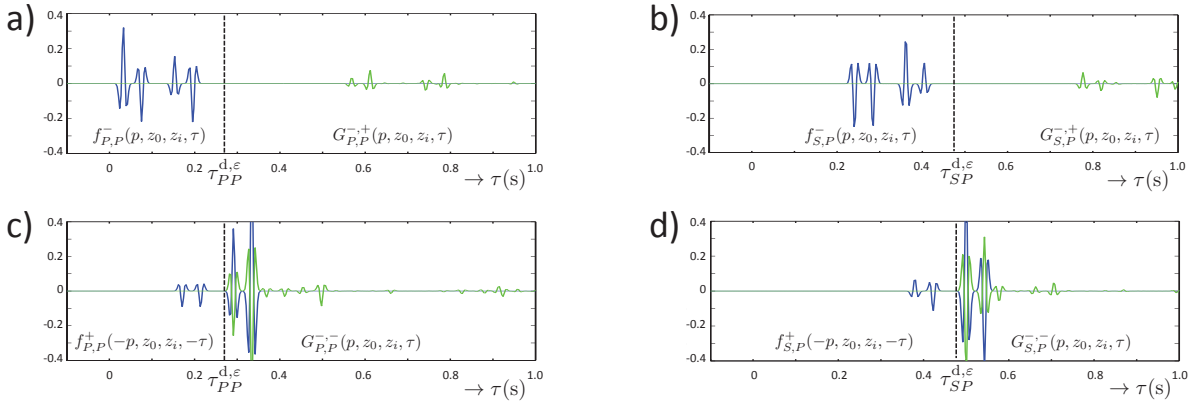

Figure 3 *Elastodynamic Green's functions (green) and focusing functions (blue) for oblique incidence.*

rayparameter intercept time (p,τ) domain (Wapenaar, 2014)

$$\mathbf{G}^{-,+}(p,z_0,z_i,\tau)+\mathbf{F}_1^{-}(p,z_0,z_i,\tau) = \int_{-\infty}^{\tau}\mathbf{R}(p,z_0,\tau-\tau')\mathbf{F}_1^{+}(p,z_0,z_i,\tau')\mathrm{d}\tau', \tag{11}$$

$$\mathbf{G}^{-,-}(p,z_0,z_i,\tau)+\mathbf{F}_1^{+}(-p,z_0,z_i,-\tau) = \int_{-\infty}^{\tau}\mathbf{R}(p,z_0,\tau-\tau')\mathbf{F}_1^{-}(-p,z_0,z_i,-\tau')\mathrm{d}\tau', \tag{12}$$

$$\mathbf{G}^{-,\pm}=\begin{pmatrix}G^{-,\pm}_{P,P} & G^{-,\pm}_{P,S}\\ G^{-,\pm}_{S,P} & G^{-,\pm}_{S,S}\end{pmatrix},\ \mathbf{F}_1^{\pm}=\begin{pmatrix}f^{\pm}_{P,P} & f^{\pm}_{P,S}\\ f^{\pm}_{S,P} & f^{\pm}_{S,S}\end{pmatrix},\ \mathbf{R}=\begin{pmatrix}R_{P,P} & R_{P,S}\\ R_{S,P} & R_{S,S}\end{pmatrix}, \tag{13}$$

with subscripts P and S standing for compressional and shear waves, respectively. We illustrate some elements of the Green's functions and focusing functions for the medium of Figure 1 in Figure 3, choosing $p=0.0002$ s/m. Note that $\tau^{\mathrm{d},\varepsilon}_{SP}=\tau^{\mathrm{d}}_{SP}-\varepsilon$, where τ^{d}_{SP} is the arrival time of the first arrival of $G^{-,-}_{S,P}(p,z_0,z_i,\tau)$, etc. (this wave starts as an upgoing P-wave at $z_i=1000$ m, converts to an S-wave at $z=400$ m and hence arrives as an S-wave at $z_0=0$ m). Figures 3c and 3d clearly show that the overlap between the functions in the left-hand side of equation (12) is not restricted to the first arrival. In Figures 3a and 3b the functions in the left-hand side of equation (11) are well-separated, but this only holds when z_i is not too close to the interface directly above it. In the following we assume this condition is fulfilled. We define a time-window matrix $\mathbf{W}(p,\tau)$, according to

$$\mathbf{W}(p,\tau)=\begin{pmatrix}H(\tau^{\mathrm{d},\varepsilon}_{PP}-\tau) & H(\tau^{\mathrm{d},\varepsilon}_{PS}-\tau)\\ H(\tau^{\mathrm{d},\varepsilon}_{SP}-\tau) & H(\tau^{\mathrm{d},\varepsilon}_{SS}-\tau)\end{pmatrix}, \tag{14}$$

where $H(\tau)$ is the Heaviside step function. Note that $\mathbf{W}(p,\tau)\circ\mathbf{G}^{-,\pm}(p,z_0,z_i,\tau)=\mathbf{O}$, where $\mathbf{O}$ is the null matrix and $\circ$ denotes Hadamard matrix multiplication (i.e., element-wise multiplication). Moreover, with the assumption made above, we have $\mathbf{W}(p,\tau)\circ\mathbf{F}_1^{-}(p,z_0,z_i,\tau)=\mathbf{F}_1^{-}(p,z_0,z_i,\tau)$. Hence, applying $\mathbf{W}(p,\tau)$ to both sides of equations (11) and (12) we obtain

$$\mathbf{F}_1^{-}(p,z_0,z_i,\tau) = \mathbf{W}(p,\tau)\circ\int_{-\infty}^{\tau}\mathbf{R}(p,z_0,\tau-\tau')\mathbf{F}_1^{+}(p,z_0,z_i,\tau')\mathrm{d}\tau', \tag{15}$$

$$\mathbf{W}(p,\tau)\circ\mathbf{F}_1^{+}(-p,z_0,z_i,-\tau) = \mathbf{W}(p,\tau)\circ\int_{-\infty}^{\tau}\mathbf{R}(p,z_0,\tau-\tau')\mathbf{F}_1^{-}(-p,z_0,z_i,-\tau')\mathrm{d}\tau'. \tag{16}$$

Note that equation (15) covers the entire function $\mathbf{F}_1^{-}(p,z_0,z_i,\tau)$, whereas equation (16) covers the function $\mathbf{F}_1^{+}(-p,z_0,z_i,-\tau)$, except its parts that overlap with $\mathbf{G}^{-,-}(p,z_0,z_i,\tau)$, as illustrated in Figures 3c and 3d. Equations (15) and (16) form a set of two equations with two unknowns, $\mathbf{F}_1^{+}$ and $\mathbf{F}_1^{-}$. Assuming the reflection response $\mathbf{R}(p,z_0,\tau)$ and the parts of $\mathbf{F}_1^{+}(-p,z_0,z_i,-\tau)$ not covered by equation (16) are known, these equations can be solved via an iterative Marchenko scheme.

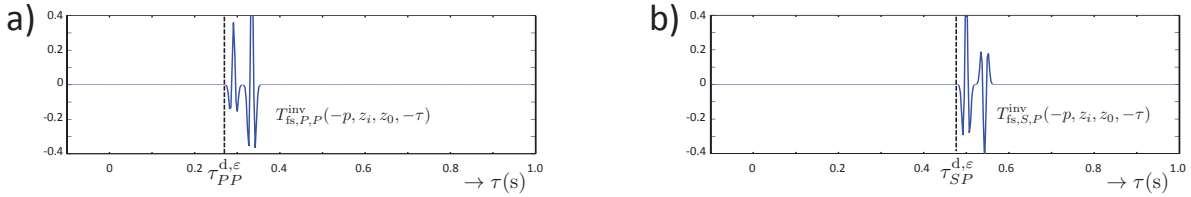

Figure 4 *Inverse of the forward-scattering transmission response. This equals the parts of the focusing functions in Figures 3c,d which overlap the Green's functions.*

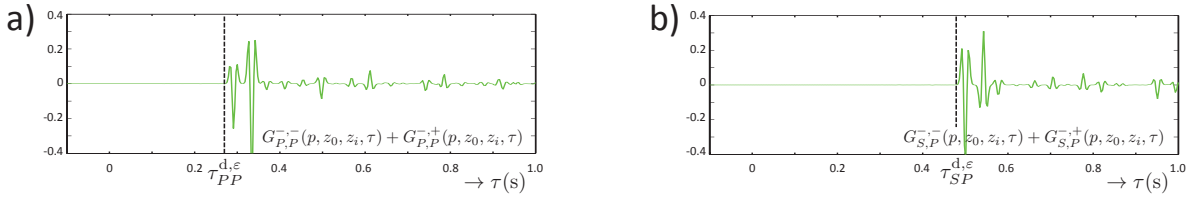

Figure 5 *Retrieved Green's functions after four iterations of the Marchenko scheme.*

We now discuss how to define the parts of $\mathbf{F}_1^+(-p,z_0,z_i,-\tau)$ that overlap the Green's functions. Similar as in the acoustic case, the focusing matrix $\mathbf{F}_1^+(p,z_0,z_i,\tau)$ is the inverse of the transmission response matrix $\mathbf{T}(p,z_i,z_0,\tau)$ of the reference medium (which, between z_0 and z_i, is equal to the actual medium). We write $\mathbf{F}_1^+(p,z_0,z_i,\tau)=\mathbf{T}^{\text{inv}}_{\text{fs}}(p,z_i,z_0,\tau)+\mathbf{M}^+(p,z_0,z_i,\tau)$, where $\mathbf{T}^{\text{inv}}_{\text{fs}}(p,z_i,z_0,\tau)$ is the inverse of the "forward-scattering" transmission response matrix $\mathbf{T}_{\text{fs}}(p,z_i,z_0,\tau)$, i.e., the part of the transmission response that includes direct and forward converted waves, but no internal multiples. $\mathbf{M}^+(p,z_0,z_i,\tau)$ represents the scattering coda (i.e., everything that is not included in $\mathbf{T}^{\text{inv}}_{\text{fs}}(p,z_i,z_0,\tau)$). Figure 4 shows some elements of $\mathbf{T}^{\text{inv}}_{\text{fs}}(-p,z_i,z_0,-\tau)$, which correspond precisely with the overlapping parts in Figures 3c and 3d. Hence, assuming $\mathbf{R}(p,z_0,\tau)$ and $\mathbf{T}^{\text{inv}}_{\text{fs}}(p,z_i,z_0,\tau)$ are known, equations (15) and (16) can be solved for the coda $\mathbf{M}^+$ (of $\mathbf{F}_1^+$) and $\mathbf{F}_1^-$ (Wapenaar and Slob, 2014). Once these are found, the elastodynamic Green's functions follow from equations (11) and (12), see Figure 5.

Conclusions

We have shown that the simple causality conditions of the acoustic single-sided Marchenko scheme do not hold for the elastodynamic situation. For the latter situation the focusing functions and Green's functions exhibit an overlap in a finite time interval. The part of the focusing functions that overlaps with the Green's functions cannot be resolved via the Marchenko method and needs to be specified separately. It is defined as the inverse of the forward-scattered part of the elastodynamic transmission response of the medium. Further research is required to investigate which approximations are allowed in the estimation of the forward-scattered transmission response.

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